Modelling Dependence of Time Series with Copulae

Barbara Choroś

Institut für Statistik and Ökonometrie
CASE - Center for Applied Statistics
and Economics
Humboldt-Universität zu Berlin

Copulae

A copula is a multivariate distribution function defined on the unit cube $[0,1]^d$, with uniformly distributed margins.

$$
\begin{aligned}
\mathrm{P}(X_1 \leq x_1, \ldots, X_n \leq x_d) &= C\left\{\mathrm{P}(X_1 \leq x_1), \ldots, \mathrm{P}(X_d \leq x_d)\right\} \\
&= C\left\{F_1(x_1), \ldots, F_d(x_d)\right\}
\end{aligned}
$$

Bivariate copulae

A *2-dimensional copula* is a function $C : [0,1]^2 \to [0,1]$ with the following properties:

1. For every $u \in [0,1]$, $C(0,u) = C(u,0) = 0$ (**grounded**)
2. For every $u \in [0,1]$, $C(u,1) = u$ and $C(1,u) = u$
3. For every $(u_1,u_2),(v_1,v_2) \in [0,1] \times [0,1]$ with $u_1 \le v_1$ and $u_2 \le v_2$: $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \ge 0$ (**2-increasing**)

Sklar's Theorem

For a distribution function F with marginals $F_{X_1} \dots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \qquad (1)$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \dots, d$. If $F_{X_1}, \dots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \dots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \dots, F_{X_d}$.

Fréchet Copulae

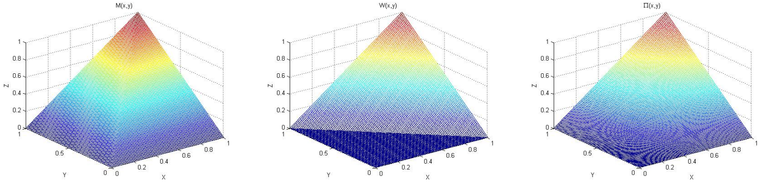
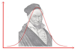


Figure 1: $M(u,v) = \min(u,v)$, $W(u,v) = \max(u+v-1,0)$ and $\Pi(u,v) = uv$

M. Fréchet on BBI: 

Gauss Copula

$$C(u_1, u_2) = \Phi_\rho\{\Phi^{-1}(u_1), \Phi^{-1}(u_2)\}$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} dx\, dy$$

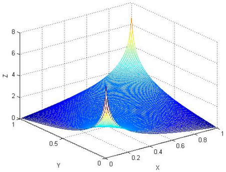
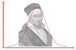


Figure 2: Gauss copula density, parameter $\rho = 0.4$.

C. Gauss on BBI: 

t-Student Copula

$$
\begin{aligned}
C(u_1,u_2) &= t_{\rho,\nu}\{t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\} \\
&= \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2} dx\, dy
\end{aligned}
$$

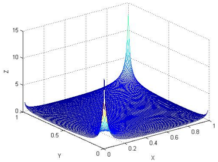
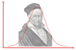


Figure 3: *t*-Student copula density, parameters $\nu = 3$ and $\rho = 0.4$.

W. Gosset on BBI: 

Archimedean Copulae

Archimedean copula:

$$C(u,v) = \psi^{[-1]}\{\psi(u) + \psi(v)\}$$

for some continuous, decreasing and convex ψ, $\psi(1) = 0$.

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t), & 0 \le t \le \psi(0), \\ 0, & \psi(0) < t \le \infty. \end{cases}$$

For $\psi(0) = \infty$: $\psi^{[-1]} = \psi^{-1}$.

Gumbel Copula

$$C(u,v) = \exp\left[-\left\{(-\log u)^{\theta} + (-\log v)^{\theta}\right\}^{\frac{1}{\theta}}\right]$$

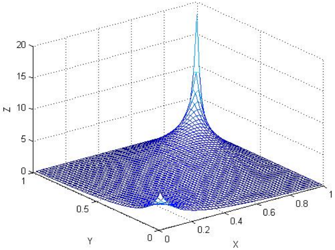
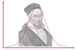


Figure 4: Gumbel copula density, parameter $\theta = 2$.

E. Gumbel on BBI: 

Clayton Copula

$$C(u,v) = \max\left\{(u^{-\theta} + v^{-\theta} - 1)^{\frac{1}{\theta}}, 0\right\}$$

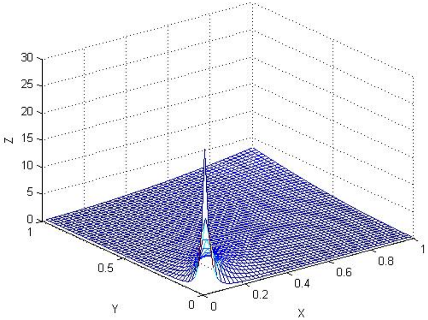


Figure 5: Clayton copula density, parameter $\theta = 2$.

Frank Copula

$$C(u,v) = -\frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right\}$$

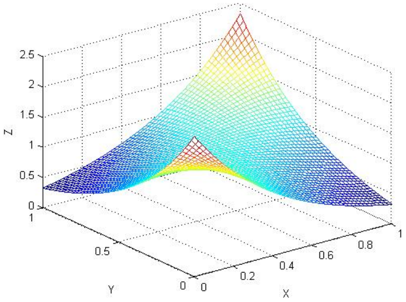


Figure 6: Frank copula density, parameter $\theta = 2$.

Multivariate Elliptical Copulae

- Gauss

$$\int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} |R|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} r^\top R^{-1} r\right) \, dr_1 \dots dr_d,$$

  where $r = (r_1, \dots, r_n)^\top$

- t-Student

$$\int_{-\infty}^{t_\nu^{-1}(u_1)} \dots \int_{-\infty}^{t_\nu^{-1}(u_d)} (2\pi)^{-\frac{d}{2}} |R|^{-\frac{1}{2}} \left(1 + \frac{r^\top R^{-1} r}{\nu}\right)^{-\frac{\nu+n}{2}} \, dr_1 \dots dr_d$$

  where $r = (r_1, \dots, r_n)^\top$

Multivariate Archimedean Copulae

- Gumbel

$$C(u_1,\dots,u_d) = \exp\left[-\left\{(-\log u_1)^\theta + \dots + (-\log u_d)^\theta\right\}^{\frac{1}{\theta}}\right]$$

- Cook-Johnson

$$C(u_1,\dots,u_d) = \left(\sum_{j=1}^{n} u_j^{-\theta} - d + 1\right)^{-\frac{1}{\theta}}$$

- Frank

$$C(u_1,\dots,u_d) = -\frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u_1}-1)\dots(e^{-\theta u_d}-1)}{(e^{-\theta}-1)^{d-1}}\right\}$$

Parameter estimation

- Maximum Likelihood (ML) method
- Method of Inference Functions for Margins (IFM)
- Canonical Maximum Likelihood (CML) method

Inference Functions for Margins

1. Estimating the parameters θ_i, $i = 1, \dots, d$ of the marginal distributions F_i using the ML method

$$\hat{\theta}_i = \arg\max l^i(\theta_i) = \arg\max \sum_{t=1}^{T} \log f_i(x_i^t; \theta_i),$$

where l^i is the log-likelihood function of the marginal distribution F_i with density f_i.

2. Estimating the copula parameters $\boldsymbol{\alpha}$,

$$\hat{\boldsymbol{\alpha}} = \arg\max l^c(\boldsymbol{\alpha}) = \arg\max \sum_{t=1}^{T} \log c(F_1(x_1^t; \theta_1), \dots, F_d(x_d^t; \theta_1); \boldsymbol{\alpha}),$$

where l^c is the log-likelihood function of the copula.

Mixture Copula

Let $C_1, \ldots, C_K$ be d-dimensional copulae and $w_1, \ldots, w_K$ positive weights so that $w_1 + \ldots + w_K = 1$.
Mixture copula:

$$C_{mix}(u_1, \ldots, u_d) = \sum_{i=1}^{K} w_i C_i(u_1, \ldots, u_d).$$

Mixture Copula

Joint density of $(X_1, \ldots, X_d)^\top$

$$f(x_1, \ldots, x_d; \psi) = \sum_{i=1}^{K} w_i f_i(x_1, \ldots, x_d; \theta_i),$$

where $\psi = (w_i, \theta_i)_{i=1}^{K}$ and

$$f_i(x_1, \ldots, x_d; \theta_i) = c_i\{F_1(x_1), \ldots, F_d(x_d); \theta_i\} \prod_{j=1}^{d} g_j(x_j),$$

where g_j, $j = 1, \ldots, d$, are densities of marginal distributions.

Description of the Data Set

1. Warsaw Stock Exchange, http://www.gpw.pl/
2. Polish stocks: KGHM Polska Miedź SA (metallurgical industry), PKN Orlen SA (fuel), Telekomunikacja Polska SA (telecommunications), ComputerLand SA (computer science), Orbis SA (tourism), Bank Pekao SA (bank)
3. 1001 closing prices (1000 log-returns)
4. time period: 02.01.2003 – 18.12.2006

Example

$$C_{mix}(u_1,\dots,u_6) = wC_{Frank}(u_1,\dots,u_6) + (1-w)C_{Cook}(u_1,\dots,u_6),$$

where $\hat{\theta}_{Frank} = 2.5594$, $\hat{\theta}_{Cook} = 1.3063$, $\hat{w} = 0.8156$.

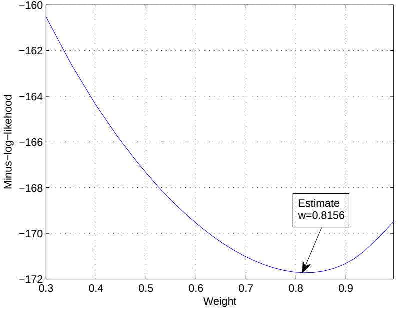


Figure 7: Parameters estimation of mixture copula with Expectation-Maximization method.

General Multivariate Archimedean Copula

Let $\varphi(t) : [0,1]^d \mapsto [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(0) = \infty$ and $\varphi(1) = 0$. Let φ^{-1} denote the inverse of φ. For all $n \geq 2$ the function $C : [0,1]^d \mapsto [0,1]$ given by

$$C(u_1, u_2, \ldots, u_d) = \varphi^{-1}\{\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_d)\}$$

is d-copula if and only if φ^{-1} is completely monotone on $[0,\infty)$.

Hierarchical Archimedean Copulae

A hierarchical Archimedean copula joins two or more ordinary bivariate or higher-dimensional Archimedean copulae by another Archimedean copula.

The fully nested HAC

$$C(u_1,u_2,u_3,u_4) = C_{31}[u_4, C_{21}\{u_3, C_{11}(u_1,u_2)\}]$$
$$= \varphi_{31}^{-1}[\varphi_{31}\{u_4\} + \varphi_{31}\{\varphi_{21}^{-1}(\varphi_{21}[u_3] + \varphi_{21}[\varphi_{11}^{-1}\{\varphi_{11}(u_1) + \varphi_{11}(u_2)\}])\}]$$

The partially nested HAC

$$C(u_1,u_2,u_3,u_4) = C_{21}\{C_{11}(u_1,u_2), C_{12}(u_3,u_4)\}$$
$$= \varphi_{21}^{-1}(\varphi_{21}[\varphi_{11}^{-1}\{\varphi_{11}(u_1) + \varphi_{11}(u_2)\}] + \varphi_{21}[\varphi_{12}^{-1}\{\varphi_{12}(u_3) + \varphi_{12}(u_4)\}])$$

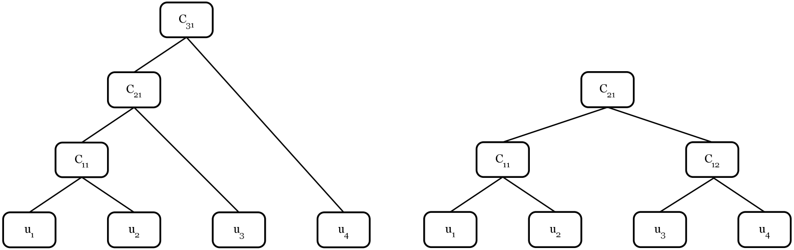


Figure 8: Fully (left) and partially nested (right) HAC.

Risk Measures

1. Value-at-Risk (negative)

$$VaR^X_{1-\alpha} = Q^X_\alpha = -q^{-X}_{1-\alpha},$$

   - $Q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) > \alpha\}$,
   - $q^X_\alpha = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$.

2. Expected Shortfall

$$ES^X_{1-\alpha} = E(X|X < VaR^X_{1-\alpha}).$$

Margins

The process $\{X_t\}_{t=1}^{T}$ of log-returns can be modeled with $GARCH(1,1)$

$$
\begin{aligned}
Y_t &= \mu + \sigma_t \varepsilon_t, \\
\sigma_t^2 &= c_0 + c_1 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + b_1 \sigma_{t-1}^2.
\end{aligned}
$$

The standardized innovations

$$\{\varepsilon_1, \dots, \varepsilon_d\}$$

are independent with distribution function

$$F(\varepsilon_1, \dots, \varepsilon_d) = C\{F_1(\varepsilon_1), \dots, F_d(\varepsilon_d)\},$$

where

- C is copula with parameter θ,
- ε_j have continuous marginal distributions F_j, $j = 1, \dots, d$.

Risk Measures with Copulae

For a sample of log-returns $\{X_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$

1. specification of innovations $\hat{\varepsilon}_{j,t}$ by fitting $GARCH(1,1)$
2. specification of marginal distributions $F_{X_j}(x_j; \delta_j)$
3. specification of copula $C(u_1, \ldots, u_d; \theta)$
4. fit of the copula C
5. generation of n Monte Carlo data $X_{T+1} \sim C\{F_1(x_1), \ldots, F_d(x_d); \hat{\theta}\}$
6. generation of a sample of portfolio profits $L_{T+1}(X_{T+1})$
7. estimation of $\widehat{VaR}_{1-\alpha}$ and $\widehat{ES}_{1-\alpha}$.

Estimation of VaR and ES

$$\widehat{VaR}^L_{1-\alpha} = L_{(\lfloor \alpha n \rfloor + 1):n},$$
$$\widehat{ES}^L_{1-\alpha} = \frac{1}{\lfloor \alpha n \rfloor} \sum_{i=1}^{\lfloor \alpha n \rfloor} L_{i:n},$$

where L is Profit and Loss function

$$L_{t+1} = \sum_{i=1}^{d} S_{i,t}(\exp(X_{i,t+1}) - 1) \tag{2}$$

and $X_{t+1} = \log S_{t+1} - \log S_t$.

Moving Window

- Specify the subsets of size $h = 250$: $\{u_{j,t}\}_{t=s-h+1}^{s}$ for $s = h, \ldots, T$.
- Obtain the sequence $\{\widehat{VaR}^{j}_{1-\alpha}\}_{j=1}^{T-h}$, $\{\widehat{ES}^{j}_{1-\alpha}\}_{j=1}^{T-h}$ and $\{\theta_j\}_{j=1}^{T-h}$.

Backtesting

The estimated VaR values are compared with true realizations $\{L_t\}$ of the Profit and Loss function. An *exceedance* occurs when L_t is smaller than $\widehat{VaR}^{t}_{1-\alpha}$.
The ratio of the number of exceedances to the number of observations gives the *exceedances ratio*:

$$\hat{p} = \frac{1}{T-h} \sum_{t=h+1}^{T} I_{\{L_t < \widehat{VaR}^{t}_{1-\alpha}\}}.$$

For Expected Shortfall we define

$$\hat{q} = \frac{1}{T-h} \sum_{t=h+1}^{T} I_{\{L_t < \widehat{ES}^{t}_{1-\alpha}\}}.$$

Kupiec test

H_0 : the exceedances ratio for Value-at-Risk is equal to α.
Likelihood ratio statistic:

$$LR = 2\log\{(1-\hat{p})^{T-h-N}\hat{p}^N\} - 2\log\{(1-\alpha)^{T-h-N}\alpha^N\} \xrightarrow{\mathcal{L}} \chi^2_1,$$

where N – number of exceedances, $\hat{p}$ – exceedances ratio.
We use 0.05 significance level for the Kupiec LR statistic.

Embrechts, Kaufmann and Patie Measure

$$V^{ES} = \frac{|V_1^{ES}| + |V_2^{ES}|}{2},$$

$$V_1^{ES} = \frac{\sum_{t=h+1}^{T} (L_t - \widehat{ES}_{1-\alpha}^{t}) I_{\{L_t < \widehat{VaR}_{1-\alpha}^{t}\}}}{\sum_{t=h+1}^{T} I_{\{L_t < \widehat{VaR}_{1-\alpha}^{t}\}}}$$

$D_t = L_t - \widehat{ES}_{1-\alpha}^{t}$, D^α – empirical α-quantile $\{D_t\}_{t=h+1}^{T}$.

$$V_2^{ES} = \frac{\sum_{t=h+1}^{T} D_t I_{\{D_t < D^\alpha\}}}{\sum_{t=h+1}^{T} I_{\{D_t < D^\alpha\}}}$$

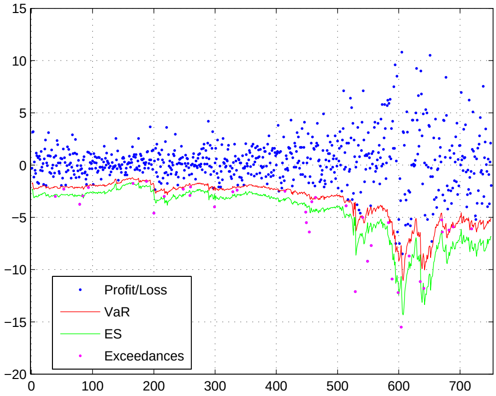


Figure 9: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with Cook-Johnson 3-copula and t-Student residuals.

Backtesting for 3-Copulae Archimedean and Elliptical

3-copula	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
Gaussian Residuals						
Gumbel	51	0.0679	4.5827	26	0.0346	0.9551
Cook-Johnson	38	0.0506	**0.0057**	18	0.0240	0.6727
Frank	42	0.0559	0.5355	20	0.0266	1.0035
Gauss	37	0.0493	0.0085	19	0.0253	0.8377
t-Student	37	0.0493	0.0085	19	0.0253	0.8148
t-Student Residuals						
Gumbel	53	0.0706	5.9666	21	0.0280	0.9022
Cook-Johnson	38	0.0506	**0.0057**	16	0.0213	**0.5331**
Frank	45	0.0599	1.4671	20	0.0266	0.9763
Gauss	40	0.0533	0.1649	19	0.0253	0.7510
t-Student	40	0.0533	0.1649	20	0.0266	0.7172

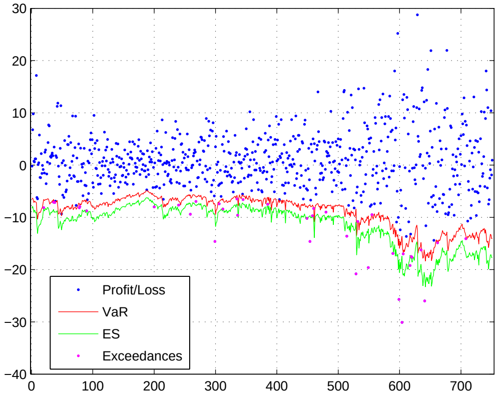


Figure 10: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with t-Student 6-copula and Gaussian residuals.

Backtesting for 6-Copulae Archimedean and Elliptical

6-copula	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
Gaussian Residuals						
Gumbel	48	0.0639	2.8245	28	0.0373	2.1217
Cook-Johnson	39	0.0519	0.0582	17	0.0226	1.0134
Frank	46	0.0613	1.8735	24	0.0320	2.0761
Gauss	38	0.0506	**0.0057**	19	0.0253	1.1093
t-Student	38	0.0506	**0.0057**	17	0.0226	0.9272
t-Student Residuals						
Gumbel	47	0.0626	2.3263	26	0.0346	2.0857
Cook-Johnson	36	0.0479	0.0682	17	0.0226	**0.5207**
Frank	43	0.0573	0.7970	26	0.0346	2.1775
Gauss	37	0.0493	0.0085	17	0.0226	0.9292
t-Student	37	0.0493	0.0085	15	0.0200	0.8213

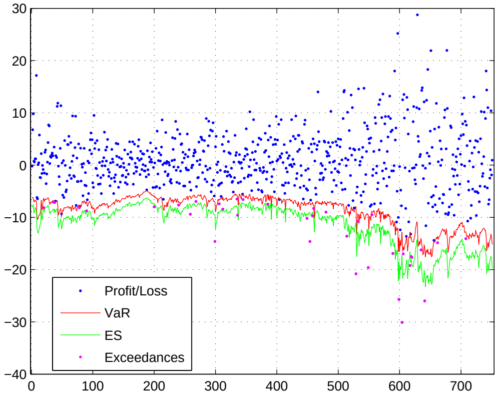


Figure 11: Value at Risk and Expected Shortfall at level $\alpha = 0.05$, P&L and exceedances estimated with mixture copula of Frank 6-copula and Cook-Johnson 6-copula and *t*-Student residuals.

Backtesting for Mixture Copulae

Mixture	N-VaR	$\hat{p}$	LR	N-ES	$\hat{q}$	V^{ES}
3-copula						
Gaussian Residuals						
Frank-CookaJohn.	35	0.0466	0.1863	18	0.0240	0.8709
Gumbel-CookaJohn.	40	0.0533	0.1649	19	0.0253	0.7083
Frank-Gumbel	45	0.0599	1.4671	21	0.0280	0.9460
t-Student Residuals						
Frank-CookaJohn.	37	0.0493	**0.0085**	18	0.0240	0.7194
Gumbel-CookaJohn.	40	0.0533	0.1649	18	0.0240	**0.6122**
Frank-Gumbel	47	0.0626	2.3263	20	0.0266	0.9198
6-copula. Gaussian Residuals						
Frank-CookaJohn.	39	0.0519	0.0582	19	0.0253	1.3383
6-copula. t-Student Residuals						
Frank-CookaJohn.	37	0.0493	**0.0085**	17	0.0226	1.0659

References

X. Chen, Y. Fan, A. Patton
Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates

P. Embrechts, F. Lindskog, A. McNeil
Modelling dependence with copulas and application to risk management

E. Giacomini, W. Härdle, V. Spokoiny
Inhomogeneous Dependency Modelling with Time Varying Copulae
JPES, in print

R. Nelsen
An intoduction to copulas
Springer, 1999

O. Okhrin, Y. Okhrin, W. Schmid
On the structure and estimation of hierarchical Archimedean copulas