Barrier options

Szymon Borak

CASE-Center for Applied
Statistics and Economics
Humboldt-Universität zu Berlin

Barrier options

Barrier are financial options that the payoff depends whether the spot crosses or touches the prespecified barrier.

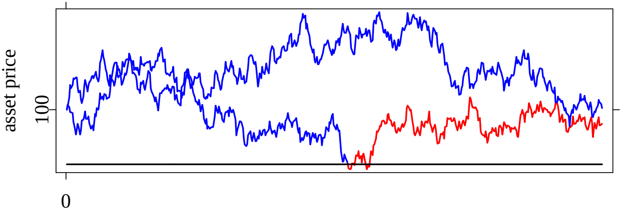
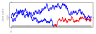


Figure 1: Example of two possible paths of asset's price. When the price hits the barrier (red) the option is no longer valid regardless further evolution of the price.



Barrier options can be classified according to several criteria:

- payoff type - put/call
- knocking type - knock out/knock in
- barrier type - up/down
- relative level of barrier and strike price

Basiswert: DAX 4.130,81 +41,68 +1,02% 11.11.2004 **Java-Applet:** aktiv Neu Starten

WKN	Typ	Bid	Zeit	Ask	Zeit	Strike	StopLoss	Währung	BV	Fälligkeit
SAL60F	Long	2.470	7:05:14 PM	2.490	7:05:14 PM	3.900,00	3.900,00	XXP	0,01	23.12.2004
SAL6UC	Long	2.650	7:05:24 PM	2.670	7:05:24 PM	3.900,00	3.900,00	XXP	0,01	24.03.2005
SAL60G	Long	2.240	7:05:14 PM	2.260	7:05:14 PM	3.925,00	3.925,00	XXP	0,01	23.12.2004
SAL6U9	Long	1.970	7:05:14 PM	1.990	7:05:14 PM	3.950,00	3.950,00	XXP	0,01	23.12.2004
SAL6UA	Long	1.730	7:05:14 PM	1.750	7:05:14 PM	3.975,00	3.975,00	XXP	0,01	23.12.2004
SAL6UB	Long	1.470	7:05:14 PM	1.490	7:05:14 PM	4.000,00	4.000,00	XXP	0,01	23.12.2004
SAL4VM	Short	0.160	7:05:14 PM	0.180	7:05:14 PM	4.150,00	4.150,00	XXP	0,01	23.12.2004
SAL4VN	Short	0.410	7:05:14 PM	0.430	7:05:14 PM	4.175,00	4.175,00	XXP	0,01	23.12.2004
SAL1S6	Short	0.660	7:05:24 PM	0.670	7:05:24 PM	4.200,00	4.200,00	XXP	0,01	23.12.2004
SAL2GN	Short	0.610	7:05:27 PM	0.630	7:05:27 PM	4.200,00	4.200,00	XXP	0,01	24.03.2005

Figure 2: Bid-/Ask information of Sal. Oppenheim's knock-out options

KOTIERUNGSINSERAT

Knock Out Call Optionen auf DAX®
KDAXA

Emittentin	Zürcher Kantonalbank, Bahnhofstrasse 9, 8001 Zürich
Anzahl	25 000 000 Optionen -mit Erhöhungsmöglichkeit-
Ausgabepreis	CHF 1.20 (EUR/CHF 1.5535, Kurs vom 11. August 2005)
Referenzkurs Basiswert **Valor**	EUR 4 955.00 998 032
Ausübungspreis	EUR 4 700.00
Knock Out Preis	EUR 4 700.00
Optionsrecht	400 Optionen berechtigen zur Auszahlung des inneren Wertes der Option basierend auf dem offiziellen Eurex Settlementkurs des DAX® am 16. Dezember 2005 (falls erhältlich; ansonsten der Schlusskurs des DAX® vom 16. Dezember 2005), umgerechnet in CHF. Ein Indexpunkt entspricht EUR 1.00. Sollte während der Laufzeit der des DAX® den Knock Out Preis erreichen oder unterschreiten, verfällt jeglicher Anspruch auf eine Auszahlung und die Optionen verfallen unmittelbar wertlos (Down and Out).
Laufzeit	11. August 2005 bis 16. Dezember 2005, 12.00 Uhr MEZ
Mindestausübungsmenge	400 Optionen oder ein Mehrfaches davon
Ausübungsstil	europäisch
Liberierung	18. August 2005
Symbol **Valor / ISIN**	**KDAXA** 2 248 317 / CH0022483173
Handel **Telefon 044 293 66 65**	Reuters: ZKBWTS Telekurs: 85,ZKB Bloomberg: ZKBW <go> Internet: www.zkb.warrants.ch
Kotierung	wird am Hauptsegment der SWX Swiss Exchange beantragt
Verbriefung	Globalurkunde auf Dauer
Verkaufsrestriktionen	U.S.A. / U.S. persons
Risiko	Abhängig vom Kursverlauf des zugrunde liegenden Index beinhaltet eine Anlage in diese Knock Out Optionen das Risiko, dass der Wert der Optionen abnimmt und die Optionen vorzeitig bei Erreichen des Knock Out Preises wertlos verfallen.

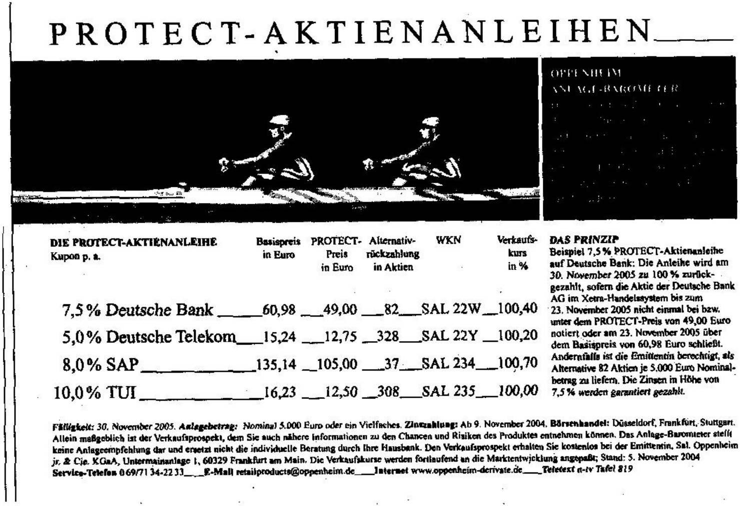
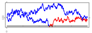


Figure 3: Newspaper advertisement of Sal. Oppenheim's knock-out options (source: Frankfurter Allgemeine Zeitung, November 2004)

Barrier options are cheaper than plain vanilla options since

knock-out + knock-in = vanilla



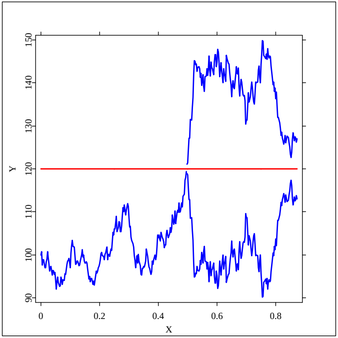
Some of the barrier options have zero value:

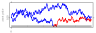
- up-and-out call if the barrier is smaller than strike
- down-and-out put if the barrier is bigger than strike

Black-Scholes model

In the Black-Scholes model the prices of the barrier options can be calculated analytically.

In the proof some properties of the Wiener process (like reflection property) are used:





The price of the barrier call options in Black-Scholes model:

up-and-out call:

$$
\begin{aligned}
C^{uo} = Se^{-d\tau} \Bigg\{ & \Phi\left(y + \sigma\sqrt{\tau}\right) - \Phi\left(y + \sigma\sqrt{\tau} + \log(K/B)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \\
& + \left(\frac{B}{S}\right)^{1+2r/\sigma^2} \Phi\left(y + \sigma\sqrt{\tau} - \log\left(\frac{S^2}{BK}\right)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \\
& - \left(\frac{B}{S}\right)^{1+2r/\sigma^2} \Phi\left(y + \sigma\sqrt{\tau} + \log\left(\frac{K^2}{B^2}\right)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \Bigg\} \\
& - Ke^{-r\tau} \Bigg\{ \Phi(y) - \Phi\left(y + \log(K/B)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \\
& - \left(\frac{B}{S}\right)^{-1+2r/\sigma^2} \Phi\left(y - \log\left(\frac{S^2}{BK}\right)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \\
& + \left(\frac{B}{S}\right)^{-1+2r/\sigma^2} \Phi\left(y + \log\left(\frac{K^2}{B^2}\right)^{\frac{1}{\sigma\sqrt{\tau}}}\right) \Bigg\}
\end{aligned}
$$

where B is a barrier level and other parameters follow the SFM notation

down-and-out call:

if $K > B$

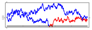
$$C^{do} = Se^{-d\tau}\left\{\Phi\left(y+\sigma\sqrt{\tau}\right) - \left(\frac{B}{S}\right)^{1+2r/\sigma^2}\Phi\left(y+\sigma\sqrt{\tau}+\log(S^2/B^2)^{\frac{1}{\sigma\sqrt{\tau}}}\right)\right\}.$$

$$-Ke^{-r\tau}\left\{\Phi\left(y\right) - \left(\frac{B}{S}\right)^{-1+2r/\sigma^2}\Phi\left(y-\log(S^2/B^2)^{\frac{1}{\sigma\sqrt{\tau}}}\right)\right\}.$$

if $K \leq B$

$$C^{do} = Se^{-d\tau}\left\{\Phi\left(y+\sigma\sqrt{\tau}+\log(K/B)^{\frac{1}{\sigma\sqrt{\tau}}}\right) - \left(\frac{B}{S}\right)^{1+2r/\sigma^2}\Phi\left(y+\sigma\sqrt{\tau}+\log(S^2/B^2)^{\frac{1}{\sigma\sqrt{\tau}}}\right)\right\}.$$

$$-Ke^{-r\tau}\left\{\Phi\left(y+\log(K/B)^{\frac{1}{\sigma\sqrt{\tau}}}\right) - \left(\frac{B}{S}\right)^{-1+2r/\sigma^2}\Phi\left(y-\log(S^2/B^2)^{\frac{1}{\sigma\sqrt{\tau}}}\right)\right\}.$$



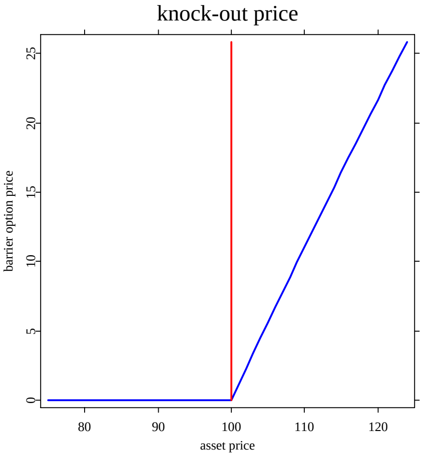


Figure 4: Barrier call option price as a function of the spot price. The strike $K = 100$, the barrier $B = 100$, interest rate $r = 0.05$, time to maturity 0.5 and volatility $\sigma = 0.4$.

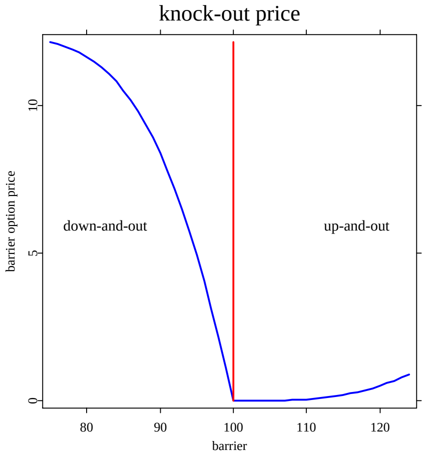
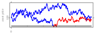


Figure 5: Barrier call option price as a function of the barrier level price. The spot and strike $S = K = 100$, interest rate $r = 0.05$, time to maturity 0.5 and volatility $\sigma = 0.4$.



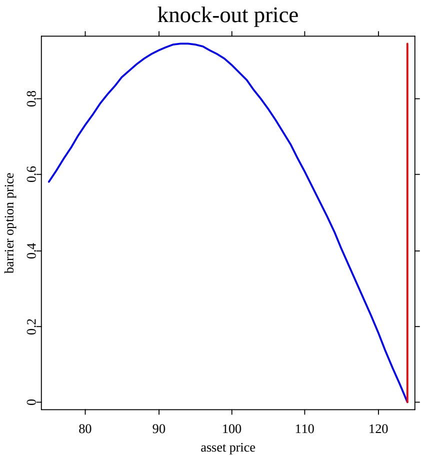
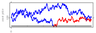


Figure 6: Up-and-out call price as a function of the spot price. The strike $K = 100$, the barrier $B = 124$, interest rate $r = 0.05$, time to maturity 0.5 and volatility $\sigma = 0.4$.



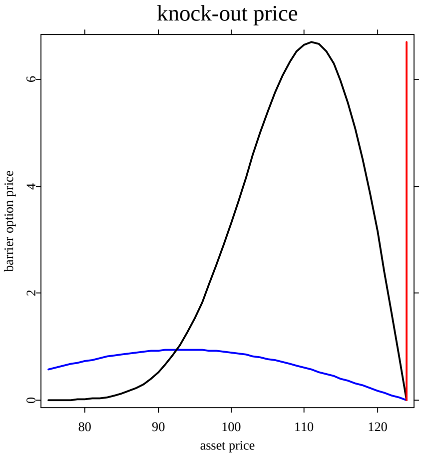


Figure 7: Up-and-out call price as a function of the spot price for the two different times to maturity. The strike $K = 100$, the barrier $B = 124$, interest rate $r = 0.05$, time to maturity $\tau_1 = 0.5$ (blue) $\tau_2 = 0.05$ (black) and volatility $\sigma = 0.4$.

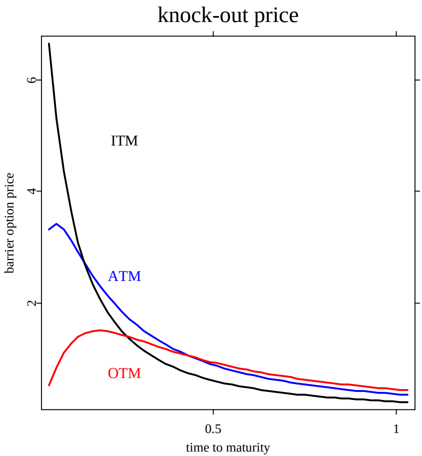


Figure 8: Up-and-out call price as a function of time to maturity for three different contracts. The barrier $B = 124$, the strike $K = 100$, ITM $S = 110$, OTM $S = 90$, interest rate $r = 0.05$, volatility $\sigma = 0.4$.

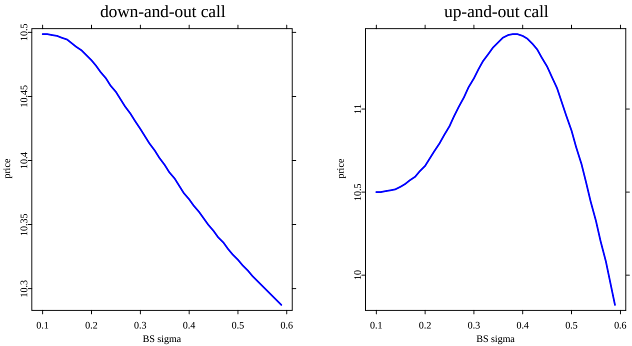
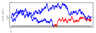


Figure 9: The barrier call option price as a function of BS-σ. Left panel: $B = K = 100$, $S = 110$. Right panel: $B = 150$, $K = 100$, $S = 110$. In both cases interest rate $r = 0.05$, time to maturity $\tau = 0.1$



Hedging

One can distinguish mainly two types of hedging

- **dynamic hedging** - constant rebalancing of the portfolio (eg. BS proof for the price of the European call)
- **static hedging** - "buy and hold" strategy (eg. proof of the put-call parity)

Binomial Models

- In the standard binomial approach the option prices are obtained by backward induction.
- The plain vanilla options can be replicated by the portfolio of stocks and bond which is adjusted in each node. It corresponds to the perfect dynamic hedging.
- Pricing of the barrier options follow the same idea with different barrier condition. The static replication can obtained with the set of plain vanilla options.

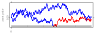
Example: Up-and-out call option

stock price S_t	230.00
exercise price K	210.00
barrier B	269.40
time of expiration τ	0.50
volatility σ	0.25
risk free interest rate r	0.04545
dividend	no
steps	5
Call/Put	European call

Table 1: Explanatory example for pricing the barrier options

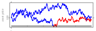
stock price					
					341.51
				315.55	
			291.56		291.56
		269.40		269.40	
	248.92		248.92		248.92
230.00		230.00		230.00	
	212.52		212.52		212.52
		196.36		196.36	
			181.44		181.44
				167.65	
					154.90
0.00	0.10	0.20	0.30	0.40	0.50

Table 2: Stock price tree



stock price	option price					
341.50558						0
315.54682					0	
291.56126				0		0
269.39890			0	——-	0	
248.92117		5.221		10.240		38.921
230.00000	6.727		10.682		20.952	
212.51708		8.351		11.240		2.517
196.36309			6.011		1.275	
181.43700				0.646		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 3: Option prices



steps	5	10	20	50	100	150	Black-Scholes
option price	6.7273	8.1060	6.4094	8.7532	7.2328	7.1742	7.1136

Table 4: Convergence of the binomial barrier option price to the BS price

Static Replication

For the static replication of the barrier options one has to assume that the full set of plain vanilla options is given.
The options can be trade without any additional costs and the fractional position can be taken.
For the static replication one applies the following procedure:

1) Buy an option portfolio which partially replicates the payoff at the expiry ie. long the option with $K = 230$ and $\tau = 0.5$.

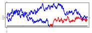
stock price	option price					
341.50558						131.506
315.54682					106.497	
291.56126				83.457		81.561
269.39890			62.237		60.351	
248.92117		44.328		40.818		38.921
230.00000	30.378		26.175		20.951	
212.51708		16.200		11.238		2.517
196.36309			6.010		1.275	
181.43700				0.646		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 5: Option prices

2) Make the value of your portfolio equal to 0 at the barrier notes by using call options with the strike equal to the barrier $K = 269.39$. Firstly short 5.3741 calls with $K = B$ and $\tau = 0.5$

stock price	option price					
341.50558						-387.5
315.54682					-254.57	
291.56126				-158.48		-119.1
269.39890			-95.256		-60.35	
248.92117		-55.842		-30.58		0.000
230.00000	-32.134		-15.496		0.000	
212.51708		-7.8521		0.000		0.000
196.36309			0.000		0.000	
181.43700				0.000		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 6: The value of the short 5.3741 calls



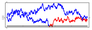
stock price	portfolio value					
341.50558						-256.000
315.54682					-148.07	
291.56126				-75.029		-37.541
269.39890			-33.014		0.000	
248.92117		-11.507		10.240		38.921
230.00000	-1.749		10.683		20.951	
212.51708		8.351		11.238		2.517
196.36309			6.015		1.275	
181.43700				0.646		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 7: The portfolio's value

Secondly long 2.9397 calls with $K = B$ and $\tau = 0.3$

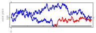
stock price	option price			
291.56126				65.152
269.39890			33.014	
248.92117		16.729		0.000
230.00000	8.4767		0.000	
212.51708		0.000		0.000
196.36309			0.000	
181.43700				0.000
time	0.00	0.10	0.20	0.30

Table 8: The value of the long 2.9397 calls



stock price	portfolio value					
341.50558						-256.000
315.54682					-148.07	
291.56126				-9.877		-37.541
269.39890			0.000		0.000	
248.92117		5.221		10.240		38.921
230.00000	6.7273		10.683		20.951	
212.51708		8.351		11.238		2.517
196.36309			6.015		1.275	
181.43700				0.646		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 9: The portfolio value



Continuous Case

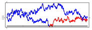
In the continues case the similar approach can be applied.
Firstly buy the option, which partially replicates the payoff at the expiry date.
By buying/selling options with strike equal to the barrier one may set the value on the boundary to zero

One needs to consider two cases:

- The barrier is not hit, the payoff of the exotic is the same like the payoff of the plain vanilla option
- The barrier is hit, the value of the exotic option is zero and the replication portfolio should be zero for the spot equal to barrier

Example: Up-and-out call option

stock price S_t	100.00
exercise price K	100.00
barrier B	120.00
time of expiration τ	1.00
volatility σ	0.25
risk free interest rate r	0.04545
Call/Put	European call



- The value of the barrier option is 0.6927
- Buy the call with strike $K = 100$ and time to maturity $\tau = 1.0$ for 12.108.
- Set the portfolio so that it has a zero value when $\tau_1 = 0.5$ and the spot is equal to the barrier - for example sell 2.3951 call with strike $K = B$ and time to maturity $\tau = 1.00$ for 11.721.

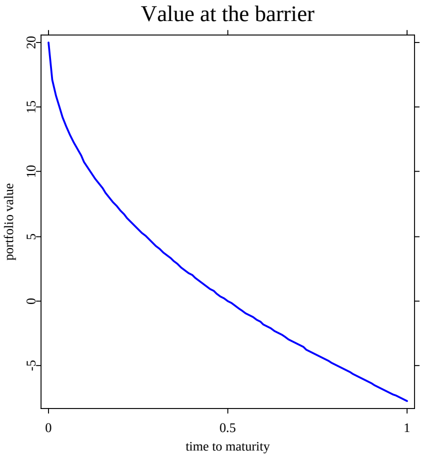


Figure 10: The value of the portfolio at the barrier for as a function of the time to maturity.

More frequent replication could be obtained by using more options with different maturities. As an example one may consider the case when the replication at the barrier is achieved at 3 months, 6 months, 9 months and expiry.

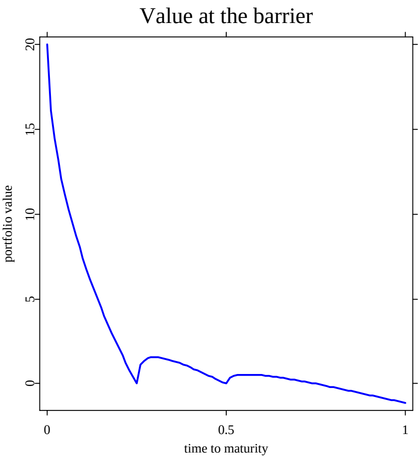


Figure 11: The value of the portfolio at the barrier for as a function of the time to maturity.

Perfect Hedge Example

Example

Consider a short position in a knock-out call option (C^{KO}) with strike 100 and barrier 90. Consider also one long position in a European call with strike 100 and a short position in 100/90 European puts with strike 81.

- if spot is at the barrier level 90 call and put would be worth the same
- if barrier was not reached before maturity the payoff of C^{KO} is equal to the payoff of the call

C^{KO} is replicated with vanilla options.

Position	Value at time t hits barrier	Value at time T doesn't hit barrier
C	$BS_{call}(K = 100)$	$(S_T - 100)^+$
$-100/90P$	$-\frac{100}{90}BS_{put}(K = 81)$	0
$-C^{KO}$	0	$-(S_T - 100)^+$
Sum	0	0

For each time t and each value of σ if $r = 0$ and $S_t = 90$ then $BS_{call}(K = 100) = \frac{100}{90}BS_{put}(K = 81)$