Estimating the quadratic covariation of asynchronously observed Itô processes under microstructure noise

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Model and assumptions

Assumption (efficient log-price processes)

The efficient processes X and Y are Itô–processes on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$:

$$dX_t = \mu_t^X\, dt + \sigma_t^X\, dB_t^X \;,$$
$$dY_t = \mu_t^Y\, dt + \sigma_t^Y\, dB_t^Y \;,$$

with $(\mathfrak{F}_t)$–adapted Standard Brownian Motions B^X and B^Y and $\rho_t = corr\left(B_t^X, B_t^Y\right)$.
The drift processes μ_t^X and μ_t^Y, spot volatilities σ_t^X and σ_t^Y and ρ_t are $(\mathfrak{F}_t)$–adapted, continuous processes.

Aim: Estimation of the quadratic covariation

$$\langle X, Y\rangle_T = \int_0^T \sigma_t^X \sigma_t^Y \rho_t dt \;.$$

Model and assumptions

Assumption (**observation schemes**)

The deterministic sampling times
$\mathcal{T}^{X,n} = \{0 \le t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} \le 1\}$ and
$\mathcal{T}^{Y,m} = \{0 \le \tau_0^{(m)} < \tau_1^{(m)} < \ldots < \tau_m^{(m)} \le 1\}$ are assumed to be regular in the following sense:

$$\delta_n^X := \sup_{i \in \{1,\ldots,n\}} (t_i - t_{i-1}) = \mathcal{O}\left(n^{-1}\right) \tag{1a}$$

$$\delta_m^Y := \sup_{j \in \{1,\ldots,m\}} (\tau_j - \tau_{j-1}) = \mathcal{O}\left(m^{-1}\right) \,. \tag{1b}$$

Furthermore, $t_0 = \mathcal{O}\left(n^{-1}\right)$ and $\tau_0 = \mathcal{O}\left(m^{-1}\right)$ as well as $(1 - t_n) = \mathcal{O}\left(n^{-1}\right)$ and $(1 - \tau_m) = \mathcal{O}\left(m^{-1}\right)$.
Number of observation are of same asymptotic orders: $m \sim n$.

Model and assumptions

Assumption (economic market microstructure noise)

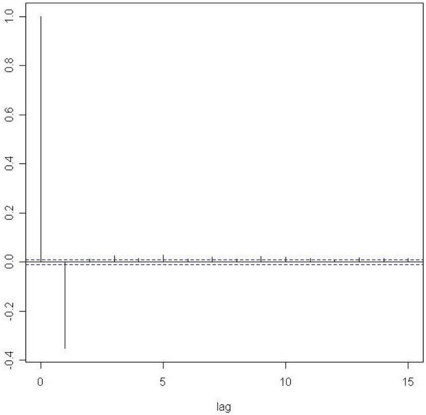
At the sampling times the efficient processes are observed with an additive microstructure noise:

$$\tilde{X}_{t_i} = X_{t_i} + \epsilon_{t_i}^X \; , \; \tilde{Y}_{\tau_j} = Y_{\tau_j} + \epsilon_{\tau_j}^Y \; .$$

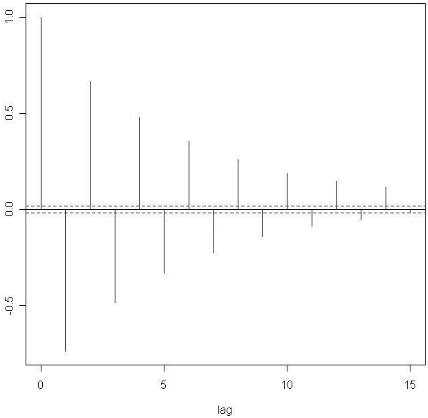
$\epsilon_{t_i}^X$ and $\epsilon_{\tau_j}^Y$ are modelled as i. i. d. centred discrete processes independent to each other and to the efficient processes. We assume $\mathbb{E}\left(\epsilon^X\right)^4, \mathbb{E}\left(\epsilon^Y\right)^4 < \infty$ where $\mathbb{E}\left(\epsilon^X\right)^k := \mathbb{E}\left(\epsilon_{t_i}^X\right)^k$ and $\mathbb{E}\left(\epsilon^Y\right)^k = \mathbb{E}\left(\epsilon_{\tau_j}^Y\right)^k$ analogously.

The noise variances are denoted by

$$\eta_X^2 := \mathbb{E}\left(\epsilon_{t_i}^X\right)^2 \quad \text{and} \quad \eta_Y^2 := \mathbb{E}\left(\epsilon_{\tau_j}\right)^2 \quad .$$



tick-by-tick data S&P 500 02.09.2004.



only transactions that led to price changes for the same data set

A lower bound for the rate of convergence

more informative model: Equidistant synchronous observations of Standard Brownian Motions X and Y with $\rho = corr(B^X, B^Y)$; Gaussian i. i. d. noise:
$\epsilon_{t_i}^X \sim \mathcal{N}\left(0, \eta_X^2\right)$ and $\epsilon_{\tau_j}^Y \sim \mathcal{N}\left(0, \eta_Y^2\right)$, $i = 0, \ldots, N$

Local Asymptotic Normality (LAN) with $N^{-1/4}$-rate:

$$\log\left(\frac{d\mathbb{P}^{2N}_{\rho+N^{-\frac{1}{4}}h_N}}{d\mathbb{P}^{2N}_{\rho}}\right) \xrightarrow{\mathbb{P}^{2N}_{\rho}} hZ\sqrt{I(\rho)} - \frac{h^2 I(\rho)}{2} \qquad (2)$$

for $h_N \to h$ where $Z \sim \mathcal{N}(0,1)$ and Fisher-Information $I(\rho)$.

(convolution + minimax) $\Rightarrow$
For any sequence of estimators ρ_N a lower bound for the rate of convergence is given by $N^{1/4}$.
Parametric efficiency bound (Cramér-Rao):
$\mathbf{AVAR}(\rho_N) \geq I(\rho)^{-1}$.

bounds for the Fisher-information ($\eta_X \geq \eta_Y$):

$$I(\rho) \geq \frac{1}{8\eta_X}\left(\frac{1}{(1+\rho)^{3/2}} + \frac{1}{(1-\rho)^{3/2}}\right) \tag{3a}$$

$$I(\rho) \leq \frac{\sqrt{2}}{8}\frac{1}{\sqrt{\eta_X^2+\eta_Y^2}}\left(\frac{1}{(1+\rho)^{3/2}} + \frac{1}{(1-\rho)^{3/2}}\right) . \tag{3b}$$

(convolution + minimax) $\Rightarrow$
For any sequence of estimators ρ_N a lower bound for the rate of convergence is given by $N^{1/4}$.
Parametric efficiency bound (Cramér-Rao):
$\textbf{AVAR}(\rho_N) \geq I(\rho)^{-1}$.

Fisher-information : for $\eta_X = \eta_Y$:

$$I(\rho) = \frac{1}{8\eta}\left(\frac{1}{(1+\rho)^{3/2}} + \frac{1}{(1-\rho)^{3/2}}\right) . \tag{3c}$$

Outline

Outline

Hayashi-Yoshida-estimator

The Hayashi-Yoshida(HY)-estimator

$$\widehat{\langle X, Y\rangle}_T^{(HY)} = \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta X_{t_i} \Delta Y_{\tau_j} \mathbb{1}_{[\min(t_i,\tau_j)>\max(t_{i-1},\tau_{j-1})]}$$

yields a consistent, asymptotically unbiased and (under certain assumptions) asymptotically normal estimator for the integrated covariance from asynchronous observations in the absence of noise.

"Translation" into an algorithm:

$$\widehat{\langle X, Y\rangle}_T^{(HY)} = \sum_{i=0}^{N} \tilde{X}^{\mathcal{H}^i} \tilde{Y}^{\mathcal{G}^i} = \sum_{i=0}^{N} \left(\sum_{t_j \in \mathcal{H}^i} \Delta \tilde{X}_{t_j} + \sum_{\tau_j \in \mathcal{G}^i} \Delta \tilde{Y}_{\tau_j} \right)$$

$$= \sum_{i=0}^{N} \left(\tilde{X}_{g_i} - \tilde{X}_{l_i}\right)\left(\tilde{Y}_{\gamma_i} - \tilde{Y}_{\lambda_i}\right)$$

pseudo-aggregation

first step:

- for $t_0 < \tau_0$ and $\mu_0 := \min\left(w \in \{1, \ldots, n\} | \tau_0 \leq t_w\}\right)$:

$$\mathcal{H}^0 = \{t_0, \ldots, t_{\mu_0}\} \text{ and } \mathcal{G}^0 = \{\tau_0\}$$

$$q_1 := \begin{cases} \mu_0 + 1 & \text{if } \tau_0 = t_{\mu_0} \\ \mu_0 & \text{if } \tau_0 < t_{\mu_0} \end{cases} \quad \text{and } r_1 := 1$$

- for $t_0 = \tau_0$:

$$\mathcal{H}^0 = \{t_0\} \text{ and } \mathcal{G}^0 = \{\tau_0\}$$

$$q_1 := 1 \quad \text{and} \quad r_1 := 1$$

- for $t_0 > \tau_0$ and $w_0 := \min\left(l \in \{1, \ldots, m\} | t_0 \leq \tau_l\}\right)$:

$$\mathcal{H}^0 = \{t_0\} \text{ and } \mathcal{G}^0 = \{\tau_0, \ldots, \tau_{w_0}\}$$

$$q_1 := 1 \text{ and } r_1 := \begin{cases} w_0 + 1 & \text{if } t_0 = \tau_{w_0} \\ w_0 & \text{if } t_0 < \tau_{w_0} \end{cases}$$

i-th step (given $\mathcal{H}^{i-1}$ and $\mathcal{G}^{i-1}$):

- for $t_{q_i} < \tau_{r_i}$ and $\mu_i := \min\left(w \in \{q_i+1,\ldots,n\} | \tau_{r_i} \le t_w\}\right)$:

$$\mathcal{H}^i = \{t_{q_i},\ldots,t_{\mu_i}\} \text{ and } \mathcal{G}^i = \{\tau_{r_i}\}$$

$$q_i \longrightarrow \begin{cases} q_{i+1} = \mu_i + 1 & \text{if } \tau_{r_i} = t_{\mu_i} \\ q_{i+1} = \mu_i & \text{if } \tau_{r_i} < t_{\mu_i} \end{cases} \quad \text{and } r_i \longrightarrow r_{i+1} = r_i + 1$$

- for $t_{q_i} = \tau_{r_i}$:

$$\mathcal{H}^i = \{t_{q_i}\} \text{ and } \mathcal{G}^i = \{\tau_{r_i}\}$$

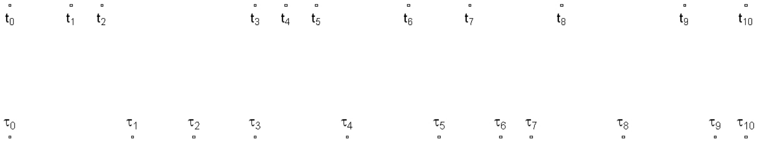
$$q_i \longrightarrow q_{i+1} = q_i + 1 \quad \text{and} \quad r_i \longrightarrow r_{i+1} = r_i + 1$$

- for $t_{q_i} > \tau_{r_i}$ and $w_i := \min\left(l \in \{r_i+1,\ldots,m\} | t_{q_i} \le \tau_l\}\right)$:

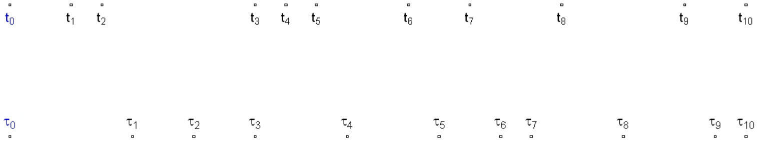
$$\mathcal{H}^i = \{t_{q_i}\} \text{ and } \mathcal{G}^i = \{\tau_{r_i},\ldots,\tau_{w_i}\}$$

$$q_i \longrightarrow q_{i+1} = q_i + 1 \text{ and } r_i \longrightarrow \begin{cases} r_{i+1} = w_i + 1 & \text{if } t_{q_i} = \tau_{w_i} \\ r_{i+1} = w_i & \text{if } t_{q_i} < \tau_{w_i} \end{cases}$$

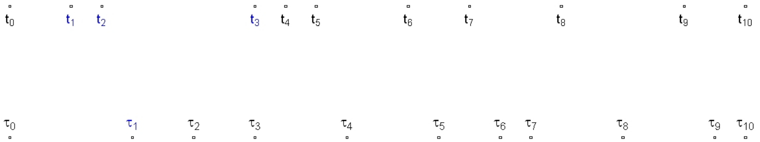
Example for synchronization



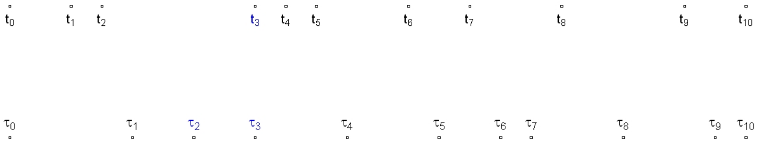
$n=m=10; N=8$



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$	$\mathcal{G}^2$	$\mathcal{G}^3$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$	$\mathcal{G}^3$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$



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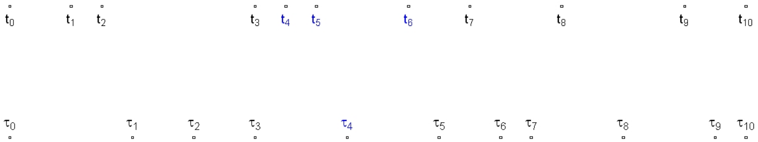

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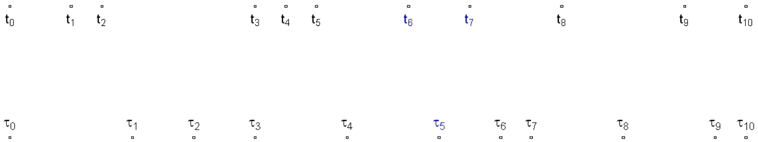
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$



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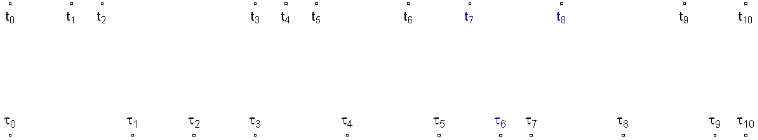

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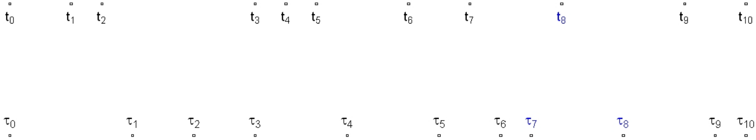
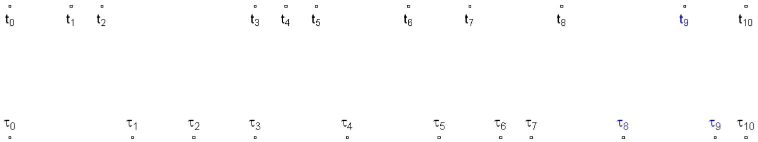
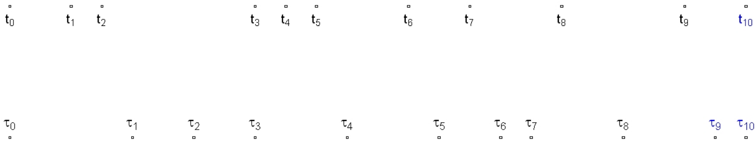
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$



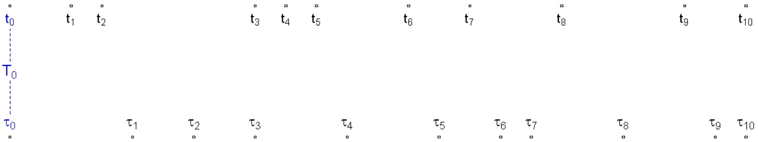
$\mathcal{H}^0$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\{t_0\}$	$\{t_1, t_2, t_3\}$	$\{t_3\}$	$\{t_4, t_5, t_6\}$	$\{t_6, t_7\}$	$\{t_7, t_8\}$	$\{t_8\}$		
$\mathcal{G}^0$	$\mathcal{G}^1$	$\mathcal{G}^2$	$\mathcal{G}^3$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$
$\{\tau_0\}$	$\{\tau_1\}$	$\{\tau_2, \tau_3\}$	$\{\tau_4\}$	$\{\tau_5\}$	$\{\tau_6\}$	$\{\tau_7, \tau_8\}$		



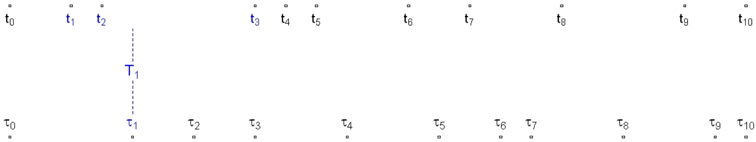
$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$ $\{t_9\}$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$ $\{\tau_7, \tau_8\}$	$\mathcal{G}^7$ $\{\tau_8, \tau_9\}$	$\mathcal{G}^8$



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$ $\{t_9\}$	$\mathcal{H}^8$ $\{t_{10}\}$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$ $\{\tau_7, \tau_8\}$	$\mathcal{G}^7$ $\{\tau_8, \tau_9\}$	$\mathcal{G}^8$ $\{\tau_9, \tau_{10}\}$



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$	$\mathcal{G}^2$	$\mathcal{G}^3$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$
T_0 $t_0 = \tau_0$	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
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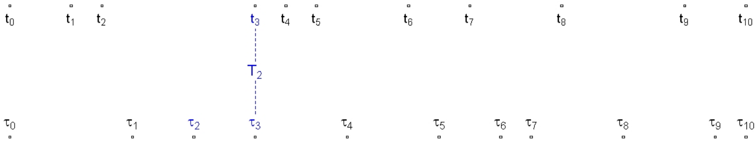
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8



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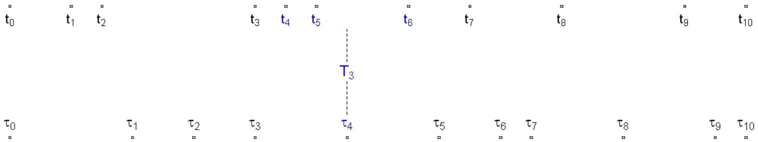
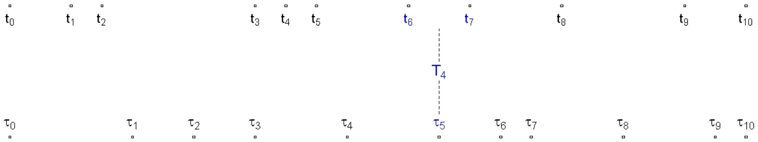
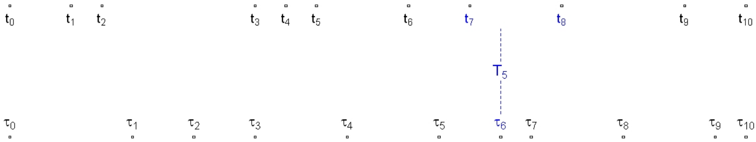
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3	T_4	T_5	T_6	T_7	T_8



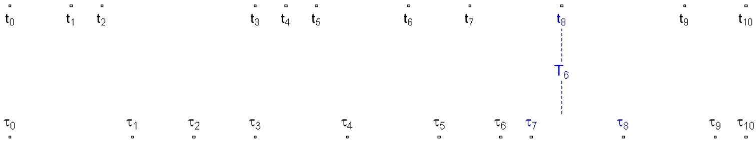
$\mathcal{H}^0$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\{t_0\}$	$\{t_1, t_2, t_3\}$	$\{t_3\}$	$\{t_4, t_5, t_6\}$					
$\mathcal{G}^0$	$\mathcal{G}^1$	$\mathcal{G}^2$	$\mathcal{G}^3$	$\mathcal{G}^4$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$
$\{\tau_0\}$	$\{\tau_1\}$	$\{\tau_2, \tau_3\}$	$\{\tau_4\}$					
T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
$t_0 = \tau_0$	τ_1	$t_3 = \tau_3$	τ_4					



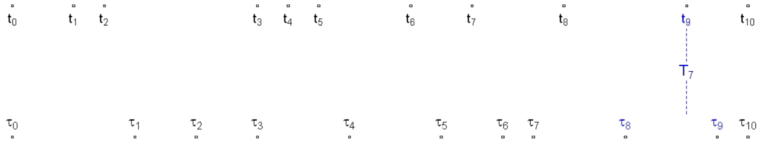
$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5	T_6	T_7	T_8



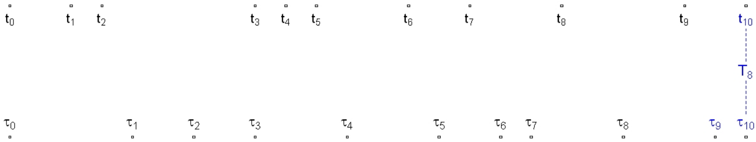
$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$	$\mathcal{G}^7$	$\mathcal{G}^8$
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5 τ_6	T_6	T_7	T_8



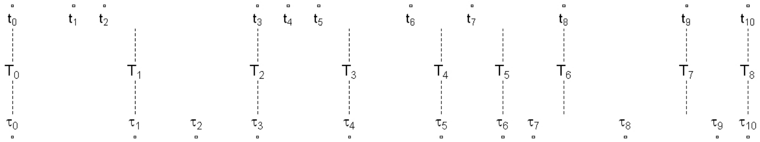
$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$ $\{\tau_7, \tau_8\}$	$\mathcal{G}^7$	$\mathcal{G}^8$
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5 τ_6	T_6 t_8	T_7	T_8



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$ $\{t_9\}$	$\mathcal{H}^8$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$ $\{\tau_7, \tau_8\}$	$\mathcal{G}^7$ $\{\tau_8, \tau_9\}$	$\mathcal{G}^8$
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5 τ_6	T_6 t_8	T_7 t_9	T_8



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$ $\{t_9\}$	$\mathcal{H}^8$ $\{t_{10}\}$
$\mathcal{G}^0$ $\{\tau_0\}$	$\mathcal{G}^1$ $\{\tau_1\}$	$\mathcal{G}^2$ $\{\tau_2, \tau_3\}$	$\mathcal{G}^3$ $\{\tau_4\}$	$\mathcal{G}^4$ $\{\tau_5\}$	$\mathcal{G}^5$ $\{\tau_6\}$	$\mathcal{G}^6$ $\{\tau_7, \tau_8\}$	$\mathcal{G}^7$ $\{\tau_8, \tau_9\}$	$\mathcal{G}^8$ $\{\tau_9, \tau_{10}\}$
T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5 τ_6	T_6 t_8	T_7 t_9	T_8 $t_{10} = \tau_{10}$



$\mathcal{H}^0$ $\{t_0\}$	$\mathcal{H}^1$ $\{t_1, t_2, t_3\}$	$\mathcal{H}^2$ $\{t_3\}$	$\mathcal{H}^3$ $\{t_4, t_5, t_6\}$	$\mathcal{H}^4$ $\{t_6, t_7\}$	$\mathcal{H}^5$ $\{t_7, t_8\}$	$\mathcal{H}^6$ $\{t_8\}$	$\mathcal{H}^7$ $\{t_9\}$	$\mathcal{H}^8$ $\{t_{10}\}$
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T_0 $t_0 = \tau_0$	T_1 τ_1	T_2 $t_3 = \tau_3$	T_3 τ_4	T_4 τ_5	T_5 τ_6	T_6 t_8	T_7 t_9	T_8 $t_{10} = \tau_{10}$

Central limit theory

Proposition

The set of refresh times $\mathcal{T}^{syn} = \{T_0, \ldots, T_N\}$ with $T_i := \min(g_i, \gamma_i),\ i = 0, \ldots, N$ defines a partition of $[0, T]$.
The refresh times are related to pseudo-aggregation as follows:

$$T_i = \min(g_i, \gamma_i) = \max(l_{i+1}, \lambda_{i+1}),\ i = 1, \ldots, N-1 \tag{4}$$

and $\delta_N^{syn} := \sup_{i\in\{1,\ldots,N\}} (T_i - T_{i-1}) = \mathcal{O}\left(N^{-1}\right)$.

Central limit theory

Split the error due to discretization

$$\sum_{i=0}^{N}\left(X_{g_i}-X_{l_i}\right)\left(Y_{\gamma_i}-Y_{\lambda_i}\right)-\int_0^1 \rho_t\sigma_t^X\sigma_t^Y\,dt=D_T^N+A_T^N$$

in a discretization error of the closest synchronous approximation

$$D_T^N:=\sum_{i=1}^{N}\left(X_{T_i}-X_{T_{i-1}}\right)\left(Y_{T_i}-Y_{T_{i-1}}\right)-\int_{T_{i-1}}^{T_i}\rho_t\sigma_t^X\sigma_t^Y\,dt$$
$$-\int_0^{t_0\wedge\tau_0}\rho_t\sigma_t^X\sigma_t^Y\,dt-\int_{t_n\wedge\tau_m}^{T}\rho_t\sigma_t^X\sigma_t^Y\,dt$$

and an error due to asynchronicity

$$A_T^N=\sum_{i=1}^{N}\left(Y_{\gamma_i}-Y_{\lambda_i}\right)\left(\left(X_{g_i}-X_{T_i}\right)\mathbb{1}_{\{T_i=\gamma_i\}}+\left(X_{T_{i-1}}-X_{l_i}\right)\mathbb{1}_{\{T_{i-1}\}=\lambda_i}\right)$$
$$+\sum_{i=1}^{N}\left(X_{g_i}-X_{l_i}\right)\left(\left(Y_{\gamma_i}-Y_{T_i}\right)\mathbb{1}_{\{T_i=g_i\}}+\left(Y_{T_{i-1}}-Y_{\lambda_i}\right)\mathbb{1}_{\{T_{i-1}=l_i\}}\right)$$

Asymptotic quadratic variation of refresh times

Definition

The sequence of functions

$$G^N(t) = \frac{N}{T} \sum_{T_i^{(N)} \leq T} \left(\Delta T_i^{(N)} \right)^2 \, , t \in [0, T]$$

is called the sequence of quadratic variations of refresh times.

Asymptotic quadratic variation of refresh times

Assumption

We assume for $T_i^{(N)}$ and the sequence of quadratic variations of refresh times the following:

1. $G^N(t) \to G(t)$ for $N \to \infty$, with $G(t)$ continuous differentiable.
2. $\frac{G^N(t+h_N)-G^N(t)}{h_N} \to G'(t)$ uniformly in $N \to \infty$ and $h_N = \mathcal{O}\left(N^{-1}\right)$.

Crucial assumption for central limit theory

Assumption

The observation schemes have to tend to "an asymptotic degree of asynchronicity", so that the sequence of variances of A_T^N converges.

$$\left(\mathcal{T}^{X,n},\mathcal{T}^{Y,m}\right)\rightarrow\left(\mathcal{T}^{X,as},\mathcal{T}^{Y,as}\right)\quad\text{for}\quad n,m\rightarrow\infty$$

or equivalently

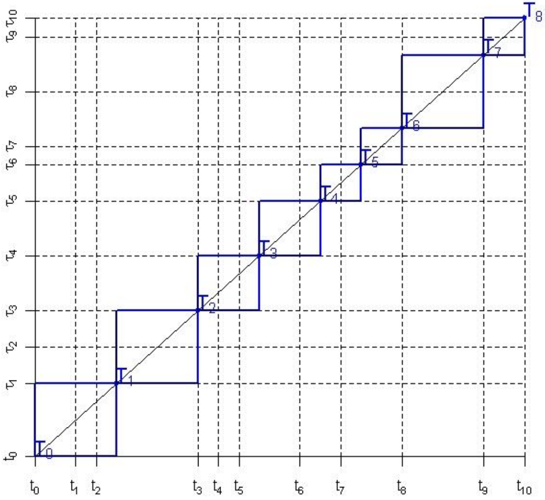
$$\bigcup_i\left(g_i^{(N)}-T_i^{(N)},\gamma_i^{(N)}-T_i^{(N)},T_{i-1}^{(N)}-l_i^{(N)},T_{i-1}^{(N)}-\lambda_i^{(N)}\right)\rightarrow$$
$$\bigcup_i\left(g_i^{(as)}-T_i^{(as)},\gamma_i^{(as)}-T_i^{(as)},T_{i-1}^{(as)}-l_i^{(as)},T_{i-1}^{(as)}-\lambda_i^{(as)}\right)\quad\text{for}\quad N\rightarrow\infty.$$

Central limit theory

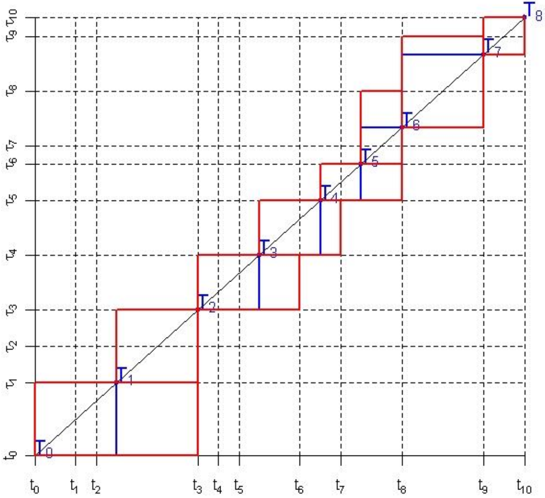
$$\sqrt{N}\left(D_T^N - \langle X, Y\rangle_T\right) \xrightarrow{\mathcal{L}_{st}} \mathbf{N}\left(0\,,\;T\int_0^T G'(t)\left(\sigma_t^X\sigma_t^Y\right)^2(\rho_t^2+1)\right)\;. \tag{5}$$

$$\sqrt{N}A_T^N \xrightarrow{\mathcal{L}_{st}} \mathbf{N}\left(0\,,\;T\int_0^T c_{as}G'(t)\left(\sigma_t^X\sigma_t^Y\right)^2\right)\;. \tag{6}$$

Synchronous refresh time approximation



Synchronous refresh time approximation



Outline

One-scale subsampling estimator

$$\widehat{\langle X, Y\rangle}_T^{sub} = \frac{1}{K_N}\sum_{i=K_N-1}^{N}\left(\tilde{X}_{g_i} - \tilde{X}_{l_{i-K_N+1}}\right)\left(\tilde{Y}_{\gamma_i} - \tilde{Y}_{\lambda_{i-K_N+1}}\right) .$$

$$\mathbb{V}\text{ar}\widehat{\langle X, Y\rangle}_T^{sub} = \frac{1}{K_N^2}\sum_{i,j}\left(\mathbb{C}\text{ov}\left(\textcircled{e}_i, \textcircled{e}_j\right) + \mathbb{C}\text{ov}\left(\textcircled{m}_i, \textcircled{m}_j\right) + \mathbb{C}\text{ov}\left(\textcircled{\nu}_i, \textcircled{\nu}_j\right) + \mathbb{C}\text{ov}\left(\textcircled{n}_i, \textcircled{n}_j\right)\right)$$

with 4 uncorrelated addends:

$$\textcircled{e}_i = \left(X_{g_i} - X_{l_{i-K_N+1}}\right)\left(Y_{\gamma_i} - Y_{\lambda_{i-K_N+1}}\right) \qquad \textcircled{m}_i = \left(X_{g_i} - X_{l_{i-K_N+1}}\right)\left(\epsilon^Y_{\gamma_i} - \epsilon^Y_{\lambda_{i-K_N+1}}\right)$$

$$\textcircled{\nu}_i = \left(\epsilon^X_{g_i} - \epsilon^X_{l_{i-K_N+1}}\right)\left(Y_{\gamma_i} - Y_{\lambda_{i-K_N+1}}\right) \qquad \textcircled{n}_i = \left(\epsilon^X_{g_i} - \epsilon^X_{l_{i-K_N+1}}\right)\left(\epsilon^Y_{\gamma_i} - \epsilon^Y_{\lambda_{i-K_N+1}}\right)$$

$$\Rightarrow \mathbb{V}\text{ar}\left(\widehat{\langle X, Y\rangle}_T^{sub}\right) = \mathcal{O}\left(\frac{K_N}{N} + \frac{\eta_X^2}{K_N} + \frac{\eta_Y^2}{K_N} + \frac{\eta_X^2\eta_Y^2 N}{K_N^2}\right)$$

One-scale subsampling estimator

- The overall asymptotic variance is minimized for $K_N = c_{sub}\, N^{2/3}$.
- No bias-correction is needed as in the univariate case (Two Scales Realized Volatility).

Illustration:
Multi Scale Realized Covariance (MSRC)

$$\widehat{\langle X,Y\rangle}_T^{mult} = \sum_{i=1}^{M_N} \alpha_{i,N} \widehat{\langle X,Y\rangle}_T^{sub,i} \quad , \quad (M_N > 2).$$

$K_i = i$							$\widehat{\langle X,Y\rangle}_T^{sub,i}$
1	(l_0, g_0),	(l_1, g_1),	$\ldots$	$\ldots$		$\ldots$	$\widehat{\langle X,Y\rangle}_T^{sub,1}$
2	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	$(\mathbf{l_2}, \mathbf{g_2})$,	(l_3, g_3),	$\ldots$	$\ldots$	$\widehat{\langle X,Y\rangle}_T^{sub,2}$
3	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	$(\mathrm{l}_2, \mathrm{g}_2)$,	$(\mathbf{l_3}, \mathbf{g_3})$,	(l_4, g_4),	$\ldots$	$\widehat{\langle X,Y\rangle}_T^{sub,3}$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
M_N							$\widehat{\langle X,Y\rangle}_T^{sub,M_N}$

Illustration:

Multi Scale Realized Covariance (MSRC)

$$\widehat{\langle X, Y\rangle}_T^{mult} = \sum_{i=1}^{M_N} \alpha_{i,N} \widehat{\langle X, Y\rangle}_T^{sub,i} \quad , \quad (M_N > 2).$$

$K_i = i$							$\widehat{\langle X, Y\rangle}_T^{sub,i}$
1	(l_0, g_0),	(l_1, g_1),	$\ldots$	$\ldots$		$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,1}$
2	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	$(\mathbf{l_2}, \mathbf{g_2})$,	(l_3, g_3),	$\ldots$	$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,2}$
3	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	$(\mathrm{l}_2, \mathrm{g}_2)$,	$(\mathbf{l_3}, \mathbf{g_3})$,	(l_4, g_4),	$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,3}$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
M_N							$\widehat{\langle X, Y\rangle}_T^{sub,M_N}$

Illustration:
Multi Scale Realized Covariance (MSRC)

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$K_i = i$							$\widehat{\langle X, Y\rangle}_T^{sub,i}$
1	(l_0, g_0),	(l_1, g_1),	$\ldots$	$\ldots$		$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,1}$
2	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	$(\mathbf{l_2}, \mathbf{g_2})$,	(l_3, g_3),	$\ldots$	$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,2}$
3	$(\mathbf{l_0}, \mathbf{g_0})$,	(l_1, g_1),	(l_2, g_2),	$(\mathbf{l_3}, \mathbf{g_3})$,	(l_4, g_4),	$\ldots$	$\widehat{\langle X, Y\rangle}_T^{sub,3}$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
$\vdots$				$\vdots$		$\vdots$	$\vdots$
M_N							$\widehat{\langle X, Y\rangle}_T^{sub,M_N}$

A generalized multi-scale approach:

$$\widehat{\langle X,Y\rangle}_T^{mult} = \sum_{i=1}^{M_N} \frac{\alpha_{i,N}}{i} \sum_{j=i-1}^{N} \left(\tilde{X}_{g_j} - \tilde{X}_{l_{j-i+1}}\right)\left(\tilde{Y}_{\gamma_j} - \tilde{Y}_{\lambda_{j-i+1}}\right)$$

with $\sum \alpha_i = 1$. Analysis of the overall variance:

$$\widehat{\langle X,Y\rangle}_T^{mult} - \langle X,Y\rangle_T = \text{error due to noise} + \text{discretization error}\,.$$

$$\text{discretization error} = \text{synchronous approximation} + \text{error due to asynchronicity}\,.$$

$$\text{error due to noise} = \sum_{i=1}^{M_N} \frac{\alpha_{i,N}}{i} \sum_{j=i-1}^{N} \left(\epsilon^X_{g_j} - \epsilon^X_{l_{j-i+1}}\right)\left(\epsilon^Y_{\gamma_j} - \epsilon^Y_{\lambda_{j-i+1}}\right) + \text{cross terms}\,.$$

Choosing the weights optimally

The leading term in the error due to noise is minimized for

$$\alpha_{i,N}{}^{opt} = \left(\frac{12i^2}{(M^3-M)} - \frac{6i}{(M^2-1)} - \frac{6i}{(M^3-M)}\right) \tag{7}$$
$$= \frac{12i^2}{M^3} - \frac{6i}{M^2}\left(1+\mathcal{O}(1)\right) \; .$$

Theorem

The generalized multi-scale estimator,

$$\widehat{\langle X\,,\,Y\rangle}_T^{mult} = \sum_{i=1}^{M_N} \alpha_{i,N}{}^{opt} \sum_{j=i-1}^{N} \left(\tilde{X}_{g_j} - \tilde{X}_{l_{j-i+1}}\right)\left(\tilde{Y}_{\gamma_j} - \tilde{Y}_{\lambda_{j-i+1}}\right)$$

converges for $M_N = c\cdot\sqrt{N}$ stably in law to a mixed Gaussian:

$$N^{1/4}\left(\widehat{\langle X\,,\,Y\rangle}_T^{mult} - \langle X\,,\,Y\rangle_T\right) \xrightarrow{\mathcal{L}_{st}} \mathbf{N}\left(0\,,\,\eta^2_{mult}\right) \tag{8}$$

with asymptotic variance

$$\mathbf{AVAR}_{mult} = \frac{(24+12\kappa_1)}{c^3}\eta_X^2\eta_Y^2 + \frac{13}{35}cT\int_0^T(\kappa_2+\rho_t^2)G'(t)(\sigma_t^X\sigma_t^Y)^2dt + \frac{12}{5}\frac{\eta_X^2\eta_Y^2}{c} + \frac{12}{5}\frac{(1+\kappa_3)}{c}\left(\eta_X^2\langle Y\rangle_T + \eta_Y^2\langle X\rangle_T\right)\,,$$

where $0\le\kappa_1\le 1$, $1\le\kappa_2\le 2$ and $0\le\kappa_3<2$.

Proposition

The one-scale subsampling estimator with $i = c \cdot N^{2/3}$

$$\widehat{\langle X, Y\rangle}_T^{sub} = \frac{1}{i} \sum_{j=i-1}^{N} \left(\tilde{X}_{g_j} - \tilde{X}_{l_{j-i+1}}\right)\left(\tilde{Y}_{\gamma_j} - \tilde{Y}_{\lambda_{j-i+1}}\right)$$

converges with rate $N^{1/6}$ stably in law:

$$N^{1/6}\left(\widehat{\langle X, Y\rangle}_T^{sub} - \langle X, Y\rangle_T\right) \xrightarrow{\mathcal{L}_{st}} \mathbf{N}\left(0\,,\,\eta_{sub}^2\right) \tag{9}$$

with the asymptotic variance

$$\mathbf{AVAR}_{sub} = \frac{4\eta_X^2\eta_Y^2}{c^2} + \frac{1}{3}cT\int_0^T (\kappa + \rho_t^2)G'(t)(\sigma_t^X\sigma_t^Y)^2 dt\,,$$

where $1 \leq \kappa \leq 2$.

Outline

Methods to deal with noise contamination

There exist three related methods to deal with the influence of noise (for univariate or multivariate synchronous data):

- one-scale subsampling estimator and multi-scale estimator(s) (Mykland, Zhang, Aït-Sahalia, Palandri)
- pre-averaged estimators (Jacod, Podolskij, Kinnebrock, Christensen, Vetter)
- kernel estimators (Barndorff-Nielsen, Hansen, Lunde, Shephard)

3 methods:

method	technique for noise
1	generalized multi-scale approach
2	kernel estimator
3	pre-averaging
method	synchronization
1	pseudo–aggregation (HY)
2	refresh times and previous–tick
3	–/refresh times /HY

BNHLS kernel estimator converges with $N^{1/5}$-rate and assures positive semi-definiteness with

$$\textbf{AVAR} = 2c\left(\int_0^1 \mathfrak{K}^2(x)dx\right) T\int_0^T g^2\left((1+\rho_s^2)(\sigma_s^X\sigma_s^Y)^2 ds\right)\circ fdu\,.$$

For synchronous observations with optimal rate.
The pre-average estimator for synchronous, equidistant observations converges with optimal rate and

$$\textbf{AVAR} = \frac{2}{\psi_2^2}\Big(\frac{151}{80640}\theta T\int_0^T (1+\rho_s^2)(\sigma_s^X\sigma_s^Y)^2 ds$$
$$+\frac{1}{96\theta}T\int_0^T \left((\sigma_s^X\eta_Y)^2+(\sigma_s^Y\eta_X)^2+2\eta_{XY}\rho_s\sigma_s^X\sigma_s^Y\right)ds$$
$$+\frac{1}{6\theta^3}\left(\eta_X^2\eta_Y^2+\eta_X^2\eta_Y^2\eta_{XY}^2\right)\Big)\,.$$

There is a version that assures positive semi-definiteness with $N^{1/5}$-rate.
There is a rate-optimal Hayashi-Yoshida version (without CLT).

Outline

Simulation model

Constant parameters $\sigma^X = \sigma^Y = 1$, $\rho = 0.5$, $\mu^X = \mu^Y = 0$ and set $T = 1$.
Sampling times: two independent Poisson–processes with $\mathbb{E}n = 30000 = \mathbb{E}m$ (then $\mathbb{E}N = 20000$)
Gaussian i. i. d. noise
estimators in the absence of noise (50 MC-iterations):

	mean	sd
Hayashi-Yoshida/pseudo-aggregation	0,506	0,037
refresh time realised covariance	0,336	0,037

$\Rightarrow$ Realised Covariance using the previous-tick refresh-time method is heavily biased downwards

Choosing the subsampling and multi-scale frequencies

Minimizing the MSEs for $K_N = c_{sub}N^{2/3}$ and $M_N = c_{multi}\sqrt{N}$ leads to:

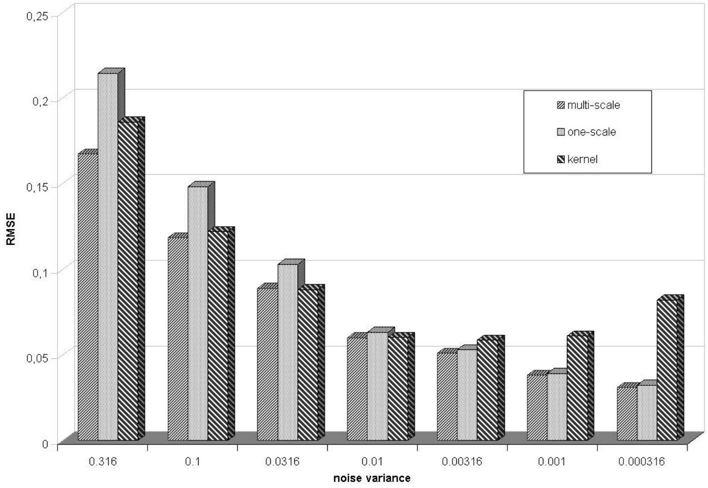
$$c_{sub}^{opt} \approx \sqrt[3]{\frac{24\eta_X^2\eta_Y^2}{\int_0^1 (1+\rho_s^2)(\sigma_s^X\sigma_s^Y)^2 dt}} = \sqrt[3]{\frac{96}{5}\eta_X^2\eta_Y^2}$$

$$c_{multi}^{opt} \approx \sqrt[4]{\frac{2520\,\eta_X^2\eta_Y^2}{52\int_0^1 (1+\rho_s^2)(\sigma_s^X\sigma_s^Y)^2 dt}} = \sqrt[4]{\frac{504\eta_X^2\eta_Y^2}{13}}\,.$$

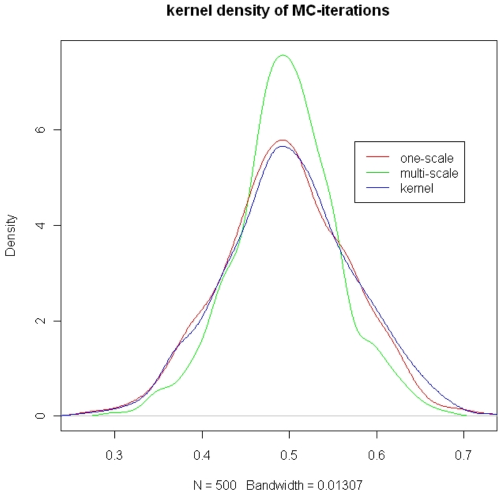
Choosing the subsampling and multi-scale frequencies

noise level $\eta_X^2 = \eta_Y^2$	K_N	M_N
$(1/\sqrt{10})$	646	216
0.1	300	122
$(1/\sqrt{10})\cdot 0.1$	139	68
0.01	65	38
$(1/\sqrt{10})\cdot 0.01$	30	22
0.001	14	12
$(1/\sqrt{10})\cdot 0.001$	6	7
0.0001	3	4

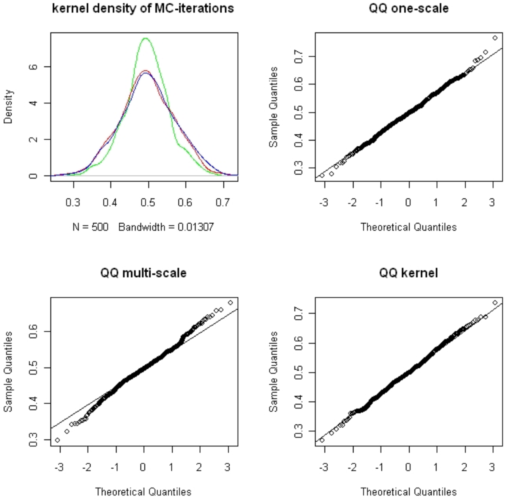
Comparison of RMSEs for different noise levels



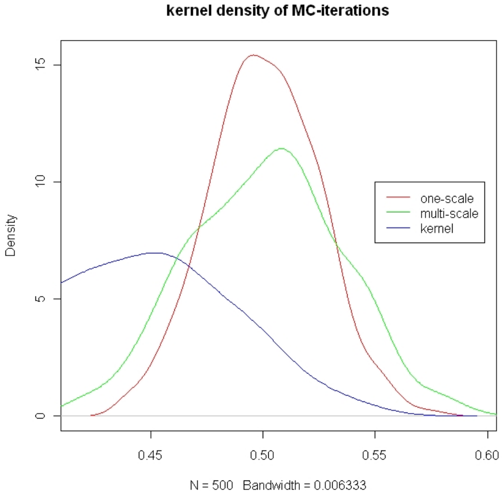
Empirical distribution $\rho = 1/2$, $\eta_X = \eta_Y = 0.1$



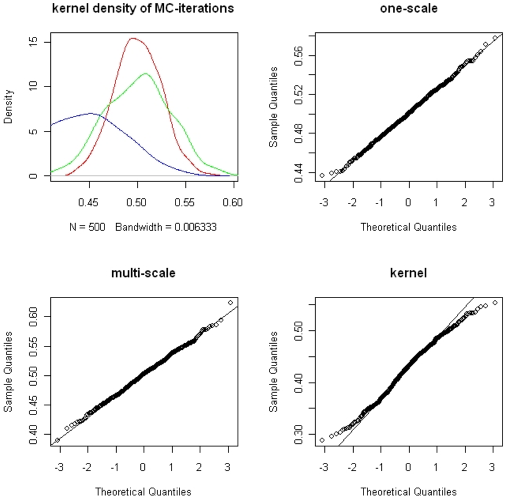
Empirical distribution $\rho = 1/2$, $\eta_X = \eta_Y = 0.1$



Empirical distribution $\rho = 1/2$, $\eta_X = \eta_Y = 0.01$



Empirical distribution $\rho = 1/2$, $\eta_X = \eta_Y = 0.01$



Outline

further tasks

- jump diffusion / general semi-martingale / Lévy-process models for the efficient processes
- non-i. i. d. noise and relaxing other assumptions
- random sampling schemes (in general, others than independent Poisson-sampling)
- unifying theory for different estimation methods

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Thank you for your attention!