Understanding the Risk of CDOs

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CDO Dynamics

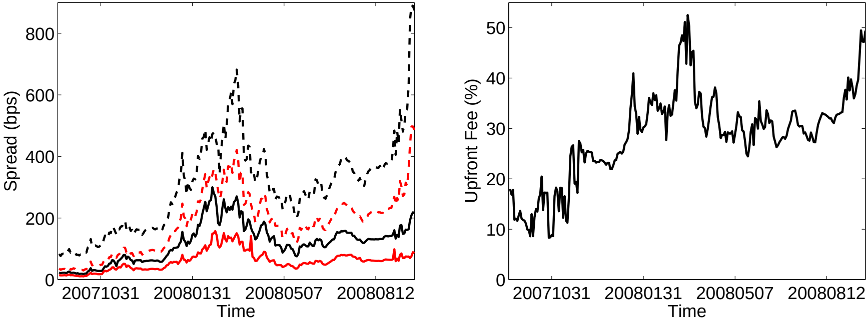


Figure 1: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.

Dependence Matters!

The normal world is not enough.

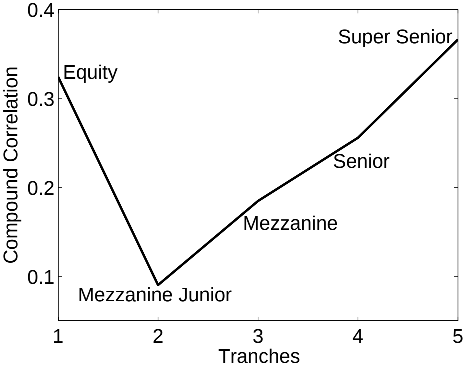


Figure 2: Gaussian one factor model with constant correlation. Data from 20071022.

Hierarchical Archimedean Copula (HAC)

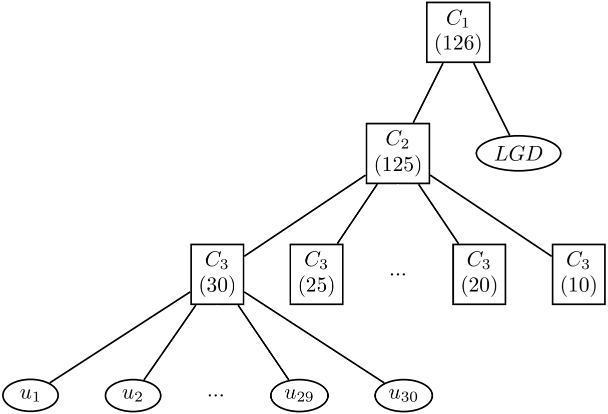


Figure 3: HAC with the random loss given default and the sector structure.

CDO and HAC: Results

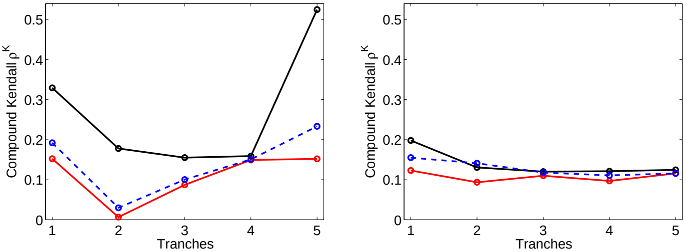
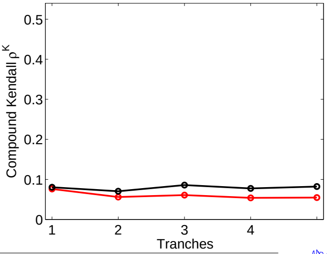


Figure 4: Implied compound Kendall's ρ^K. Upper panel: copula models with the constant LGD, Gaussian (left) and Gumbel (right). Lower panel: copula model with the random LGD, without the industry structure. Two-parameter models (ρ_1^K, $\rho_\mathbf{2}^\mathbf{K}$), one-parameter models (ρ_1^K).



Dependence Modelling

- How to reduce the space dimension?
- How to model varying dependency?
- How to extend the static approach into a dynamic approach?

Outline

1. Motivation ✓
2. CDO Problem Review
3. Copula Models
4. Base Correlations

Cash-Flow Structure

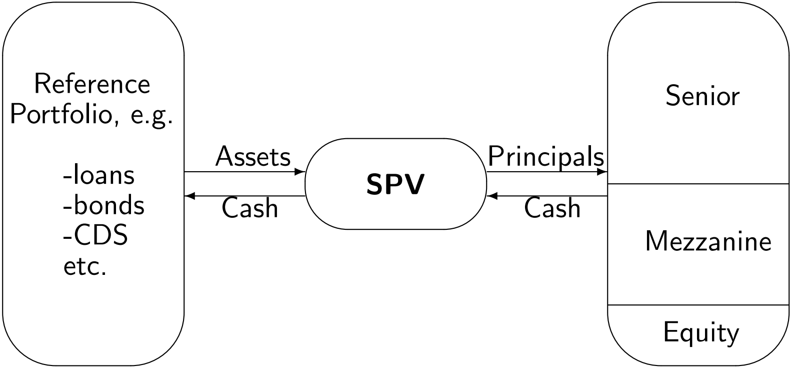


Figure 5: Illustration of a CDO cash flow mechanism.

iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10)
- Maturities: 5Y, 7Y, 10Y

Notation

t	time (in years) passed since CDS, CDO were originated
t_0	date of the trade
T	maturity
Δt	time between t and the nearest preceding payment day
M	initial value of contracts
$\beta(t_0, t)$	discount factor from t to t_0
$s_i(t_0)$	spread of the ith CDS, $i = 1, \ldots, d$
R_i	recovery rate of the ith CDS
$s_j(t_0)$	the premium of tranche j, $i = 1, \ldots, J$
u_j	upper attachment point of tranche j
l_j	lower attachment point of tranche j

Portfolio Loss

- Loss variable of i-th firm until $t \in [t_0, T]$

$$\Gamma_i(t) = \mathbf{1}(\tau_i < t)$$

- Portfolio loss process

$$L(t) = \frac{1}{d} \sum_{i=1}^{d} (1 - R_i)\Gamma_i(t), \quad t \in [t_0, T]. \tag{1}$$

Loss of the Tranche

- Loss of the tranche j at time t

$$
\begin{aligned}
L_j(t) &= \min\{\max(0, L(t) - l_j); u_j - l_j\} \quad (2) \\
&= \begin{cases} 0, & L(t) < l_j, \\ L(t) - l_j, & l_j \le L(t) \le u_j, \\ u_j - l_j, & L(t) > u_j. \end{cases}
\end{aligned}
$$

- Outstanding notional of tranche j

$$F_j(t) = (u_j - l_j) - L_j(t),\ j = 1, \ldots, J. \quad (3)$$

Valuation of CDO

1. Fixed (premium) leg – the payments tranche holders receive

$$PL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) s_j(t_0) \Delta t \, \mathsf{E}\{F_j(t) + F_j(t - \Delta t)\} M/2 \quad (4)$$

2. Floating (protection) leg – the payments tranche holders pay

$$DL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) \, \mathsf{E}\{L_j(t) - L_j(t - \Delta t)\} M \quad (5)$$

Premium

The premium $s_j(t_0)$ is chosen in such a way that (4) and (5) are equal:

$$PL_j(t_0) = DL_j(t_0) \text{ for } j = 2, \ldots, J.$$

This leads to:

$$s_j(t_0) = \frac{\sum_{t=t_1}^{T} \beta(t_0,t)\,\mathrm{E}\{L_j(t) - L_j(t-\Delta t)\}}{\sum_{t=t_1}^{T} \beta(t_0,t)\Delta t\,\mathrm{E}\{F_j(t) + F_j(t-\Delta t)\}/2}. \tag{6}$$

We observe this in the market!

Default Times

Time to default variable τ_i of the ith company, $i = 1, \ldots, d$,

$$F_i(t) = \mathrm{P}(\tau \leq t)$$

Intensity model

Default probability:

$$p_i(t) = F_i(t) = 1 - \exp\left\{ - \int_0^t \lambda_i(u) du \right\}. \tag{7}$$

Survival probability:

$$\bar{p}_i(t) = 1 - p_i(t) = \exp\left\{ - \int_0^t \lambda_i(u) du \right\},$$

where $\lambda_i(t)$ is the intensity function.

Probability of Default

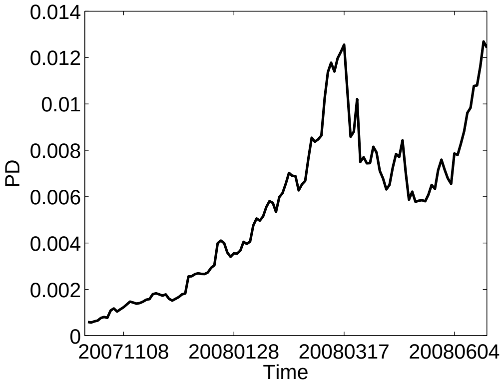


Figure 6: Probabilities of default (7) with constant intensity of Deutsche Bank, time period 20071022-20080630.

Copula Model

The ith obligor survives until t if and only if

$$U_i \le \bar{p}_i(t),$$

where U_i is a $\mathrm{U}[0,1]$ rv called a trigger.
Hence the time to default variable is

$$\tau_i = \inf\{t \ge t_0 : \bar{p}_i(t) \le U_i\}. \tag{8}$$

The joint and marginal distributions of the triggers satisfy:

$$\begin{aligned} C\{\bar{p}_1(t), \ldots, \bar{p}_d(t)\} &= \mathrm{P}\{U_1 \le \bar{p}_1(t), \ldots, U_d \le \bar{p}_d(t)\}, \\ \mathrm{P}\{U_i \le \bar{p}_i(t)\} &= \bar{p}_i(t). \end{aligned}$$

Gaussian 2-Factor Model, 3-parameters

$$
\Sigma = \left(\begin{array}{ccccc}
\boxed{\begin{matrix}1 & \cdots & \rho_2 \\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}} & \begin{matrix}\rho_1 & \cdots & \cdots \\ \vdots & & \\ \rho_1 & & \end{matrix} & & & \begin{matrix}\cdots & \cdots & \rho_1 \\ & & \vdots \\ & & \vdots\end{matrix} \\
\begin{matrix}\rho_1 & \cdots & \rho_1 \\ \vdots & & \\ \vdots & & \end{matrix} & \boxed{\begin{matrix}1 & \cdots & \rho_2 \\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}} & & & \\
 & & \ddots & & \\
 & & & \boxed{\begin{matrix}1 & \cdots & \rho_3 \\ & \ddots & \\ \rho_3 & \cdots & 1\end{matrix}} & \begin{matrix} & & \vdots \\ & & \vdots \\ \rho_1 & \cdots & \rho_1\end{matrix} \\
\begin{matrix}\vdots & & \\ \vdots & & \\ \rho_1 & \cdots & \cdots\end{matrix} & & & \begin{matrix} & & \rho_1 \\ & & \vdots \\ \cdots & \cdots & \rho_1\end{matrix} & \boxed{\begin{matrix}1 & \cdots & \rho_3 \\ & \ddots & \\ \rho_3 & \cdots & 1\end{matrix}}
\end{array}\right)
$$

HAC Model

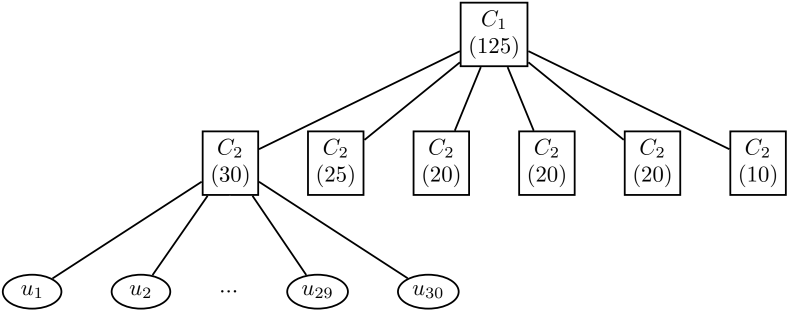
Partially nested hierarchical Archimedean copula:

$$C(u_1, \ldots, u_d; \theta) = C_1\{ C_2(u_1, \ldots, u_{m_1}; \theta_2), C_2(u_{m_1+1}, \ldots, u_{m_1+m_2}; \theta_2), \ldots, C_2(u_{m_1+\ldots+m_5+1}, \ldots, u_d; \theta_2); \theta_1\},$$

where $\theta = (\theta_1, \theta_2)^\top$ and m_k, $k = 1, \ldots, 6$, indicates a number of the companies in kth industry sector.

- C_2 models the dependency in the industry sector,
- C_1 models the dependency between the industry sectors.

HAC Model for CDO



Monte-Carlo Simulations

- estimate intensities $\lambda_i, i = 1, \ldots, d$ from CDS spreads
- generate N samples of random numbers $(U_1, \ldots, U_d) \sim C$, where C is a copula
- compute times of default τ_i (8)
- compute portfolio's loss process L_t (1) and for each tranche $L_{t,j}$ (2), $F_{t,j}$ (3)
- calculate the model spread s_j^c (6) for $j = 1, \ldots, 5$.

Calibration

Minimise the cumulative relative deviations of the model spreads s_j^c from the market spreads s_j^m with respect to the copula parameters θ:

$$D(t_0) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \frac{|s_j^c(t_0) - s_j^m(t_0)|}{s_j^m(t_0)} \rightarrow \min \text{ w.r.t. } \theta, \tag{9}$$

where s_1^m denotes the market UFF and s_j^m, $j = 2, \ldots, J$, are the market spreads.

Types of Correlation

- **Compound correlation** of a given tranche j makes the tranche spread computed by the model equal to its observed market value.
- **Base correlation**
  Loss $L_{(l_j,u_j)}$ of a tranche (l_j, u_j) can be represented as a difference between losses of the two fictive equity tranches $(0, u_j)$ and $(0, l_j)$:

  $$\mathsf{E}\{L_{(l_j,u_j)}\} = \mathsf{E}\{L_{(0,u_j)}\} - \mathsf{E}\{L_{(0,l_j)}\},\ j = 2, \ldots, J.$$

  Fix a single correlation to price the $(0, l_j)$ tranche, then look for a second correlation to price the $(0, u_j)$ tranche such that the spread difference is consistent with the observed (l_j, u_j) tranche spread.

Base Correlations on the Market

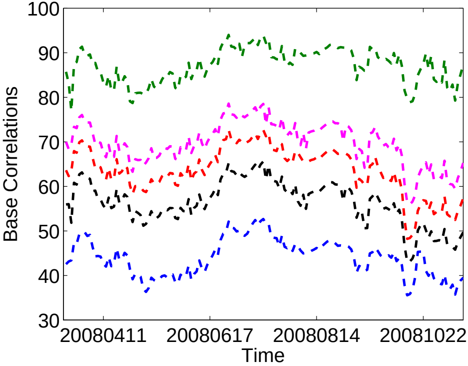


Figure 7: Base correlations implied from tranche: 1, **2**, 3, 4, 5. Series 9, 20080321-20081113.

Homogeneous Gaussian One Factor Model

For each CDS, $i = 1, \ldots, d$, define:

$$X_i = \sqrt{\rho}Y + \sqrt{1-\rho}Z_i,$$

where Y (systematic risk factor), $\{Z_i\}_{i=1}^d$ (idiosyncratic risk factors) are i.i.d. $N(0,1)$. Hence:

$$(X_1, \ldots, X_d)^\top \sim N(0, \Sigma),$$

with

$$\Sigma = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}$$

Which model is used by the market to imply base correlations (BCs)?

Compare

- Homogeneous large pool Gaussian copula (HLPGC) model,
- One-factor Gaussian copula model with the sequence of different intensity parameters $\{\lambda_i\}_{i=1}^d$

for series 9, 20080321-20081113, $r = 0$, $R = 40\%$.

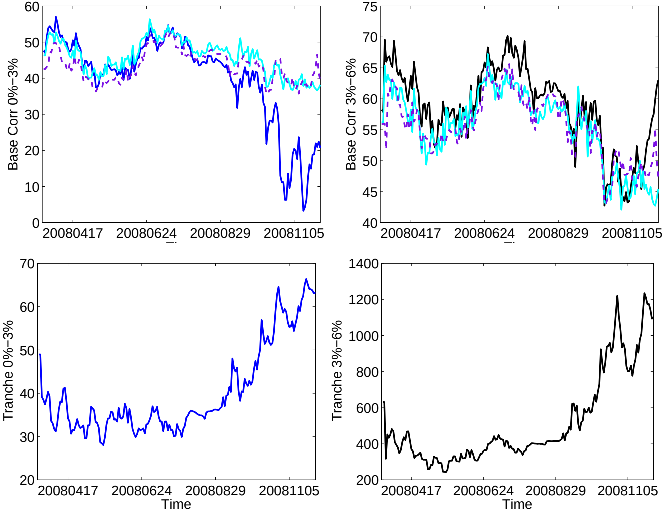


Figure 8: Tranche BCs (up) and UFF/spreads (down). Implied BSs: 0% − 3%, **3%** − **6%**, HLPGC, market.

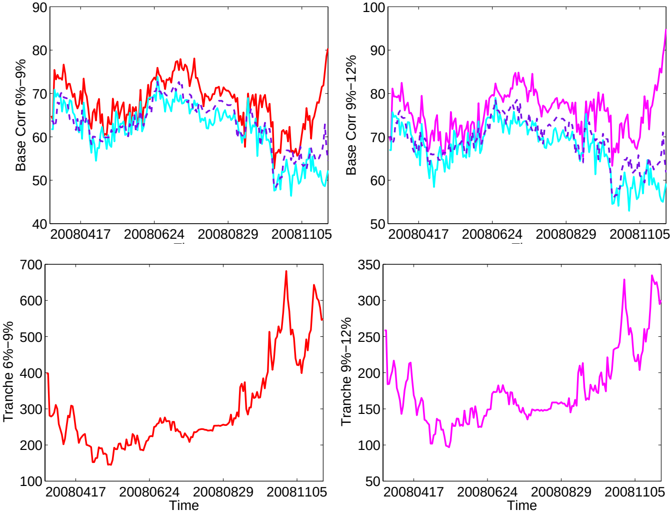


Figure 9: Tranche BCs (up) and spreads (down). Implied BSs: 6% – 9%, 9% – 12%, HLPGC, market.

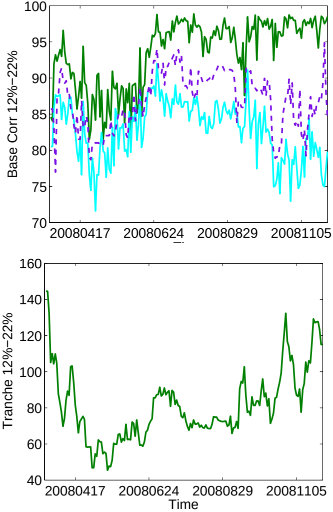


Figure 10: Tranche BCs (up) and spreads (down). Implied BSs: 12% − 22%, HLPGC, market.

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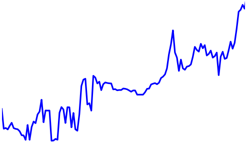

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