Adaptive Interest Rate Modeling

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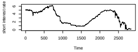
Interest Rate

- Essential for pricing bonds and interest rate derivatives.
- A signal of macroeconomic activity.
- Influenced by macroeconomic variables.
- Follow unstable dynamic process.



Classical One-factor Short Rate Models

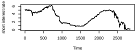
Vasicek Model

$$dr(t) = a\{b - r(t)\}dt + \sigma dW_t$$

CIR Model

$$dr(t) = a\{b - r(t)\}dt + \sigma \sqrt{r(t)} dW_t$$

- $r_t \geq 0$
- Mean reversion



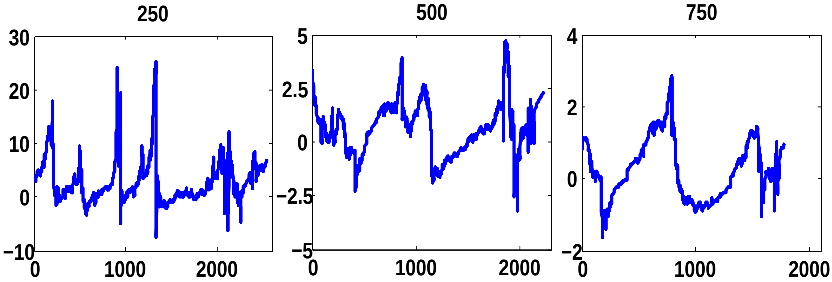
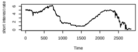


Figure 1: Moving window estimator $\hat{a}$ with window sizes 250, 500 and 750



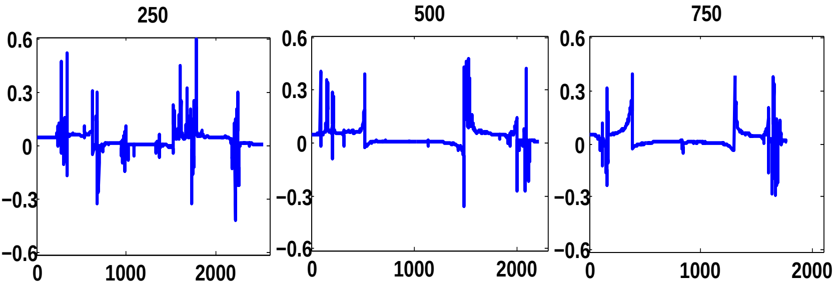
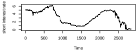


Figure 2: Moving window estimator $\hat{b}$ with window sizes 250, 500 and 750



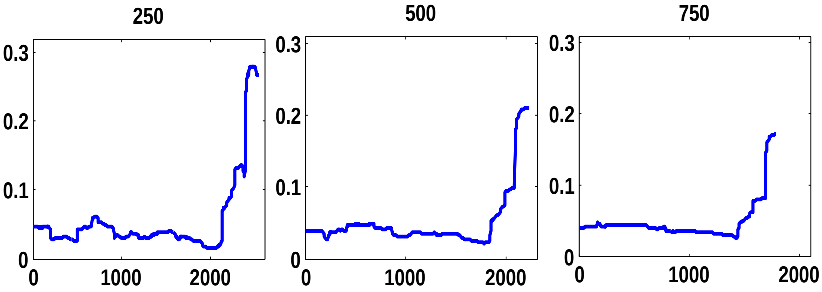
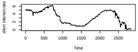


Figure 3: Moving window estimator $\hat{\sigma}$ with window sizes 250, 500 and 750

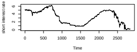


Extended Short Rate Models

Two types of extended interest rate models:

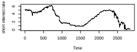
- Jump-Diffusion Models
- Regime-Switching Models

Nonparametric method to test the coefficients in CIR model.



Outline

1. Motivation ✓
2. Local Adaptive Method for Interest Rate
3. Empirical Study
4. Conclusion



Model

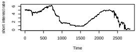
The time-varying interest rate model with parameter set $\theta_t = (a_t, b_t, \sigma_t)^\top$:

$$dr(t) = a_t\{b_t - r(t)\}dt + \sigma_t\sqrt{r(t)}dW_t \tag{1}$$

Discretization:

$$r_{t+\Delta t} - r_t = a_t\{b_t - r_t\}\Delta t + \sigma_t\sqrt{r_t}\varepsilon_t \tag{2}$$

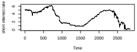
$\{\varepsilon_t\} \sim N(0, \Delta t)$



Local Change Point Method

Given time point T, go back and split time series into K intervals, $I_0 \subset I_1 \subset \cdots \subset I_K$, with $I_k = [T - m_k + 1, T]$.

- Accept the smallest interval I_0 without change point (i.e. homogeneous interval).
- Sequentially check the historical intervals to search the change point.



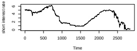
Local Change Point Method

Goal: Find an unknown change point τ closest to time point T.

1. Determine $\tilde{\theta}_0$.
2. Increase interval to I_k, $k \geq 1$. Get $\tilde{\theta}_{I_k}$.
3. Compare test statistics with critical value. If test statistic is accepted go to step 4, otherwise go to step 5.
4. Let $\hat{\theta} = \tilde{\theta}_{I_k}$, and set $k = k + 1$, repeat step 2.
5. Detect the change point τ in I_k, $I^* = I_{k-1}$ without change point.

Why we use LCP?

- Capture the time-varying parameters in interest rate models.
- Allow for structural breaks and jumps in parameter values.
- Detect the change points in the dynamic process.



Test Statistic

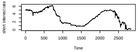
Test statistics $\mathfrak{T}_{I_{k+1},t}$:

$$\mathfrak{T}_{I_{k+1},t} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_{I_{k+1}}(\tilde{\theta}_{I_{k+1}}) \tag{3}$$

where $J = [t+1, T]$, and $J^c = [T - m_{k+1}, t]$, and $t \in J_k = I_k \backslash I_{k-1}$.

Consider the supremum of the test statistics over interval J_k:

$$\mathfrak{T}_k = sup_{t \in J_k} \mathfrak{T}_{I_{k+1},t} \tag{4}$$



Test Algorithm

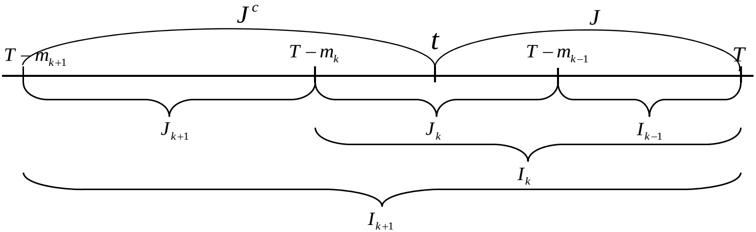
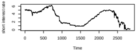


Figure 4: Construction of the Test Statistics in the Local Change Points Test

- The criteria for testing homogeneous intervals:

$$\mathfrak{T}_k \leq \mathfrak{z}_k, \qquad \text{for} \quad k \leq k^* \tag{5}$$

  and $\mathfrak{T}_{k+1} > \mathfrak{z}_{k^*+1}$.
- I_{k^*} is the longest time homogeneous interval for time point T, $\tau \in J_{k^*+1}$ is the change point.
- $\hat{\theta}_T = \tilde{\theta}_{I_{k^*}}$.
- $\mathfrak{z}_k$ is the critical value, obtained by Monte-Carlo simulations.

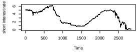


Critical Value

- Sequential choice of critical values $\mathfrak{z}_k$.
- Change point detected at step $\ell \le k$
- $\mathfrak{B}_\ell$: rejection at step ℓ.

$$\mathfrak{B}_\ell = \{\mathfrak{T}_1 \le \mathfrak{z}_1, \cdots, \mathfrak{T}_{\ell-1} \le \mathfrak{z}_{\ell-1}, \mathfrak{T}_\ell > \mathfrak{z}_\ell\}$$

and $\hat{\theta}_{I_k} = \tilde{\theta}_{I_{\ell-1}}$ on $\mathfrak{B}_\ell, \quad \ell = 1, 2, \cdots, k$.



Critical Value

To determine $\mathfrak{z}_1$,

$$\mathbb{E}_{\theta_0}|L(\tilde{\theta}_{I_K}, \tilde{\theta}_{I_0})|^r \mathbf{1}(\mathfrak{B}_1) \leq \rho \mathfrak{R}_r(\theta_0)/K$$

$\mathfrak{B}_\ell$ only depends on $\mathfrak{z}_1, \cdots, \mathfrak{z}_\ell$, controlled by $\mathfrak{z}_\ell$. The minimal value ensures

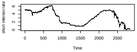
$$max_{k \geq l} \mathbb{E}_{\theta_0}|L(\tilde{\theta}_{I_k}, \tilde{\theta}_{I_{\ell-1}})|^r \mathbf{1}(\mathfrak{B}_\ell) = \rho \mathfrak{R}_r(\theta_0)/K \tag{6}$$

$\mathfrak{R}_r(\theta_0)$ is parametric risk bound.

$$\mathbb{E}_{\theta_0}|L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r \leq \mathfrak{R}_r(\theta_0) \tag{7}$$

Choice of the Length of Interval

- I_0 with length m_0, reflect the flexibility of the parametric model.
- Interval I_k: $m_k = [m_0 a^k]$ with $a > 1$.
- Results not sensitive to the choice of a.
- r=0.5, power of the loss function.
- ρ= 0.2, level of the test

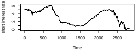


Data

Yield of 3M US T-Bill from the Federal Reserve Bank of St. Louis from 19980102 to 20090513.

	Mean	*SD*	*Skewness*	*Kurtosis*
r_t	0.0319	0.0176	-0.1159	-1.4104
dr_t	-1.764*10^{-5}	0.0006	-0.7467	34.4856

Table 1: Statistical Summary of 3-month T-Bill



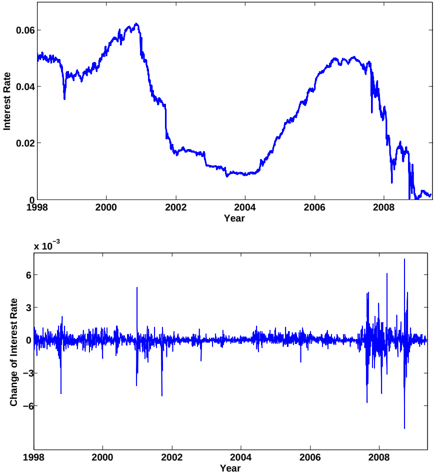
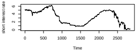


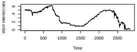
Figure 5: 3-month Treasure Bill Rate: 19980102—20090513. Top panel: Daily yields. Bottom panel: Changes of daily yields.



MLE Estimator of CIR model

Sample Size	$\hat{a}$	$\hat{b}$	$\hat{\sigma}$
19980102–20090513	0.2657	0.0153	0.0944
19980102–20070731	0.1424	0.0252	0.0428
20070801–20090513	3.6792	0.0081	0.2280

Table 2: Estimated parameters of CIR model using MLE



Critical Value

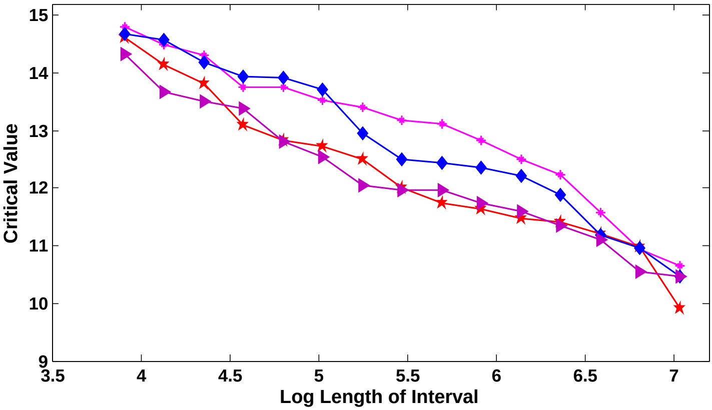
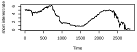


Figure 6: Critical values with $m_0 = 40$, K=15



Homogeneous Intervals

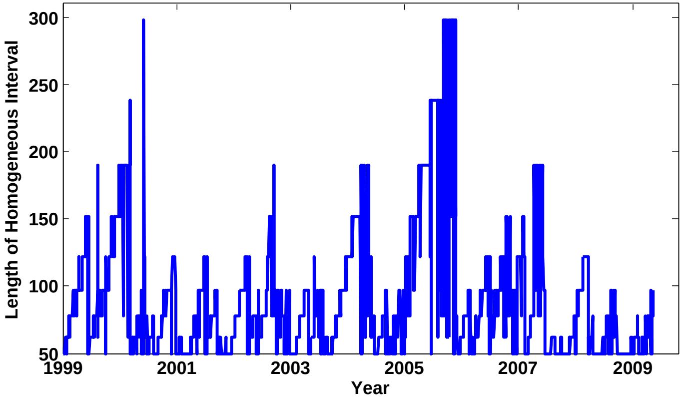
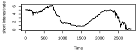


Figure 7: Test result using the real data with $\rho = 0.2$, and $r = 0.5$



Estimator *a*

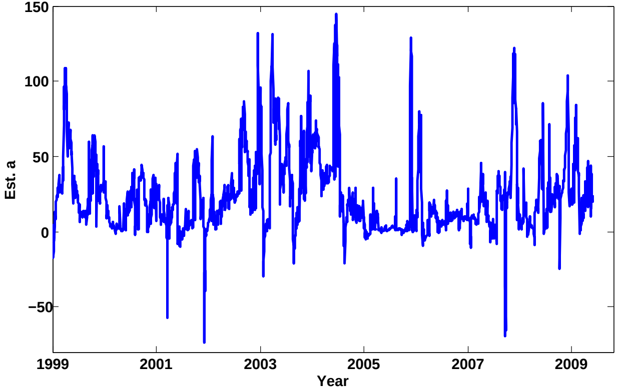
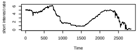


Figure 8: Estimated *a* by LCP method



Estimator b

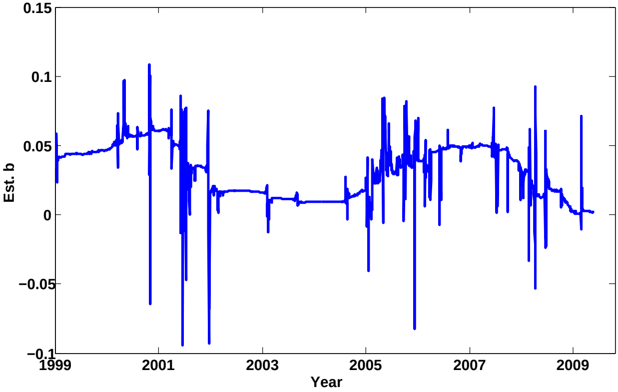
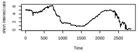


Figure 9: Estimated b by LCP method



Estimator σ

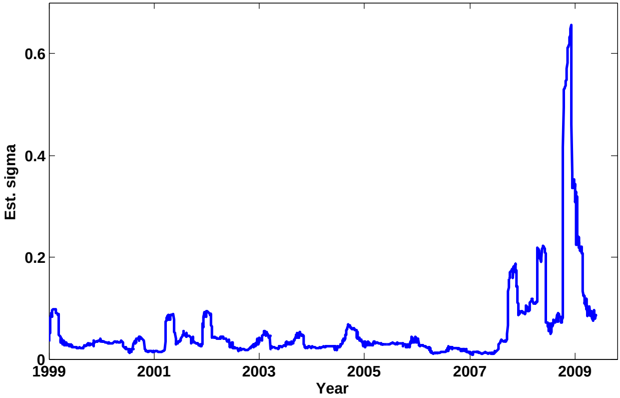
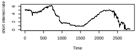


Figure 10: Estimated σ by LCP method



Comparison with CIR Model

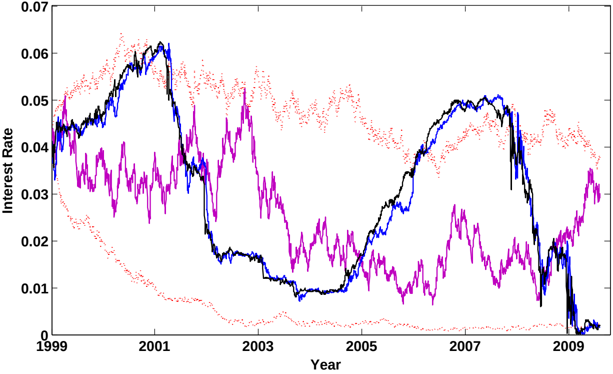


Figure 11: Confidence Interval (Red); Real Data (Black); LCP CIR (Blue); CIR (Purple)

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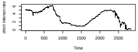

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