Asymptotic normality in the calibration of exponential Lévy models

Jakob Söhl

Institut für Mathematik
Humboldt-Universität zu Berlin

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SFB 649 ÖKONOMISCHES RISIKO

Chart of Schering stock 2002-2008

Lévy Processes

Definition

A càdlàg stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties:

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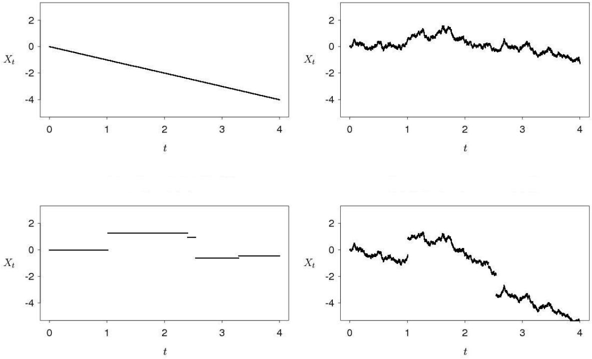
- Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.
- Stochastic continuity: $\forall \epsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.

Some Examples



Pure Drift, Brownian Motion,
Pure Jump, Sum of previous

Characterization and Finite Intensity Case

Lévy processes are characterized by their Lévy triplets (σ^2, γ, ν), with volatility $\sigma \geq 0$, drift $\gamma \in \mathbb{R}$, jump measure ν and intensity $\lambda = \nu(\mathbb{R})$.

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In the case of finite intensity $\lambda < \infty$ we have:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

with Brownian motion W_t, independent random variables $Y_i \sim \nu/\lambda$ and Poisson process N_t with intensity λ (all independent).

Exponential Lévy Model

Let $(e^{-rt}S_t, t \geq 0)$ be a martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t))$, where $r \geq 0$ is the riskless interest rate.

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Aim: Estimation of the Lévy triplet (σ^2, γ, ν) from option data

Restriction: Let (X_t) have finite intensity and an absolutely continuous jump measure.

Options

European Call Option $\mathcal{C}(K, T)$: right, but not the obligation to buy an underlying asset for the price K at time T

European Put Option $\mathcal{P}(K, T)$: right, but not the obligation to sell an underlying asset for the price K at time T

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Substitute K by the negative log-forward moneyness x:

$$x := \log(K/(Se^{rT}))$$

Put-call parity:

$$\mathcal{C}(x, T) - \mathcal{P}(x, T) = S(1 - e^x)$$

Observations

Define:

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$\mathcal{O}$ is observed in the Gaussian white noise model:

$$\mathrm{d}\mathcal{O}_\epsilon(x) = \mathcal{O}(x)\mathrm{d}x + \epsilon\, \epsilon_0(x)\mathrm{d}W(x),$$

with Brownian motion W, $\epsilon_0 \in L^2(\mathbb{R})$ and $\epsilon > 0$.

Description of the method

Putting $\mu(x) := e^x \nu(x)$, the following equality holds:

$$\begin{aligned} \psi_{\mathcal{O}}(u) &:= \frac{1}{T}\log\Big(1 + iu(1+iu)\mathcal{F}\mathcal{O}(u)\Big) \\ &= -\frac{\sigma^2}{2}u^2 + i(\sigma^2+\gamma)u + (\frac{\sigma^2}{2} + \gamma - \lambda) + \mathcal{F}\mu(u). \end{aligned}$$

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1. Substitute $\mathcal{F}\mathcal{O}(u)$ by its empirical counterpart $\mathcal{F}(\mathrm{d}\mathcal{O}_\epsilon)(u) = \mathcal{F}\mathcal{O}(u) + \epsilon \int_{-\infty}^{\infty} e^{iux}\epsilon_0(x)\mathrm{d}W(x)$.
2. Calculate $\psi_{\mathcal{O}_\epsilon}(u)$ with $\mathcal{F}(\mathrm{d}\mathcal{O}_\epsilon)$ instead of $\mathcal{F}\mathcal{O}$.
3. Determine $\hat{\sigma}^2$, $\hat{\gamma}$, $\hat{\lambda}$ from the coefficients of the quadratic polynomial. $\mathcal{F}\hat{\mu}$ is given by the remainder.

Known Results

There are estimators for discrete observations such that (Belomestny & Reiß 2006):

- The Lévy triplet is estimated consistently.
- In general the rates are logarithmic. If $\sigma = 0$ is known, the rates are polynomial.
- The rates are optimal in the minimax sense.

Continuous Complex Logarithm

We need to calculate $\psi_{\mathcal{O}_\epsilon} : \mathbb{R} \to \mathbb{C}$ with

$$\psi_{\mathcal{O}_\epsilon}(u) = \frac{1}{T} \log \Big(\underbrace{1 + iu(1+iu)\big(\mathcal{F}\mathcal{O}(u) + \epsilon \int_{-\infty}^{\infty} e^{iux} \epsilon_0(x) \mathrm{d}W(x)\big)}_{=:\varphi(u)} \Big).$$

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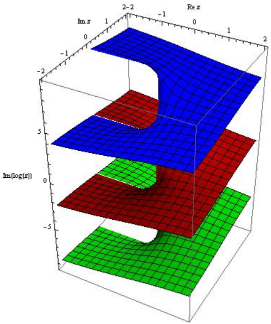
We have $\mathrm{Re}(\log(re^{i\theta})) = \log(r)$. The real part is defined for all $\varphi(u) \neq 0$. For all $u \in \mathbb{R}$ almost surely $\varphi(u) \neq 0$.

Imaginary Part of the Complex Logarithm

We have $\mathrm{Im}(\log(re^{i\theta})) = \theta + 2\pi k$ with $k \in \mathbb{Z}$.

To define $\psi_{\mathcal{O}_\epsilon}(v) = (1/T)\log(\varphi(v))$ the whole path $\varphi : [0, v] \to \mathbb{C}$ needs to be distinct from zero.



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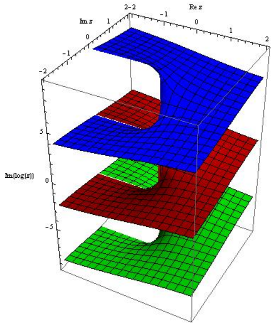

If $\int_{-\infty}^{\infty}(1 + |x|)^p \epsilon_0(x)^2 \mathrm{d}x < \infty$ with $p > 1$, then almost surely for all u in $\mathbb{R}$ $\varphi(u) \neq 0$ (S. 2010).

$\int_{-\infty}^{\infty} e^{iux}\epsilon_0(x)\mathrm{d}W(x)$ is a Gaussian process. Use methods from Gaussian processes.

Confidence sets

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Method: Study the asymptotic behavior of the estimators. Obtain confidence sets from quantiles of the asymptotic distribution.

First steps towards confidence sets

$$\psi_{\mathcal{O}}(u) := \frac{1}{T}\log\Big(1 + iu(1+iu)\mathcal{F}\mathcal{O}(u)\Big)$$

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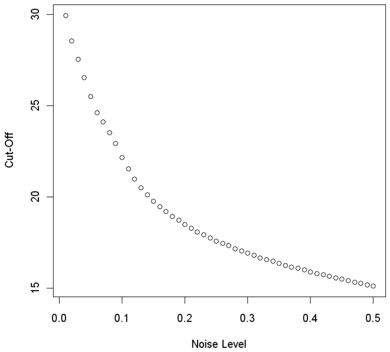
Let $\mathcal{B}$ be the Borel σ-algebra with respect to the topology of uniform convergence on compact sets. Then

$$\frac{1}{\epsilon}(\psi_{\mathcal{O}_\epsilon}(u)-\psi_{\mathcal{O}}(u)) \to \frac{iu(1+iu)}{T(1+iu(1+iu)\mathcal{F}\mathcal{O}(u))}\int_{-\infty}^{\infty} e^{iux}\epsilon_0(x)\mathrm{d}W(x)$$

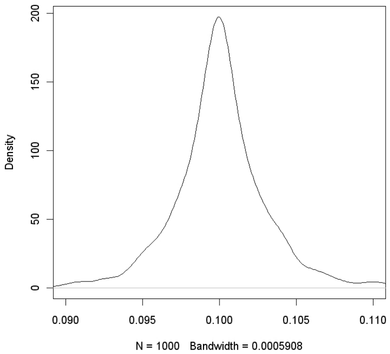
in distribution with respect to $\mathcal{B}$ as $\epsilon \to 0$.

Cut-Off Parameter



Dependence between oracle cut-off and noise level, noise in multiples of $\mathcal{O}(x)$, Kuo model, oracle cut-off without noise: 68.4

Empirical Density



Density of $\hat{\sigma}$ after 1000 Monte Carlo Simulations, $[\mathbb{E}(\hat{\sigma}-\sigma)^2]^{1/2} = 0.00341$, $\sigma = 0.1$, oracle cut-off, noise $0.1\,\mathcal{O}(x)$, Kuo model

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J. Söhl.
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Thank you for your attention!