CHANNEL SYSTEMS: WHY IS THERE A POSITIVE SPREAD?

ALEKSANDER BERENTSEN AND ALESSANDRO MARCHESIANI

Abstract. An increasing number of central banks implement monetary policy via two standing facilities: a lending facility and a deposit facility. All set a non-zero interest rate spread. In this paper we show that it is socially optimal to implement a non-zero interest rate spread. We proof this result in a dynamic general equilibrium model where market participants have heterogenous liquidity needs and where the central bank requires government bonds as collateral. We also discuss how an change in the outstanding stock of collateral - as widely seen in the recent crisis where central banks increased the number of eligible assets - affects the economy.

JEL classification #: E52, E58, E59

Key words: monetary policy, open market operations, standing facilities

Very preliminary draft

Please do not quote

1. Introduction

In a channel system, a central bank offers two facilities: a lending facility whereby it is ready to supply money overnight at a given lending rate against collateral and a deposit facility whereby banks can make overnight deposits to earn a deposit rate. The interest-rate corridor is chosen to keep the overnight interest rate in the money market close to the target rate. In a pure channel system, a change in policy is implemented by simply changing the interest-rate corridor without any open market operations. Implementing monetary policy through standing facilities thus differs substantially from open market operations where the central bank intervenes directly in money markets by trading government bonds for central bank money to keep the money market rate close to the policy rate.

An increasing number of central banks are using channel systems or at least some features of the channel system for implementing monetary policy.[1] Despite its popularity, we know little

[1]Versions of a channel system are operated by the Bank of Canada, the Bank of England, the European Central Bank, the Reserve Bank of Australia, and the Reserve Bank of New Zealand. The US Federal Reserve System

about optimal policy in a channel system. For example, why do all central banks set a non-zero interest-rate corridor? Or, if a central bank wants to "tighten" or "loosen" its policy should it shift the interest-rate corridor, change the spread, or both? Are there any implications for real allocations if a central bank implements monetary policy via standing facilities rather than via open market operations? Finally, how do collateral requirements and in particular a change in eligible assets affects equilibrium outcomes.

Furthermore, we know little why so many central banks changed their operating procedures in the first place. Central bankers often argue that standing facilities help them to control money market rates. But then, why not simply choose a zero corridor? For example, in its press release from October 8, 2008, the Federal Reserve argues that "paying interest on excess balances should help to establish a lower bound on the federal funds rate." However, in the same press release the Federal Reserve also motivates its change in policy by efficiency considerations by writing that "paying interest on required reserve balances should essentially eliminate the opportunity cost of holding required reserves, promoting efficiency in the banking sector."

recently modified the operating procedures of its discount window facility in such a way that it now shares elements of a standing facility. Prior to 2003, the discount window rate was set below the target federal funds rate, but banks faced penalties when accessing the discount window. In 2003, the Federal Reserve decided to set the discount window rate 100 basis points above the target federal funds rate and eased access conditions to the discount window. The resulting framework was similar to a channel system, where the deposit rate is zero and the lending rate 100 basis points above the target rate. Furthermore, since October 2008, the Federal Reserve pays interest on required and excess reserve balances.

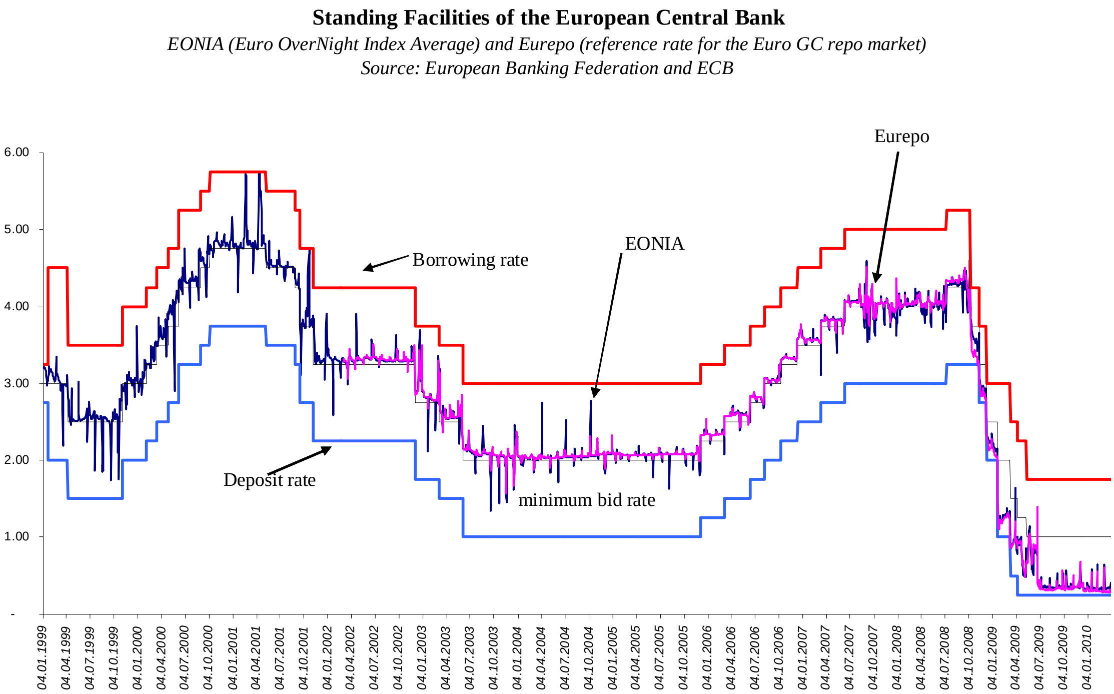

Figure 1: Standing Facilities of the ECB

The purpose of this paper is to shed some light on these questions. In particular, we want to investigate the allocative implications of implementing monetary policy via standing facilities. For this purpose, we construct a dynamic stochastic general equilibrium model with microfoundations for the demand for central bank money. Market participants face two types of idiosyncratic liquidity shocks: early and late shocks. A market participant with a early liquidity shock, can use the money market and the standing facilities to readjust its portfolio. Late liquidity shocks arise after the money market's close implying that for these shocks only the standing facilities can be used to readjust portfolios.

2. Related Literature

Berentsen and Monnet (2008) were the first to study optimal policy in channel system in a general equilibrium framework. Their key result is that it is socially optimal for a central bank to implement a strictly positive interest-rate spread. In their the collateral is a real asset. Furthermore, due to its liquidity premium the social return of this real asset is lower than the marginal return to an agent. From a social point of view, this results in an overaccumulation of the collateral and by implementing a positive spread the central bank can discourage the use of its lending facility and thereby reduce the stock of collateral in the economy.

In contrast, in our model, the central bank's borrowing facility accepts government bonds as collateral. Government bonds are essentially pieces of paper that are costless to produce and so there is no social waste in its use. Nevertheless, we show that it can be optimal to implement a strictly positive interest-rate spread. The optimality of a non-zero corridor arises if it affects the distribution of liquid assets in welfare improving way. In order to show this we extend the Berentsen and Monnet (2008) to introduce idiosyncratic liquidity shocks and show that a non-zero band improve the allocation of central bank money in the economy. [2]

With the exception of Berentsen and Monnet (2008) and Martin and Monnet (2009) previous studies on channel systems or aspects of channel systems were conducted in partial equilibrium models. Berentsen and Monnet (2008) contains a discussion of these models. In a similar paper, Martin and Monnet (2009) compare standing facilities and open market operations in the Lagos-Wright framework. Our model, however, differs from Martin-Monnet (2009) in that we have a richer type structure that they do.

The way we model shocks and the functioning of the secondary bonds market is similar to Berentsen and Waller (2009). They compare two economies, one economy with outside bonds and one with inside bonds, and show that any allocation in the inside-bonds economy can be replicated in the outside-bonds economy, but not vice versa. Our model differs from Berentsen and Waller (2009), however, in that we allows banks to access a standing facility system to meet their liquidity needs after the shock, while they don't.

The structure of the paper is the following. Section 3 describes the environment. The household decisions are described in Section 4. Section 5 studies symmetric stationary equilibria in the case of perfect sterilization policy. The extension to no-sterilization policy is analyzed in Section 6. The conclusions end the paper. All proofs are in the Appendix.

[2] See Berentsen and Monnet (2008) who developed this modified framework.

3. Environment

Our framework builds on the divisible money model of Lagos and Wright (2005). Their framework is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable. Our changes to their framework are motivated by the functioning of existing channel systems.[3] In each day, three markets open sequentially. The first market is a settlement market, where all claims from the previous day are settled. The second market is a secondary bond market. Finally, at the beginning of the third market, after the close of the secondary bond market, the standing facility opens.[4]

The economy is populated by two types of infinitely-lived households: buyers and sellers. Each type of household has measure 1. There are two nonstorable and perfectly divisible consumption goods at each date: market 1 goods and market 3 goods. Nonstorable means that they cannot be carried from one market to the next. Buyers consume in market 3 and consume and produce in market 1. Sellers produce in market 3 and consume in market 1. Furthermore, the buyers receive an idiosyncratic preference shock which determines their liquidity need in the goods market. A fraction n receive the preference shock at the beginning of the bonds market; we call them the $e-$buyers. The remaining buyers receive the preference shock at the beginning of the goods market after the bonds market has closed; we call them the $\varepsilon-$buyers.[5]

[3]See Berentsen and Monnet (2008) who developed this modified framework.

[4]For example, as shown in Berentsen and Monnet (2008), the key features of the ECB's implementation framework and of the Euro money market are the following. First, any outstanding overnight loans at the ECB are settled at the beginning of the day. Second, most lending in the Euro money market and all credit obtained at the ECB's standing facility is collateralized. Third, the Euro money market operates between 7 am and 5 pm. Fourth, after the money market has closed, market participants can access the ECB's facilities for an additional 30 minutes. This means that after the close of the money market, the ECB's lending facility is the only possibility for obtaining overnight liquidity. Also, any late payments received can still be deposited at the deposit facility of the ECB.

[5]The interpretation for these shocks in the ECB-framework described above is that $e-$buyers are banks that receive liquidity shocks during the day and so can use the bond market to readjust their portfolio. In contrast, the $\varepsilon-$buyers are banks that receive liquidity shocks so late in the day that they have to rely on the ECB's SF to readjust their portfolio.

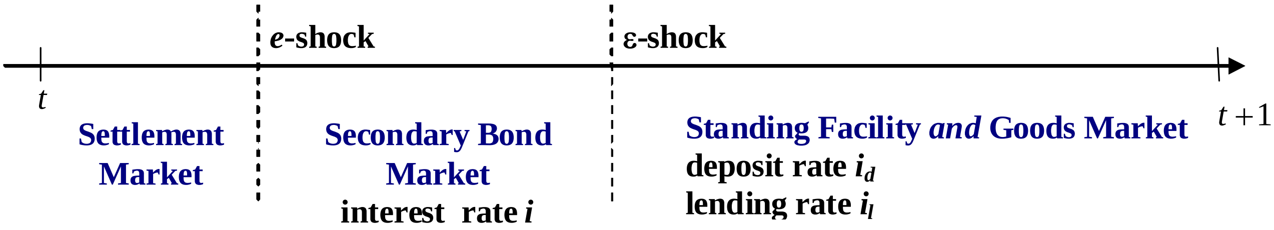

Figure 2: Sequence of events

We now discuss these markets in detail, starting with the settlement market, which opens at the beginning of the period. The discount factor across periods is $\beta = (1+r)^{-1} < 1$ where r is the time rate of discount. For notational ease, we omit the period t superscript, and use the terms "$+$" and "$-$" to denote next period variables and previous period variables, respectively. In the settlement market, buyers produce general goods, repay loans, redeem deposits, and adjust their money balances. General goods are produced solely from inputs of labor according to a constant return to scale production technology where one unit of the consumption good is produced with one unit of labor generating one unit of disutility. Thus, producing h units of the general good implies disutility $-h$, while consuming h units gives utility h. Sellers do not produce in the settlement market but they can consume. Their utility of consuming x units of general goods is $U(x) = x$.[6]

At the beginning of the secondary bond market, with probability n a buyer receives the idiosyncratic preference shocks $e \in [0, \infty)$ which determines the marginal utility of consumption in the goods market: we refer to these households as $e-$buyers. With probability $1-n$ he does not learn yet his preference for the consumption good in the current period. After buyers learn their type, they trade bonds. The preference shock e has a continuous distribution $F(e)$ with support $[0, \infty)$, is iid across $e-$buyers and serially uncorrelated. At the beginning of the goods market,

[6]The idiosyncratic preference shocks play a similar role as random matching and bargaining in Lagos and Wright (2005). Depending on the realization of these shocks, households will spend different amounts of money in the goods market. Then, without quasilinear preferences and unbounded hours in market 3, the preference shocks would generate a nondegenerate distribution of money holdings since the money holdings of individual households would depend on their history of shocks. For tractability, we therefore impose assumptions, as in Lagos and Wright (2005), that generate a degenerate distribution of money holdings at the beginning of a period.

$\varepsilon-$buyers receive their preference shock $\varepsilon \in [0, \infty)$. The preference shock ε has a continuous distribution $F(\varepsilon)$ with support $[0, \infty)$, is iid across $\varepsilon-$buyers and serially uncorrelated.

In the goods market, an $e-$buyer ($\varepsilon-$buyer) gets utility $eu(q_e)$ ($\varepsilon u(q_\varepsilon)$) from q_e (q_ε) consumption in the third market, where $u(q) = \log(q)$. Sellers incur a utility cost $c(q_s) = q_s$ from producing q_s units of output.

3.1. **First-best allocation.** We assume without loss in generality that the planner treats all sellers symmetrically. He also treats all buyers experiencing the same preference shock symmetrically. Given this assumption, the weighted average of expected steady state lifetime utility of households and firms at the beginning of the settlement market can be written as follows

$$\begin{aligned} (1-\beta)\mathcal{W} &= n\int_0^\infty [eu(q_e) - h_e]\, dF(e) \\ &\quad + (1-n)\int_0^\infty [\varepsilon u(q_\varepsilon) - h_\varepsilon]\, dF(\varepsilon) + x - q_s. \end{aligned} \tag{3.1}$$

where h_e (h_ε) is hours worked by an $e-$household ($\varepsilon-$household) in the settlement market and q_e (q_ε) is consumption of an $e-$household ($\varepsilon-$household) in the goods market. The planner maximizes (3.1) subject to the feasibility constraints

$$n\int_0^\infty q_e dF(e) + (1-n)\int_0^\infty q_\varepsilon dF(\varepsilon) - q_s \leq 0. \tag{3.2}$$

$$x - n\int_0^\infty h_e dF(e) - (1-n)\int_0^\infty h_\varepsilon dF(\varepsilon) \leq 0 \tag{3.3}$$

where x is consumption by a seller in the settlement market. The first-best allocation satisfies

$$\begin{aligned} eu'(q_e^*) &= 1 \text{ for all } e. \\ \varepsilon u'(q_\varepsilon^*) &= 1 \text{ for all } \varepsilon. \end{aligned} \tag{3.4}$$

These are the quantities chosen by a social planner who could dictate production and consumption in the goods market.

3.2. **Information frictions, money and bonds.** There are two perfectly divisible financial assets: money and one-period, nominal discount bonds. Both are intrinsically useless since they are neither arguments of any utility function nor are they arguments of any production function. Both assets are issued by the "central bank" as described below. New bonds are issued in the settlement market. They are payable to the bearer and default free. One bond pays off one unit of currency in the settlement market of the following period. The central bank is assumed to have a record-keeping technology over bond trades and bonds are book-keeping entries – no physical object exists. This implies that households are not anonymous to the central bank. Nevertheless, despite having a record-keeping technology over bond trades, the central bank has no record-keeping technology over goods trades.

At time t, the central bank sells one-period, nominal discount bonds in market 1 and redeems bonds that were sold in $t-1$. Then households trade bonds and money in market 2. The central bank acts as the intermediary for these trades, recording purchases/sales of bonds and redistributes money. Private households are anonymous to each other and cannot commit to honor inter-temporal promises. Since bonds are intangible objects, they are incapable of being used as media of exchange in market 3, hence they are illiquid.[7] Since households are anonymous and cannot commit, a household's promise in market 2 to deliver bonds to a seller in market 3 is not credible.

To motivate a role for fiat money, search models of money typically impose three assumptions on the exchange process (Shi 2008): a double coincidence problem, anonymity, and costly communication. First, our preference structure creates a single-coincidence problem in market 3 since households do not have a good desired by sellers. Second, agents in market 3 are anonymous which rules out trade credit between individual buyers and sellers. Third, there is no public communication of individual trading outcomes (public memory), which in turn eliminates the use of social punishments in support of gift-giving equilibria. The combination of these frictions implies that sellers require immediate compensation from buyers. In short, there must be immediate settlement with some durable asset and money is the only durable asset. These are the micro-founded frictions that make money essential for trade in market 3. Araujo (2004), Kocherlakota (1998), Wallace (2001), and Aliprantis, Camera and Puzzello (2007) provide a more detailed discussion of the features that generate an essential role for money. In contrast, in the settlement

[7]The beneficial role of illiquid bonds has been studied by Camera and Boel (2006), Kocherlakota (2003) and Shi (2008). More recent models with illiquid assets include, Lagos and Rocheteau (2008), Lagos (2009), Lester, Postlewaite and Wright (2009), and many others.

market all agents can produce for their own consumption or use money balances acquired earlier. In this market, money is not essential for trade.[8]

3.3. **Central bank policy and the money supply process.** At the beginning of the goods market, after all preference shocks are observed, the central bank offers a borrowing and a deposit facility. The central bank operates at zero cost and offers nominal loans ℓ at an interest rate i_ℓ and promises to pay interest rate i_d on nominal deposits d with $i_\ell \geq i_d$. Let $\rho_d = 1/(1+i_d)$ and $\rho_\ell = 1/(1+i_\ell)$. Since we focus on facilities provided by the central bank, we restrict financial contracts to overnight contracts. An agent who borrows ℓ units of money from the central bank in market 3 repays $(1+i_\ell)\,\ell$ units of money in market 1 of the following period. Also, an agent who deposits d units of money at the central bank in market 3 receives $(1+i_d)\,d$ units of money in market 1 of the following period.

In a channel system, then the money stock evolves as follows

$$M^+ = M - i_\ell L + i_d D - \rho_S B^+ + B + T \tag{3.5}$$

where M and B are the stock of money, respectively the stock of bonds, at the beginning of the current-period treasury market, M^+ and B^+ the stock of money, respectively the stock of bonds, at the beginning of the next-period treasury market, T the lump sum transfer, ρ_S the price of bonds in the settlement market. In the settlement market, total loans (L) are repaid and total deposits (D) are redeemed. Since interest-rate payments by the agents are $i_\ell L$, the stock of money shrinks by this amount. Interest payments by the central bank on total deposits are $i_d D$. The central bank simply prints additional money to make these interest payments, causing the stock of money to increase by this amount. The central bank also issues new one-period bonds which it sells at discount ρ_S. This shrinks the amount of money by $\rho_S B^+$. In the settlement market, it also redeems the stock of bonds it issued in the previous period (B) which increases the stock of money by B. Finally, the central bank can also change the stock of money via lump-sum transfers $T = \tau M$ in the settlement market.

[8]One can think of agents being able to barter perfectly in this market. Obviously in such an environment, money is not needed.

4. Household decisions

The money price of goods in the settlement market is P, implying that the goods price of money in the settlement market is $\phi = 1/P$. Let p be the money price of goods in the goods market; ρ_S the money price of newly issued bonds in the settlement market; and ρ_T the money price of bonds in the treasury market.

4.1. **Settlement market.** $V_S(m, b, \ell, d)$ denotes the expected value of entering the settlement market with m units of money, b bonds, ℓ loans, and d deposits. $V_T(m, b)$ denotes the expected value from entering the bond market with m units of money and b collateral. For notational simplicity, we suppress the dependence of the value function on the time index t.

In the settlement market, the problem of the representative buyer is:

$$V_S(m, b, \ell, d) = \max_{h, m\prime, b\prime} -h + V_T\left(m', b'\right)$$

$$s.t. \quad \phi m' + \phi\rho_S b' = h + \phi m + \phi b + \phi\left(1 + i_d\right) d - \phi\left(1 + i_\ell\right)\ell + \phi T.$$

where h is hours worked in market 1, m' is the amount of money brought into the bond market, and b' is the amount of bonds brought into the bond market. Using the budget constraint to eliminate h in the objective function, one obtains the first-order conditions

$$V_T^{m'} \leq \phi \;(= \text{ if } m' > 0\;) \tag{4.1}$$

$$V_T^{b'} \leq \phi\rho_S \;(= \text{ if } b' > 0\;) \tag{4.2}$$

$V_T^{m'} \equiv \frac{\partial V_T(m',b')}{\partial m'}$ is the marginal value of taking an additional unit of money into the bond market. Since the marginal disutility of working is one, $-\phi$ is the utility cost of acquiring one unit of money in the settlement market. $V_T^{b'} \equiv \frac{\partial V_T(m',b')}{\partial b'}$ is the marginal value of taking additional bonds into the bond market. Since the marginal disutility of working is 1, $-\phi\rho_S$ is the utility cost of acquiring one unit of bonds in the settlement market. The implication of (4.1) and (4.2) is that all agents enter the bond market with the same amount of money and the same quantity of bonds (which can be zero).

The envelope conditions are

$$V_S^m = V_S^b = \phi; V_S^d = \phi\left(1+i_d\right); V_S^\ell = -\phi\left(1+i_\ell\right) \tag{4.3}$$

where V_S^j is the partial derivative of $V_S(m, b, \ell, d)$ with respect to $j = m, b, \ell, d$.

4.2. **Goods Market.** We first consider the problem solved by sellers, then the one solved by the $\varepsilon-$buyers, and finally the one solved by the $e-$buyers. During the goods market the central bank operates SF which allows all buyers to borrow at rate i_ℓ and deposit excess money at rate i_d.

4.2.1. *Decisions by sellers.* Sellers produce goods in the goods market with linear cost $c\left(q\right) = q$ and consume in the settlement market obtaining linear utility $U(x) = x$. It is straightforward to show that that sellers are indifferent as to how much they sell in the goods market if

$$p\beta\phi^+\left(1+i_d\right) = 1. \tag{4.4}$$

Since we focus on a symmetric equilibrium, we assume that all sellers produce the same amount. With regard to bond holdings, it is straightforward to show that, in equilibrium, firms are indifferent to holding any bonds if the Fisher equation holds and will hold no bonds if the yield on the bonds does not compensate them for inflation or time discounting. Thus, for brevity of analysis, we assume sellers carry no bonds across periods.

It is also clear that sellers always deposit their proceeds from sales at the deposit facility since they can earn the interest rate i_d.

4.2.2. *Decisions by $\varepsilon-$buyer.* The indirect utility function of a $\varepsilon-$buyer in the goods market is

$$\begin{gathered} V_G(m, b|\,\varepsilon) = \max_{q_\varepsilon, d_\varepsilon, \ell_\varepsilon} \varepsilon u\left(q_\varepsilon\right) + \beta V_S\left(m + \ell_\varepsilon - pq_\varepsilon - d_\varepsilon, b, \ell_\varepsilon, d_\varepsilon|\,\varepsilon\right) \\ \text{s.t.} \quad m + \ell_\varepsilon - pq_\varepsilon - d_\varepsilon \geq 0, \text{ and } \frac{b}{1+i_\ell} - \ell_\varepsilon \geq 0 \end{gathered}$$

where d_ε is the amount of money a ε-type household deposits at the central bank and ℓ_ε is the loan received from the central bank. The first inequality is the buyer's budget constraint. The second inequality is the collateral constraint. Let $\beta\phi^+\lambda_\varepsilon$ denote the Lagrange multiplier for the first inequality and denote $\beta\phi^{+}\lambda_{\ell}$ the Lagrange multiplier of the second inequality. Then, using (4.3) to replace V_S^m, V_S^{ℓ} and V_S^d, the first-order conditions for q_ε, d_ε, and ℓ_ε can be written as follows:

$$
\begin{aligned}
\varepsilon u'\left(q_{\varepsilon}\right)-\beta p\phi^{+}\left(1+\lambda_{\varepsilon}\right) &= 0 \\
i_{d}-\lambda_{\varepsilon} &\leq 0 \qquad (=0 \text{ if } d_{\varepsilon}>0) \\
-i_{\ell}+\lambda_{\varepsilon}-\lambda_{\ell} &\leq 0 \qquad (=0 \text{ if } \ell_{\varepsilon}>0)
\end{aligned} \tag{4.5}
$$

Lemma 1 below characterizes the optimal borrowing and lending decisions by a $\varepsilon-$buyer and the quantity of goods obtained by the $\varepsilon-$buyer:

Lemma 1. *There exist critical values ε_d, ε_ℓ, $\varepsilon_{\bar{\ell}}$, with $0 \leq \varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar{\ell}}$, such that the following is true: if $0 \leq \varepsilon < \varepsilon_d$, a buyer deposits money at the central bank; if $\varepsilon_\ell < \varepsilon \leq \varepsilon_{\bar{\ell}}$, he borrows money and the collateral constraint is nonbinding; if $\varepsilon_{\bar{\ell}} \leq \varepsilon$, he borrows money and the collateral constraint is binding; and if $\varepsilon_d \leq \varepsilon \leq \varepsilon_\ell$, he neither borrows nor deposits money. The critical values solve:*

$$
\varepsilon_{d}=\left(1+i_{d}\right)\beta\phi^{+}m,\ \varepsilon_{\ell}=\left(1+i_{\ell}\right)\beta\phi^{+}m,\ \textit{and}\ \varepsilon_{\bar{\ell}}=\left(1+i_{\ell}\right)\beta\phi^{+}m+\beta\phi^{+}b. \tag{4.6}
$$

In any equilibrium, the amount of borrowing and depositing by a buyer with a taste shock ε and the amount of goods purchased by the buyer satisfy:

$$
\begin{array}{llll}
q_{\varepsilon}=\varepsilon, & d_{\varepsilon}=p\left(\varepsilon_{d}-\varepsilon\right), & \ell_{\varepsilon}=0, & \textit{if } 0\leq\varepsilon\leq\varepsilon_{d} \\
q_{\varepsilon}=\varepsilon_{d}, & d_{\varepsilon}=0, & \ell_{\varepsilon}=0, & \textit{if } \varepsilon_{d}\leq\varepsilon\leq\varepsilon_{\ell} \\
q_{\varepsilon}=\Delta\varepsilon, & d_{\varepsilon}=0, & \ell_{\varepsilon}=p\left(\Delta\varepsilon-\varepsilon_{d}\right), & \textit{if } \varepsilon_{\ell}\leq\varepsilon\leq\varepsilon_{\bar{\ell}}, \\
q_{\varepsilon}=\Delta\varepsilon_{\bar{\ell}}, & d_{\varepsilon}=0, & \ell_{\varepsilon}=\frac{b}{1+i_{\ell}}, & \textit{if } \varepsilon_{\bar{\ell}}\leq\varepsilon,
\end{array} \tag{4.7}
$$

where $\Delta \equiv \frac{1+i_d}{1+i_\ell}$.

The optimal borrowing and lending decisions follow the cut-off rules according to the realization of the taste shock. The cut-off levels, ε_d, ε_ℓ, and $\varepsilon_{\bar{\ell}}$ partition the set of taste shocks into three regions. For shocks lower than ε_d, a buyer deposits money at the SF; for shocks higher than ε_ℓ, the buyer borrows at the SF. For values between ε_d and ε_ℓ,, the buyer does not use the central bank's SF. Finally, the cutoff value $\varepsilon_{\bar{\ell}}$ determines whether a buyer's collateral constraint is binding or not.

Then, using (4.3) to replace V_S^m, V_S^ℓ and V_S^d, the envelope conditions for $\varepsilon-$buyers can be written as

$$
\begin{aligned}
V_G^m(m, b|\,\varepsilon) &= \beta\phi^+\left(1+\lambda_\varepsilon\right) \\
V_G^b(m, b|\,\varepsilon) &= \beta\phi^+\left(1+\tfrac{\lambda_\ell}{1+i_\ell}\right)
\end{aligned}
\tag{4.8}
$$

4.2.3. *Decisions by $e-$buyer.* The indirect utility function of a $e-$buyer in the goods market is

$$
\begin{gathered}
V_G(m, b|\,e) = \max_{q_e, d_e, \ell_e} eu\left(q_e\right) + \beta V_S\left(m - pq_e + \ell_e - d_e, b, \ell_e, d_e|\,e\right) \\
\text{s.t.} \quad m - pq_e + \ell_e - d_e \geq 0, \text{ and } \frac{b}{1+i_\ell} - \ell_e \geq 0
\end{gathered}
$$

where q_e is the quantity of goods he consumes, and d_e the amount of money he deposits at the central bank. The first constraint means that they cannot deposit more money than what they carry from the previous period minus what he spend in goods purchases plus what he borrows from the central bank. The second constraint means that he cannot borrow more money than what his collateral holdings allow for. Denote $\beta\phi_+\lambda_e$ the multiplier on the first constraint and $\beta\phi_+\lambda_\ell$ the one on the second constraint. Then, using (4.3), the first-order conditions can be written as follows:

$$
\begin{aligned}
eu'\left(q_e\right) - \beta p\phi^+\left(1+\lambda_e\right) &= 0 \\
i_d - \lambda_e &\leq 0 \qquad (= 0 \text{ if } d_e > 0) \\
-i_\ell + \lambda_e - \lambda_\ell &\leq 0 \qquad (= 0 \text{ if } \ell_e > 0)
\end{aligned}
\tag{4.9}
$$

Note that in equilibrium $e-$buyers never deposit money at the central bank nor do they borrow from the central bank, that is $d_e = \ell_e = 0$ implying that

$$i_d \leq \lambda_e \leq i_\ell$$

Lemma 2 below characterizes the optimal borrowing and lending decisions by a $e-$buyer and the quantity of goods obtained by the $e-$buyer:

LEMMA 2. *There exists a critical value $\tilde{e}$ with $0 < \tilde{e} < \infty$, such that the following is true: for any e the buyer neither deposits nor borrows money at the central bank. The critical value solve:*

$$\tilde{e} = (1 + i_T)\,\beta\phi^{+}m + \beta\phi^{+}b. \tag{4.10}$$

In any equilibrium, the amount of borrowing and depositing by a buyer with a taste shock e and the amount of goods purchased by the buyer satisfy:

$$\begin{aligned} q_e &= e\Delta_T, \quad d_e = 0, \quad \ell_e = 0, \quad \textit{if} \quad 0 \leq e \leq \tilde{e} \\ q_e &= \tilde{e}\Delta_T, \quad d_e = 0, \quad \ell_e = 0, \quad \textit{if} \quad \tilde{e} \leq e \end{aligned} \tag{4.11}$$

where $\Delta_T \equiv \frac{1+i_d}{1+i_T}$ *and*

$$\tilde{e} = (1 + i_T)\,\beta\phi^{+}m + \beta\phi^{+}b. \tag{4.12}$$

Replacing V_S^m, V_S^b and $\beta\phi^{+}\lambda_e$ using (4.3) and (4.9), we obtain the following envelope condition for a $e-$buyer:

$$V_G^m(m, b|\, e) = \frac{eu'(q_e)}{p} \text{ and } V_G^b(m, b|\, e) = \beta\phi^{+} \tag{4.13}$$

$$\begin{aligned} V_G^m(m, b|\, e) &= \beta\phi^{+}(1 + \lambda_e) = \frac{eu'(q_e)}{p} \\ V_G^b(m, b|\, e) &= \beta\phi^{+}\left(1 + \frac{\lambda_\ell}{1 + i_\ell}\right) = \frac{eu'(q_e)}{p(1 + i_\ell)} \end{aligned}$$

4.3. **Secondary bonds market.** In the secondary bond market households are subject to an idiosyncratic preference and technology shock. With probability n they learn their type at the beginning of the bond market, and with probability $1 - n$ they learn their type at the beginning of the goods market. The indirect expected utility of a buyer entering the bond market with a portfolio (m, b) is

$$V_T(m, b) = nV_T(m, b|\, e) + (1 - n)\,V_T(m, b|\, \varepsilon)$$

where[9]

$$V_T\left(m, b|\, e\right) = \max_{y_e} \int_0^\infty V_G\left(m - \rho_T y_e, b + y_e|\, e\right) dF\left(e\right) \tag{4.14}$$
$$m - \rho_T y_e \geq 0 \text{ and } b + y_e \geq 0$$

and

$$V_T\left(m, b|\, \varepsilon\right) = \max_{y} \int_0^\infty V_G\left(m - \rho_T y, b + y|\, \varepsilon\right) dF\left(\varepsilon\right) \tag{4.15}$$
$$\text{s.t. } m - \rho_T y \geq 0 \text{ and } b + y \geq 0$$

A household who learns its preference shock solves problem (4.14) while a household who receives the preference shock later solves (4.15). In both programs, the first inequality reflects the fact that a buyer cannot sell more money for bonds that what he brought into the bond market, and the second inequality reflects the fact that he cannot sell more bonds for money that what he brought into the bond market.

5. Equilibrium

We focus on symmetric stationary equilibria where money is used as a medium of exchange and there is a positive demand for government bonds. Such equilibria meet the following requirements: (i) Households' decisions are optimal given prices; (ii) The decisions are symmetric across all sellers and symmetric across all buyers with the same preference shocks; (iii) The goods and bond markets clear; (iv) All real quantities are constant across time; (v) The law of motion for the stock of money (3.5) holds in each period.

Point (iv) requires that the real stock of money is constant; i.e.,

$$\phi M = \phi^+ M^+. \tag{5.1}$$

This implies that $\phi/\phi^+ = M^+/M \equiv \gamma$ where γ is the gross steady-state money growth rate. Symmetry requires $m = M^+$ and $b = B^+$. The restriction that there is a positive demand for money and bonds requires that Lemma 3 holds.

[9]Note that $V_T\left(m, b|\, e\right)$ is the lifteime expected utility of a buyer who knows that he is a early buyer yet does not know the realization of the $e-$shock.

Lemma 3. *In any equilibrium*

$$\rho_S = \rho_T$$

Lemma 3 relies on an arbitrage argument. The prices for bonds must be the same in the two markets. Otherwise agents could buy bonds in one of the markets and sell them at a profit in the other market. For what follows we ignore the index and write ρ for the price of bonds in markets 1 and 2. Moreover, in a stationary equilibrium the bonds price ρ has to be constant. This can be seen for example from (4.11), where a changing ρ involves a non-stationary path for consumption. A constant bond price then implies that the bond-money ratio has to be constant, and this can be only achieved when the growth rates of money and bonds are equal. We assume there are positive initial stocks of money M_0 and government bonds B_0.[10]

Market clearing in the secondary bonds market and in the goods market require

$$n\int_0^{\infty} y_e dF\left(e\right) + \left(1-n\right)\int_0^{\infty} y dF\left(\varepsilon\right) = 0 \tag{5.2}$$

$$q_s - n\int_0^{\infty} q_e dF\left(e\right) - \left(1-n\right)\int_0^{\infty} q_\varepsilon dF\left(\varepsilon\right) = 0 \tag{5.3}$$

where q_s is aggregate production by sellers in the goods market.

Lemma 4. *In equilibrium, the critical values solve:*

$$\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell}, \varepsilon_{\bar{\ell}} = \varepsilon_d \frac{\rho_d}{\rho_\ell}\left(1+\rho_\ell \frac{B}{M}\right), \text{ and } \tilde{e} = \varepsilon_d \frac{\rho_d}{\rho_T}\left(1+\rho_T \frac{B}{M}\right) \tag{5.4}$$

In the Appendix we show the following

[10] Since the assets are nominal objects, the central bank can start the economy off by one-time injections of cash M_0 and bonds B_0.

PROPOSITION 1. *An equilibrium is a policy (i_d, i_ℓ, γ) and endogenous variables (ρ, ε_d) satisfying*

$$\begin{aligned}\frac{\rho\gamma}{\beta} &= n\left[\int_0^{\tilde{e}} dF(e) + \int_{\tilde{e}}^{\infty} \frac{e}{\tilde{e}} dF(e)\right] \\ &+\rho(1+i_d)(1-n)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon)\right]\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\int_0^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) \\ &= \rho(1+i_\ell)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon)\right]\end{aligned} \tag{5.6}$$

A detailed derivation of (5.5)-(5.6) can be found in the Appendix. Equation (5.5) is obtained from the choice of money holdings (4.1). Equation (5.6) is obtained from the $\varepsilon-$buyer's optimization problem. In equilibrium, they must be indifferent to whether they use the settlement market or the secondary bonds market to adjust their portfolio.

One can easily solve these two equations numerically for ρ and ε_d. All remaining endogenous variables can then be calculated as follows: The critical values are obtained from Lemma 4. The amount of borrowing and depositing by a buyer and the amount of goods purchased by the buyer are obtained from (4.7) and (4.11); from (4.6), the real stock of money is $\phi M = \rho_d \varepsilon_d / \beta$ and the real stock of bonds is $\phi B = (\varepsilon_{\bar{\ell}} - \varepsilon_\ell)/\beta$. Finally, from the low of motion of money holdings (3.5) one obtains the value of τ that is consistent with the policy choice γ.

6. RESULTS

We derive two types of results: analytical and numerical results.

6.1. **Analytical results.** Analytical results are presented for the case $n = 0$. If all households receive their idiosyncratic shocks after the secondary bonds market has closed, they can only use the central bank facilities to readjust their portfolio. This case is simpler because the two equilirium equations (5.5) and (5.6) are recursive for $n = 0$ as can be seen below:

$$\frac{\rho_d \gamma}{\beta} = \int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon), \tag{6.1}$$

$$\frac{\rho \gamma}{\beta} = \int_0^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} dF(\varepsilon) \tag{6.2}$$

We can first solve (6.1) for ε_d and then we get ρ from (6.2). For this case, we find the following results.

Proposition 2. *Assume n=0. An equilibrium exists and is unique if* $\gamma/\beta > 1/\rho_d$.

The existence proof involves showing that the right-hand side of (6.1) is decreasing in ε_d. Furthmore we show that the right-hand side is approaching infinity for $\varepsilon_d \to 0$ and is approaching 1 for $\varepsilon_d \to 1$. It is then clear that there exists a vlue for ε_d that solves (6.1) if $\gamma/\beta > 1/\rho_d$. Then, given the equilibrium value for ε_d then from (6.2) one obtains ρ. Note that $\rho \leq 1$ which immediately follows from inspection of (5.6) when $n = 0$.[11]

Proposition 3. *Assume n=0. If* $\gamma/\beta > 1/\rho_d$, *it is optimal to choose a non-zero spread.*

In the proof of Proposition we show that if $\gamma/\beta > 1/\rho_d$ and the central bank sets $\rho_d = \rho_\ell$, then increasing the loan rate marginally (decreasing ρ_ℓ) is strictly welfare improving. Thus, it is never optimal to have a zero spread. This result is driven by the reallocation of consumption that occurs from increasing the loan rate as depicted in the following figure:

[11]We also have a proof of existence for $n > 0$. The proof is more complicated since in involves showing that there exists a pair (ε_d, ρ) that solves (5.5) and (5.6) simulataneously.

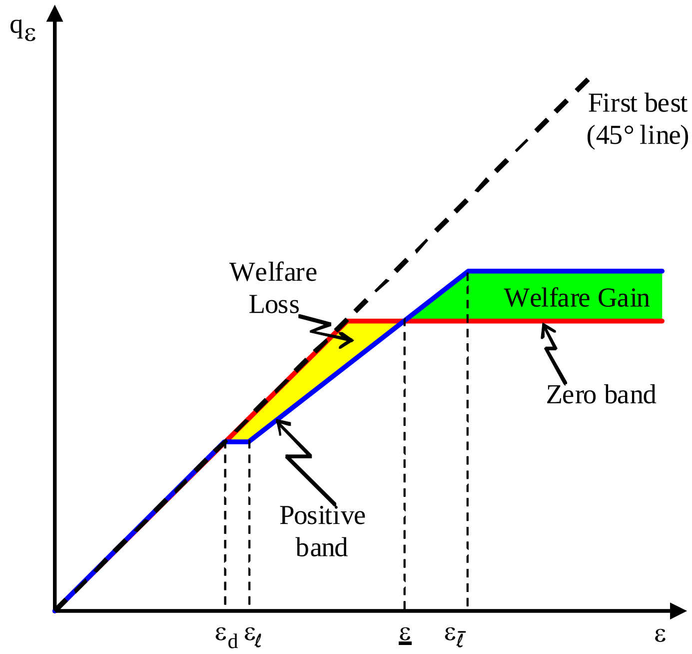

Figure 3: Welfare effects

Figure 3 indicates of why a non-zero corridor can be welfare improving if $\gamma/\beta > 1/\rho_d$. The black dotted linear curve plots the first-best consumption quantities. The red curve plots the quantities for some given inflation $\gamma > \beta/\rho_d$ and a zero corridor; i.e., $\rho_d = \rho_\ell$. Up to some treshhold for ε, the buyers receive the first-best consumption quantities after which the collateral constraint is binding as indicated by the consumption quantities that are independent of the idiosyncratic liquidity shock ε. The blue curve plots the quantities for some given inflation $\gamma > \beta/\rho_d$ and a non-zero spread; i.e., $\rho_d > \rho_\ell$. Up to the critical value ε_d the buyer consumes the first-best quantity. In this region, he deposits and excess central bank money at the deposit facility. In the region between ε_d and ε_ℓ, he neither deposits nor he borrows money. He simply spends all money brought into the period and consumes $q_\varepsilon = \varepsilon_d$. In the region between ε_ℓ and $\varepsilon_{\bar{\ell}}$ the buyers borrows but his collateral constraint is non-binding. Finally, for $\varepsilon > \varepsilon_{\bar{\ell}}$ the collateral constraint is binding.

As indicated by Figure 3, the welfare gain from increasing the borrowing rate i_ℓ marginally arises because it reallocates consuption from medium ε-buyers to high-ε buyers. In particular, there is a welfare loss for medium ε-buyers (as indicated by the yellow region) since buyers between ε_d and $\underline{\varepsilon}$ reduce their consumption. In contrast, buyers with $\varepsilon \geq \underline{\varepsilon}$ increase their consumption. In the proof of Proposition 2, we show that if $i_d = i_\ell$ and if we marginally increase i_ℓ the second effect always dominates. Furthermore, for numerical examples we find that welfare is hump-shaped in i_ℓ so that there is a unique welfare maximizing non-zero spread if $\gamma/\beta > 1/\rho_d$.

Proposition 4. *Assume n=0. The optimal policy is to set $\gamma \to \beta/\rho_d$. For this policy, the first-best allocation is achieved.*

The optimal policy makes holding money costless. Note that such a policy means that the central bank's deposit rate exactly compensate market participants for their impatience and for inflation. To see this let $\gamma = \beta/\rho_d$ and rewrite it to get $1 + i_d = (1 + \pi)(1 + r)$ which is the Fisher equation except for the fact that i_d is not a market interest rate but the central bank's deposit rate.

The first-best quantities are attained because if money is costless to hold, buyers will hold enough such that they can afford the efficient consumption quantities for any value of ε. In fact, in the proof of Proposition 4 we show that when $\gamma \to \beta/\rho_d$, then $\varepsilon_d \to \infty$ implying that all buyers consum the efficient quantities. Furthermore, it is irrelevant whether there is a spread or not since no buyers is ever using the borrowing facility.

Since under the optimal policy, households are indifferent of how much money they bring into the period, they will bring any amount that is necessary to get the efficient consumption quantities. Accordingly, since the nominal quantity of money is finite and since ε is unbounded, the real value of money approaches infinity; i.e., the price level approaches zero.

Note also that there is an indeterminacy of the optimal policy since the level of i_d is irrelevant. In particular, the central bank can set $i_d = 0$ in which case we get $\gamma = \beta$ as the optimal policy. This condition for the optimal policy frequently arises in models with microfoundation for the demand for central bank money and is called the Friedman rule. It is then clear that the policy $\gamma \to \beta/\rho_d$ is a modified Friedman rule which occurs if the central bank is allowed to pay interest on money.

From the law of motion of money holding we can calculate the lump-sum transfers that are necessary to implement $\gamma \to \beta/\rho_d$ this equilibrium. As shown in the proof of Proposition 4, the lump-sum transfers required to implement the optimal policy are

$$\tau = (\beta - 1)\left(\frac{B}{M} + 1 + i_d\right) < 0.$$

Thus, the optimal policy requires that the central bank is able to lump-sum tax households. One might think that it is unrealistic to assume that the central bank is able to lum-sum tax agents since in reality it has no fiscal authority. Therefore, in Section 7 we consider the case where the central bank cannot lum-sum tax households; i.e., we set $\tau = 0$ and consider the optimal policy for this case. Note when the central bank cannot use lump-sum taxes or lump-sum transfers, that the inflation rate is endogenous since the choice of i_d and i_ℓ will determine the money growth rate and hence inflation via (3.5).

In many channels systems, the central bank still uses open market operations to affects the amount of liquidity in the system. For example, the ECB conducts a weekly tender to change the quantity of central bank money in the Euro system. We define an open market operation as a one-time change in the bonds to money ratio.

PROPOSITION 5. *An open market operation (an increase in the bonds to money ratio) has the following effects: It decreases ε_d and ε_ℓ, while the effect on $\varepsilon_{\bar{\ell}}$ is ambiguous. Furthermore, if $\gamma/\beta > 1/\rho_d$, an increase in the ratioB/M from $B/M = 0$ is strictly welfare improving.*

The last result shows that it is strictly welfare improving to have illiquid bonds in this economy. We can only show that increasing B/M is welfare improving if we evaluate the total derivative $\frac{dW}{dB/M}$ at $B/M = 0$. If $B/M > 0$ the derivative is ambiguous. In particular as shown below for some numerical examples welfare is hum-shaped in B/M.

6.2. **Numerical examples.** In progress....

7. Extensions

So far we have assumed that the central bank chooses an inflation goal γ which it attains by setting the lump-sum transfers τ such that (**??**) is satisfies. Suppose to the contrast that the central bank can not use any sump-sum transfers or subsidies; i.e. $\tau = 0$. In this case, the equilibrium satisfies

PROPOSITION 6. *A steady state equilibrium without sterilization is a policy* (i_d, i_ℓ) *and endogenous variables* $(\gamma, \rho, \varepsilon_d)$ *satisfying* $(5.5), (5.6)$, *and*

$$\gamma = 1 + i_d + \frac{(1-n)(i_d - i_\ell)}{\varepsilon_d}\Psi + \frac{B(1-\rho\gamma)}{M} \tag{7.1}$$

where

$$\Psi \equiv \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} (\Delta\varepsilon - \varepsilon_d)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\Delta\varepsilon_{\bar{\ell}} - \varepsilon_d)\, dF(\varepsilon)$$

Equation (7.1) is obtained from the law of motion of money (3.5). Note that we can reduce these three equations to two equations in ε_d and ρ by using (7.1) to replace γ in (5.5). The resulting expressions are very hard to analyze analytically. Nevertheless we can easily solve these two equations numerically for ρ and ε_d. All remaining endogenous variables can then be calculated as follows: The critical values are obtained from Lemma 4. The amount of borrowing and depositing by a buyer and the amount of goods purchased by the buyer are obtained from (4.7) and (4.11); from (4.6), the real stock of money is $\phi M = \rho_d \varepsilon_d / \beta$ and the real stock of bonds is $\phi B = (\varepsilon_{\bar{\ell}} - \varepsilon_\ell)/\beta$. Finally, from (7.1) we obtain the endogenous value of γ.

In what follows we present a "reasonable" numerical example and show how various policy measure affect the equilibrium.

8. Conclusions

This paper studied optimal policy when central banks implement monetary policy with standing facilities. We construct a general equilibrium model with agents heterogeneity, and microfoundations for the demand of money. The main result of the paper is that the optimal spread is nonzero and we show how an increase in the outstanding stock of collateral - as widely seen in the recent crisis where central banks increased the number of eligible assets - affects the economy.

Appendix

Proof of Lemma 1. We first derive the cutoff values ε_d and ε_ℓ. For this proof, to the notation of the consumption level of a buyer, we add a subscript d if the buyer deposits money at the central bank, a subscript ℓ if the buyer takes out a loan and the collateral constraint is nonbinding, a subscript $\bar{\ell}$ if the buyer takes out a loan and the collateral constraint is binding,and a subscript 0 if the buyer does neither.

From (4.5), the consumption level of a buyer who enters the loan market satisfies:

$$q_d(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+(1+i_d)}, \quad q_\ell(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+(1+i_\ell)}. \tag{8.1}$$

A buyer who does not use the deposit facilities will spend all his money on goods – If he anticipated that he will have idle cash after the goods trade, it would be optimal to deposit the idle cash in the intermediary, provided $i_d > 0$. Thus, consumption of such a buyer is:

$$q_0(\varepsilon) = \frac{m}{p}. \tag{8.2}$$

At $\varepsilon = \varepsilon_d$, the household is indifferent between depositing and not depositing. We can write this indifference condition as:

$$\varepsilon_d u(q_d) - \beta\phi^+(pq_d - i_d d) = \varepsilon_d u(q_0) - \beta\phi^+ pq_0.$$

By using (8.1), (8.2), and $d = m - pq_d$, we can write the equation further as

$$\varepsilon_d \ln\left[\frac{\varepsilon_d}{\beta\phi^+(1+i_d)m}\right] = \varepsilon_d - (1+i_d)\beta\phi^+ m.$$

The unique solution to this equation is $\varepsilon_d = (1+i_d)\beta\phi^+ m$, which implies that $\beta\phi^+ m < \varepsilon_d$.

At $\varepsilon = \varepsilon_\ell$, the household is indifferent between borrowing and not borrowing. We can write this indifference condition as

$$\varepsilon_\ell u(q_\ell) - \beta\phi^+(pq_\ell + i_\ell \ell) = \varepsilon_\ell u(q_0) - \beta\phi^+ pq_0.$$

Using (8.1), (8.2) and $\ell = pq_\ell - m$, we can write this equation further as

$$\varepsilon_\ell \ln\left[\frac{\varepsilon_\ell}{(1+i_\ell)\beta\phi^+ m}\right] = \varepsilon_\ell - (1+i_\ell)\beta\phi^+ m$$

The unique solution to this equation is $\varepsilon_\ell = (1+i_\ell)\beta\phi^+ m$. Using the expression for ε_d we get

$$\varepsilon_\ell = \varepsilon_d/\Delta \tag{8.3}$$

We now calculate $\varepsilon_{\bar{\ell}}$. There is a critical buyer who enters the goods market and wants to take out a loan who's collateral constraint is just binding. From (4.5), for this buyer we have the following equilibrium conditions

$$q_{\bar{\ell}} = \frac{\rho_\ell \varepsilon_{\bar{\ell}}}{\beta\phi^+ p} \text{ and } pq_{\bar{\ell}} = m + \rho_\ell b$$

Eliminating $q_{\bar{\ell}}$ we get

$$\varepsilon_{\bar{\ell}} = (1 + i_\ell)\,\beta\phi^+ m + \beta\phi^+ b$$

Using (8.3) we get

$$\varepsilon_{\bar{\ell}} = \left(\varepsilon_d \frac{\rho_d}{\rho_\ell}\right)\left(1 + \rho_\ell \frac{b}{m}\right)$$

It is then evident that

$$0 \leq \varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar{\ell}}$$

□

Proof of Lemma 2. For easier reference we replicate the $e-$buyer's optimization problem in the secondary bond market here:

$$\max_{y_e} \int_0^\infty V_G\left(m - \rho_T y_e, b + y_e |\, e\right) dF\left(e\right)$$

$$m - \rho_T y_e \geq 0 \text{ and } b + y_e \geq 0$$

$$\max_{y_e} \int_0^\infty u\left(q_e\right) + V_S\left(m - \rho_T y_e - pq_e + \ell_e - d_e, b + y_e, \ell_e, d_e |\, e\right) dF\left(e\right)$$

$$b + y_e \geq 0 \text{s.t.} \quad m - \rho_T y_e - pq_e + \ell_e - d_e \geq 0, \text{ and } \tfrac{b+y_e}{1+i_\ell} - \ell_e \geq 0$$

$$\begin{aligned}
0 &= u'\left(q_e\right) - p\beta\phi^+ - p\beta\phi^+\lambda_e \\
0 &= -\rho_T\beta\phi^+ + \beta\phi^+ + \beta\phi^+\lambda - \beta\phi^+\rho_T\lambda_e + \frac{\beta\phi^+}{1+i_\ell}\lambda_\ell \\
0 &\geq -\beta\phi^+ i_\ell + \beta\phi^+\lambda_e - \beta\phi^+\lambda_\ell \\
0 &\geq \beta\phi^+ i_d - \beta\phi^+\lambda_e
\end{aligned}$$

$$\begin{aligned}
0 &= u'(q_e) - \frac{1+\lambda_e}{1+i_d} \\
0 &= -\rho_T (1+\lambda_e) + 1 + \lambda + \frac{1}{1+i_\ell}\lambda_\ell \\
0 &\geq -i_\ell + \lambda_e - \lambda_\ell \\
0 &\geq \beta\phi^+ i_d - \beta\phi^+ \lambda_e
\end{aligned}$$

Suppose he would deposit, then $u'(q_e) = 1$ which is a contradiction. Suppose he would take out a loan, then

$$\begin{aligned}
0 &= (\rho_\ell - \rho_T)(1+\lambda_e) + \lambda \\
0 &= u'(q_e) - \frac{\rho_d \lambda}{\rho_T - \rho_\ell}
\end{aligned}$$

which is a contradiction. Hence, the borrowing constraint is nonbinding which yields

$$\begin{aligned}
1+\lambda_e &= (1+\lambda)/\rho_T \\
0 &= u'(q_e) - \frac{1+i_T}{1+i_d}
\end{aligned}$$

Because of the Inada condition on consumption utility, the first constraint is never binding if $i_\ell > i_T$ and, therefore, we ignore it. Denote $\beta\phi^+\lambda$ the multiplier for the second inequality. Then, the FOC is

$$-\rho_T V_G^m(m - \rho_T y_e, b + y_e | e) + V_G^b(m - \rho_T y_e, b + y_e | e) + \beta\phi^+\lambda = 0 \tag{8.4}$$

Use (4.13) to replace $V_G^m(m - \rho_T y_e, b + y_e | e)$ and $V_G^b(m - \rho_T y_e, b + y_e | e)$ to get

$$-\rho_T \beta\phi^+ (1+\lambda_e) + \beta\phi^+ + \beta\phi^+\lambda = 0$$

$$\begin{aligned}
V_G^m(m, b | e) &= \beta\phi^+(1+\lambda_e) = \frac{eu'(q_e)}{p} \\
V_G^b(m, b | e) &= \beta\phi^+\left(1 + \frac{\lambda_\ell}{1+i_\ell}\right) = \beta\phi^+\left(\frac{1+\lambda_e}{1+i_\ell}\right) = \frac{eu'(q_e)}{p(1+i_\ell)}
\end{aligned}$$

$$\begin{aligned}-\rho_T\beta\phi^+(1+\lambda_e)+\beta\phi^+\left(\frac{1+\lambda_e}{1+i_\ell}\right)+\beta\phi^+\lambda &= 0\\ \left[-\rho_T+\rho_\ell\right](1+\lambda_e)+\lambda &= 0\end{aligned}$$

which implies that

$$1+\lambda_e=(1+\lambda)(1+i_T)$$

where $i_T \equiv (1-\rho_T)/\rho_T$. We can use this relation to rewrite the first-order condition in the goods market as follows

$$\begin{aligned}eu'(q_e)-\beta p\phi^+(1+\lambda_e) &= 0\\ eu'(q_e)-\frac{\beta p\phi^+\lambda}{\rho_T-\rho_\ell} &= 0\end{aligned}$$

$$\begin{aligned}eu'(q_e)-\beta p\phi^+(1+\lambda)(1+i_T) &= 0\\ eu'(q_e)-\beta p\phi^+(1+\lambda)(1+i_T) &= 0\end{aligned}$$

Finally, use the sellers' first-order condition to get

$$eu'(q_e)-(1+\lambda)/\Delta_T=0$$

The optimal buying and selling decisions in the bond market follow the cut-off rule according to the realization of the taste shock. The cut-off level $\tilde{e}$ partitions the set of taste shocks into two regions. If $e<\tilde{e}$, the short-selling constraint on bonds is non-binding, i.e., $\lambda=0$, implying that

$$eu'(q_e)-1/\Delta_T=0$$

If $e\geq\tilde{e}$, the short-selling constraint on bonds is binding. The e-buyer sells all bonds for money and consumes

$$pq_e=m+\rho_Tb$$

Note that for a very high e the buyer would like to borrow at the SF but he can't because he has no collateral left. □

Proof of Lemma 3. For easier reference, we replicate the $\varepsilon-$buyer's optimization problem in the bond market here:

$$\max_{y} \int_0^\infty V_G\left(m - \rho_T y, b + y \middle| \varepsilon\right) dF\left(\varepsilon\right)$$

$$\text{s.t. } m - \rho_T y \geq 0 \text{ and } b + y \geq 0$$

Note that an ε-buyer will never sell all his money for bonds or all his bonds for money which implies that in any equilibrium the following first-order condition holds

$$\rho_T = \frac{\int_0^\infty V_G^b\left(m - \rho_T y, b + y \middle| \varepsilon\right) dF\left(\varepsilon\right)}{\int_0^\infty V_G^m\left(m - \rho_T y, b + y \middle| \varepsilon\right) dF\left(\varepsilon\right)} \tag{8.5}$$

Differentiate $V_T\left(m, b\right)$ with respect to m and b to get

$$\begin{aligned} V_T^m\left(m, b\right) &= n \int_0^\infty V_G^m\left(m - \rho_T y_e, b + y_e \middle| e\right) dF\left(e\right) \\ &\quad + \left(1 - n\right) \int_0^\infty V_G^m\left(m - \rho_T y, b + y \middle| \varepsilon\right) dF\left(\varepsilon\right) \end{aligned}$$

and

$$\begin{aligned} V_T^b\left(m, b\right) &= n \int_0^\infty \left[V_G^b\left(m - \rho_T y_e, b + y_e \middle| e\right) + \beta\phi^+\lambda\right] dF\left(e\right) \\ &\quad + \left(1 - n\right) \int_0^\infty V_G^b\left(m - \rho_T y, b + y \middle| \varepsilon\right) dF\left(\varepsilon\right) \end{aligned}$$

respectively. Use (8.4) to eliminate $\beta\phi^+\lambda$ and (8.5) in the second equality to eliminate $V_G^b\left(m - \rho_T y, b + y \middle| \varepsilon\right)$ to get

$$\rho_T V_T^m\left(m, b\right) = V_T^b\left(m, b\right).$$

Use (4.1) and (4.2) to replace $V_T^m\left(m, b\right)$ and $V_T^b\left(m, b\right)$ to get

$$\rho \equiv \rho_T = \rho_S.$$

$\square$

Proof of Proposition 1. In this proof we derive two equations that solve for the endogenous variables ρ and ε_d. We first derive equation (5.5). Differentiate $V_T(m,b)$ with respect to m to get

$$\begin{aligned} V_T^m(m,b) &= n\int_0^\infty V_G^m(m-\rho_T y_e, b+y_e|\, e)\,dF(e) \\ &\quad +(1-n)\int_0^\infty V_G^m(m-\rho_T y, b+y|\,\varepsilon)\,dF(\varepsilon) \end{aligned}$$

Use equations (4.13) and (4.8) to replace $V_G^m(m-\rho_T y_e, b+y_e|\, e)$ and $V_G^m(m-\rho_T y, b+y|\,\varepsilon)$ to get

$$V_T^m(m,b) = n\int_0^\infty \frac{eu'(q_e)}{p}dF(e) + (1-n)\int_0^\infty \frac{\varepsilon u'(q_\varepsilon)}{p}dF(\varepsilon)$$

Finally, use the first-order condition for q_s to get:

$$V_T^m(m,b) = \beta\phi^+(1+i_d)\left[n\int_0^\infty eu'(q_e)\,dF(e) + (1-n)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)\right]$$

This equation can be transformed into an equation in the two endogenous variables ρ and ε_d. Using (4.1) to replace $V_T^m(m,b)$ and (5.1) to replace the ϕ^+/ϕ yields

$$\frac{\rho\gamma}{\beta} = \rho(1+i_d)\left[n\int_0^\infty eu'(q_e)\,dF(e) + (1-n)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)\right]$$

Finally, note that $u'(q) = 1/q$ and replace q_ε using Lemma 2 and q_e using Lemma 1 to get

$$\begin{aligned} &\Theta(\rho,\varepsilon_d) \equiv -\rho\gamma/\beta + n\left[\int_0^{\tilde e} dF(e) + \int_{\tilde e}^\infty \tfrac{e}{\tilde e}dF(e)\right] \\ &+\rho(1+i_d)(1-n)\left[\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \tfrac{\varepsilon}{\varepsilon_d}dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar\ell}} \tfrac{\rho_d}{\rho_\ell}dF(\varepsilon) + \int_{\varepsilon_{\bar\ell}}^\infty \tfrac{\varepsilon}{\varepsilon_{\bar\ell}}\tfrac{\rho_d}{\rho_\ell}dF(\varepsilon)\right] \\ &= 0 \end{aligned} \tag{8.6}$$

Note that since $\varepsilon_\ell = \varepsilon_d\frac{\rho_d}{\rho_\ell}$, $\varepsilon_{\bar\ell} = \left(\varepsilon_d\frac{\rho_d}{\rho_\ell}\right)\left(1+\rho_\ell\frac{B}{M}\right)$, and $\tilde e = \left(\varepsilon_d\frac{\rho_d}{\rho_T}\right)\left(1+\rho_T\frac{B}{M}\right)$ the cutoff values are functions of i_ℓ and ε_d only. Hence, the first-order condition is a function of i_ℓ and ε_d only.

We obtain a second equation in ρ and ε_d from (8.5). Use equation (4.8) to replace $V_G^b(m-\rho y, b+y|\,\varepsilon)$ and $V_G^m(m-\rho y, b+y|\,\varepsilon)$ to get

$$\rho\int_0^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon)=\int_0^\infty\left(1+\frac{\lambda_\ell}{1+i_\ell}\right)dF(\varepsilon)$$

Use (4.5) to replace λ_ℓ and rearrange to get

$$\rho(1+i_\ell)\int_0^\infty (1+\lambda_\varepsilon)\,dF(\varepsilon)=\int_0^{\varepsilon_{\bar\ell}}(1+i_\ell)\,dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty(1+\lambda_\varepsilon)\,dF(\varepsilon)$$

Use (4.5) to replace $(1+\lambda_\varepsilon)$ and rearrange to get

$$\rho(1+i_\ell)\int_0^\infty \frac{\varepsilon u'(q_\varepsilon)}{\beta\phi^+p}dF(\varepsilon)=\int_0^{\varepsilon_{\bar\ell}}(1+i_\ell)\,dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon u'(q_\varepsilon)}{\beta\phi^+p}dF(\varepsilon)$$

Multiply by $\beta\phi^+p$ and rearrange to get

$$\rho(1+i_\ell)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon)=\int_0^{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty\varepsilon u'(q_\varepsilon)\,dF(\varepsilon)$$

Note that $u'(q_\varepsilon)=1/q_\varepsilon$ and replace q_ε using Lemma 1 to get

$$\rho(1+i_\ell)\left[\int_0^{\varepsilon_d}dF(\varepsilon)+\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon}{\varepsilon_d}dF(\varepsilon)+\int_{\varepsilon_\ell}^{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon}{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)\right]$$
$$=\int_0^{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon}{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)\tag{8.7}$$

Note that since $\varepsilon_\ell=\varepsilon_d\frac{\rho_d}{\rho_\ell}$, $\varepsilon_{\bar\ell}=\left(\varepsilon_d\frac{\rho_d}{\rho_\ell}\right)\left(1+\rho_\ell\frac{B}{M}\right)$ the cutoff values are functions of i_ℓ and ε_d only. Hence, Ψ is a function of ρ and ε_d only. □

Proof of Proposition 2. For easier reference we restate (6.1) here:

$$\frac{\gamma\rho_d}{\beta}=\int_0^{\varepsilon_d}dF(\varepsilon)+\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon}{\varepsilon_d}dF(\varepsilon)+\int_{\varepsilon_\ell}^{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)+\int_{\varepsilon_{\bar\ell}}^\infty\frac{\varepsilon}{\varepsilon_{\bar\ell}}\frac{\rho_d}{\rho_\ell}dF(\varepsilon)$$

Denote the right-hand side by rhs_1. Then, rhs_1 is a strictly decreasing function in ε_d. Furthermore, we have $rhs_1 \to 1$ as $\varepsilon_d \to \infty$ and $rhs_1 \to \infty$ as $\varepsilon_d \to 0$ since $\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}} = \varepsilon_d \frac{\rho_d}{\rho_\ell}\left(1+\rho_\ell \frac{B}{M}\right)$. Accordingly, there exists a unique ε_d that solves (6.1) if $\frac{\gamma\rho_d}{\beta} > 1$. $\square$

Proof of Proposition 3. Assume $n = 0$. Then, the welfare function is

$$\mathcal{W} = \int_0^{\varepsilon_d} \left[\varepsilon u\left(q_\varepsilon\right) - q_\varepsilon\right] dF\left(\varepsilon\right) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u\left(q_\varepsilon\right) - q_\varepsilon\right] dF\left(\varepsilon\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u\left(q_\varepsilon\right) - q_\varepsilon\right] dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u\left(q_\varepsilon\right) - q_\varepsilon\right] dF\left(\varepsilon\right) \tag{8.8}$$

To show that it is never optimal to choose a zero band, we calculate $d\mathcal{W}/d\rho_\ell$, evaluate it $\rho_\ell = \rho_d = \rho$, and then show that $d\mathcal{W}/d\rho_\ell|_{\rho_\ell=\rho_d=\rho} < 0$.

Note that ρ_ℓ affects $\mathcal{W}$ directly and indirectly via ε_d; that is

$$\frac{d\mathcal{W}}{d\rho_\ell} = \frac{\partial\mathcal{W}}{\partial\varepsilon_d}\frac{d\varepsilon_d}{d\rho_\ell} + \frac{\partial\mathcal{W}}{\partial\rho_\ell}$$

We get the term $\frac{d\varepsilon_d}{d\rho_\ell}$ by taking the total derivative of the equilibrium equation (5.5) which we replicate here for easier reference:

$$\frac{\rho_d\gamma}{\beta} = \int_0^{\varepsilon_d} dF\left(\varepsilon\right) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF\left(\varepsilon\right) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_{\bar{\ell}}\rho_\ell} dF\left(\varepsilon\right)$$

From this equation we get

$$\frac{d\varepsilon_d}{d\rho_\ell} = -\frac{\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{(\rho_\ell)^2} dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\frac{b}{m}}{\varepsilon_d\left(1+\rho_\ell\frac{b}{m}\right)^2} dF\left(\varepsilon\right)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{(\varepsilon_d)^2} dF\left(\varepsilon\right) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{(\varepsilon_d)^2\left(1+\rho_\ell\frac{b}{m}\right)} dF\left(\varepsilon\right)} < 0$$

since $\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}} = \varepsilon_d \frac{\rho_d}{\rho_\ell}\left(1+\rho_\ell \frac{b}{m}\right)$.

The partial derivativ $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \varepsilon_d} &= \int_0^{\varepsilon_d} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon)\end{aligned}$$

Using (4.7) we can write this partial derivative as follows:

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_d} = \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \left(1 + \rho_\ell \frac{b}{m}\right) dF(\varepsilon) \tag{8.9}$$

Note that $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is strictly positive.

For $\frac{\partial \mathcal{W}}{\partial \rho_\ell}$ we get

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \rho_\ell} &= \int_0^{\varepsilon_d} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon)\end{aligned}$$

which using (4.7) can be written as

$$\frac{\partial \mathcal{W}}{\partial \rho_\ell} = \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{\varepsilon}{\rho_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \varepsilon_d \frac{b}{m} dF(\varepsilon)$$

which is also positive. This implies that $\frac{d\mathcal{W}}{d\rho_\ell}$ can be either positive or negative. Increasing ρ_ℓ (decreasing i_ℓ) has a positive effect on welfare through its direct effect $\frac{d\mathcal{W}}{d\rho_\ell}$ but an negative effect through its indirect effect $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}\frac{d\varepsilon_d}{d\rho_\ell}$.

We now evaluate these derivatives at $\rho_\ell = \rho_d = \rho$. We get

$$\begin{aligned}
\left.\frac{d\varepsilon_d}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} &= -\frac{\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\rho\varepsilon\frac{b}{m}}{\varepsilon_{\bar{\ell}}\left(1+\rho\frac{b}{m}\right)} dF(\varepsilon)}{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\rho\varepsilon}{\varepsilon_d\varepsilon_{\bar{\ell}}} dF(\varepsilon)} \\
\left.\frac{\partial\mathcal{W}}{\partial\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} &= \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \varepsilon_d \frac{b}{m} dF(\varepsilon) \\
\left.\frac{\partial\mathcal{W}}{\partial\varepsilon_d}\right|_{\rho_\ell=\rho_d=\rho} &= \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \left(1+\rho\frac{b}{m}\right) dF(\varepsilon)
\end{aligned}$$

since $\varepsilon_d = \varepsilon_\ell$ and $\varepsilon_{\bar{\ell}} = \varepsilon_d\left(1+\rho\frac{b}{m}\right)$ at $\rho_\ell = \rho_d = \rho$. Use these expressions to write $\left.\frac{d\mathcal{W}}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho}$ as follows

$$\left.\frac{d\mathcal{W}}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} = -\int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) \frac{(\varepsilon_{\bar{\ell}})^2 \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} dF(\varepsilon)}{\rho \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)} < 0.$$

Hence, a marginal decrease of ρ_ℓ (marginal increase of i_ℓ) from $\rho_\ell = \rho_d = \rho$ is welfare improving. It follows that a zero band is not optimal policy if $\gamma/\beta > \rho_d$. □

Proof of Proposition 5. Assume $n = 0$. Then, welfare is given by (8.8). To show that it is optimal to choose a non-zero bond-to-money ratio, we first calculate $d\mathcal{W}/d\frac{B}{M}$, then evaluate it $\frac{B}{M} = 0$, and then show that $d\mathcal{W}/d\frac{B}{M}\big|_{B/M} > 0$. Note that B/M affects $\mathcal{W}$ directly and indirectly via ε_d; that is

$$\frac{d\mathcal{W}}{dB/M} = \frac{\partial\mathcal{W}}{\partial\varepsilon_d}\frac{d\varepsilon_d}{dB/M} + \frac{\partial\mathcal{W}}{\partial B/M} \tag{8.10}$$

We get the term $\frac{d\varepsilon_d}{dB/M}$ by taking the total derivative of the equilibrium equation (5.5). It is

$$\frac{d\varepsilon_d}{dB/M} = -\frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)} \tag{8.11}$$

We have already calculated the partial derivativ $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ in the proof of Proposition 3. It is (8.9). The partial derivative of the welfare functon with respect to B/M is

$$\frac{\partial \mathcal{W}}{\partial B/M} = \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] \varepsilon_d \rho_\ell dF\left(\varepsilon\right) \tag{8.12}$$

Using (8.12), (8.11), and (8.9) we can rewrite (8.10) as

$$\begin{aligned}\frac{d\mathcal{W}}{dB/M} = & -\int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] dF\left(\varepsilon\right) \frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)} \\ & -\int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] dF\left(\varepsilon\right) \left[\frac{\left(1+\rho_\ell\frac{b}{m}\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)} - \varepsilon_d\rho_\ell\right]\end{aligned}$$

Simplifying yields

$$\begin{aligned}\frac{d\mathcal{W}}{dB/M} = & -\int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] dF\left(\varepsilon\right) \frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)} \\ & -\int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] dF\left(\varepsilon\right) \left[\frac{\left(1+\rho_\ell\frac{b}{m}\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon) - \varepsilon_d\rho_\ell \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) - \varepsilon_d\rho_\ell \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)}\right]\end{aligned}$$

or

$$\frac{d\mathcal{W}}{dB/M} = -\int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'\left(q_\varepsilon\right) - 1\right] dF\left(\varepsilon\right) \frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} dF(\varepsilon)} + + \frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \rho_\ell \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon)}{\frac{\varepsilon}{\varepsilon_d} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \frac{\rho_d}{\varepsilon_{\bar{\ell}}\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)}$$

or

$$\frac{d\mathcal{W}}{dB/M} = -\frac{\int_{\varepsilon_d}^{\varepsilon_\ell} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \frac{\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)}{\frac{1}{\varepsilon_d^2} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \frac{\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)} + \frac{\int_{\varepsilon_{\bar{\ell}}}^{\infty} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) \rho_\ell \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon)}{\frac{1}{\varepsilon_d} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \frac{\rho_d}{\varepsilon_{\bar{\ell}}\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)}$$

or

$$\frac{d\mathcal{W}}{dB/M} = -\frac{\frac{\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon) - \int_{\varepsilon_d}^{\varepsilon_\ell} dF(\varepsilon) \frac{\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)}{\frac{1}{\varepsilon_d^2} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \frac{\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)} + \frac{\frac{\rho_d}{\varepsilon_{\bar{\ell}}\varepsilon_d} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon) \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) - \int_{\varepsilon_{\bar{\ell}}}^{\infty} dF(\varepsilon) \frac{\rho_\ell}{\varepsilon_d} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon)}{\frac{1}{\left(\varepsilon_d\right)^2} \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon dF(\varepsilon) + \frac{\rho_d}{\varepsilon_{\bar{\ell}}\rho_\ell\varepsilon_d} \int_{\varepsilon_{\bar{\ell}}}^{\infty} \varepsilon dF(\varepsilon)}$$

or

$$\frac{d\mathcal{W}}{dB/M}=\frac{\frac{\rho_d}{\varepsilon_{\bar{\ell}}}\left(\frac{\varepsilon_{\bar{\ell}}\rho_\ell-\varepsilon_d\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell}\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF(\varepsilon)\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF(\varepsilon)+\frac{\varepsilon_d\rho_d^2}{\varepsilon_{\bar{\ell}}^2\rho_\ell}\int_{\varepsilon_d}^{\varepsilon_\ell}dF(\varepsilon)\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF(\varepsilon)-\int_{\varepsilon_{\bar{\ell}}}^{\infty}dF(\varepsilon)\frac{\rho_\ell}{\varepsilon_d}\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF(\varepsilon)}{\frac{1}{\varepsilon_d^2}\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF(\varepsilon)+\frac{\rho_d}{\varepsilon_d\varepsilon_{\bar{\ell}}\rho_\ell}\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF(\varepsilon)}$$

For the limiting case $b/m=0$ we have $\varepsilon_{\bar{\ell}}=\varepsilon_\ell$we get

$$\frac{d\mathcal{W}}{dB/M}=\frac{\rho_\ell\varepsilon_d\left[\int_{\varepsilon_d}^{\varepsilon_\ell}dF\left(\varepsilon\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF\left(\varepsilon\right)-\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF\left(\varepsilon\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty}dF\left(\varepsilon\right)\right]}{\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF\left(\varepsilon\right)}$$

which is positive if $\int_{\varepsilon_d}^{\varepsilon_\ell}dF\left(\varepsilon\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF\left(\varepsilon\right)-\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon dF\left(\varepsilon\right)\int_{\varepsilon_{\bar{\ell}}}^{\infty}dF\left(\varepsilon\right)>0.$ □

Proof of Proposition 6. We now derive the third equilibrium equation which comes from the law of motion of money. The first step of the proof is to derive an expression for the price of goods p. From the $\varepsilon-$buyer with $\varepsilon_d\le\varepsilon\le\varepsilon_\ell$, we know that $d_\varepsilon=\ell_\varepsilon=0$. Hence, from their budget cash constraint in the goods market we know that they spend all their money; i.e., $pq_\varepsilon=m$. From Lemma 1, $q_\varepsilon=\varepsilon_d$ and so

$$p=\frac{M}{\varepsilon_d}. \tag{8.13}$$

The second step of the proof is to derive aggregate loans, L. From Lemma 1, we know that the agents who borrow are buyers with a shock $\varepsilon\ge\varepsilon_\ell$. Thus aggregate lending is $L=(1-n)\int_{\varepsilon_\ell}^{\infty}\ell_\varepsilon dF\left(\varepsilon\right)$. From Lemma 1, we also know that $\ell_\varepsilon=p\left(\Delta\varepsilon-\varepsilon_d\right)$ if $\varepsilon_\ell\le\varepsilon\le\varepsilon_{\bar{\ell}}$, and $\ell_\varepsilon=b/\left(1+i_\ell\right)=p\left(\Delta\varepsilon_{\bar{\ell}}-\varepsilon_d\right)$ if $\varepsilon\ge\varepsilon_{\bar{\ell}}$. Thus, splitting the integral and replacing ℓ_ε, the last expression can be rewritten as

$$L=\left(1-n\right)\frac{M}{\varepsilon_d}\Psi \tag{8.14}$$

where

$$\Psi\equiv\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}\left(\Delta\varepsilon-\varepsilon_d\right)dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar{\ell}}}^{\infty}\left(\Delta\varepsilon_{\bar{\ell}}-\varepsilon_d\right)dF\left(\varepsilon\right)$$

and where we have used (8.13) to replace p. The above equation is a function of ε_d.

The third step of the proof is to derive aggregate deposits, D. From Lemma 1, we know that depositors are: (i) sellers and (ii) buyers with a shock $\varepsilon < \varepsilon_d$. We also know that sellers deposit all their proceeds from sales in the goods markets. Hence aggregate deposits are

$$D = pq_s + (1-n)\int_0^{\varepsilon_d} d_\varepsilon dF(\varepsilon)$$

From Lemma 1, we also know that $d_\varepsilon = p(\varepsilon_d - \varepsilon)$. Thus, replacing d_ε, the last expression can be written as

$$D = pq_s + (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d} (\varepsilon_d - \varepsilon)\, dF(\varepsilon),$$

where we have also eliminated p using (8.13). Clearing condition in the goods market implies

$$q_s = n\int_0^{\infty} q_e dF(e) + (1-n)\int_0^{\infty} q_\varepsilon dF(\varepsilon)$$

Splitting the integrals and replacing the terms q_e and q_ε using Lemma 1 and Lemma 2, the above expression can be rewritten as

$$\begin{aligned} q_s &= n\int_0^{\infty} q_e dF(e) \\ &+ (1-n)\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

Replacing q_s using the last equation, aggregate deposits are

$$\begin{aligned} D &= (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d} (\varepsilon_d - \varepsilon)\, dF(\varepsilon) + n\int_0^{\infty} pq_e dF(e) \\ &+ (1-n)\frac{M}{\varepsilon_d}\left[\int_0^{\varepsilon_d} \varepsilon dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

where we have also used (8.13) to pin down p. Simplifying terms, the first expression can be rewritten as

$$\begin{aligned} D &= (1-n)\frac{M}{\varepsilon_d}\int_0^{\varepsilon_d}\varepsilon_d F(\varepsilon) + n\int_0^{\infty} pq_e dF(e) \\ &\quad + (1-n)\frac{M}{\varepsilon_d}\left[\int_{\varepsilon_d}^{\varepsilon_\ell}\varepsilon_d dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}\Delta\varepsilon dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty}\Delta\varepsilon_{\bar{\ell}} dF(\varepsilon)\right] \end{aligned}$$

Replacing $\int_0^{\varepsilon_d}\varepsilon_d F(\varepsilon)$ with $\varepsilon_d - \int_{\varepsilon_d}^{\infty}\varepsilon_d F(\varepsilon)$ and rearranging terms, the last expression can be rewritten as

$$D = (1-n)M + n\int_0^{\infty} pq_e dF(e) + (1-n)\frac{M}{\varepsilon_d}\Psi$$

Note that the amount of money spent by a $e-$buyer in the goods market is given by the amount of money he enters the treasury market minus the amount of money spent in bonds purchases, that is $pq_e = M - \rho y_e$. Replacing pq_e into the above equation yields

$$D = (1-n)M + n\int_0^{\infty}(M-\rho y_e)\,dF(e) + (1-n)\frac{M}{\varepsilon_d}\Psi$$

Since $\varepsilon-$buyers do not trade bonds in the treasury market, clearing condition implies

$$\int_0^{\infty} y_e dF(e) = 0$$

Replacing $\int_0^{\infty} y_e dF(e) = 0$ into the aggregate deposits we obtain

$$D = M + (1-n)\frac{M}{\varepsilon_d}\Psi \tag{8.15}$$

which is an expression in ε_d. The last equation means that the total amount of deposits is equal to the sum of two components. The first component is the total amount of money, M, held by buyers at the beginning of the goods market, and the second component is the total amount of lending, L.

Divide both sides of (3.5) by M and get

$$\gamma = 1 + \frac{i_d D - i_\ell L}{M} + \frac{B\,(1-\rho\gamma)}{M} + \tau$$

Eliminating D and L using (8.15), respectively (8.14), the last expression can be rewritten as

$$\gamma = 1 + i_d\left[1 + \frac{(1-n)}{\varepsilon_d}\Psi\right] - \frac{(1-n)\,i_\ell}{\varepsilon_d}\Psi + \frac{B\,(1-\rho\gamma)}{M} + \tau$$

Simplifying terms, the last equation can be rewritten as

$$\gamma = 1 + i_d + \frac{(1-n)\,(i_d - i_\ell)}{\varepsilon_d}\Psi + \frac{B\,(1-\rho\gamma)}{M} + \tau \tag{8.16}$$

which is a function in γ and ε_d. □

References

[1] Aliprantis, C., Camera, G. and Puzzello, D. 2007. Anonymous markets and monetary trading. Journal of Monetary Economics 54, 1905-28.

[2] Araujo, L. 2004. Social norms and money. Journal of Monetary Economics 51, 241-256.

[3] Berentsen, A. and Monnet, C. 2008. Monetary policy in a channel system. Journal of Monetary Economics, 55, 1067-1080

[4] Berentsen, A. and Waller, C.J. 2009. Outside versus inside bonds. Mimeo, University of Notre Dame

[5] Camera, G. and Boel, P. 2006. Efficient monetary allocations and the illiquidity of bonds. Journal of Monetary Economics 53, 1693-1715

[6] Kocherlakota, N. 1998. Money is memory. Journal of Economic Theory 81, 232-251.

[7] Kocherlakota, N. 2003. Societal benefits of illiquid bonds. Journal of Economic Theory, 108, 179-193.

[8] Lagos, R. 2009. Money spots. Mimeo, New York University

[9] Lagos, R. and Rocheteau, G. 2008. Money and capital as competing media of exchange. Journal of Economic Theory 142, 247-258

[10] Lagos, R. and Wright, R. 2005. A unified framework for monetary theory and policy analysis. Journal of Political Economy 113, 463-484.

[11] Lester, B, Postlewaite, A. and Wright, L. 2009. Information and liquidity. Mimeo, University of Pennsylvania

[12] Martin, A. and Monnet, C. 2009. Monetary policy implementation frameworks: A Comparative Analysis. Mimeo, Federal Reserve Bank of Philadelphia

[13] Shi, S. 1997. A divisible search model of fiat money. Econometrica 65, 75-102.

[14] Shi, S. 2008. Efficiency improvement from restricting the liquidity of nominal bonds. Journal of Monetary Economics, 55, 1025-1037.

[15] Wallace, N. 2001. Whither monetary economics? International Economic Review 42, 847-869.

(Aleksander Berentsen) Department of Business and Economics, University of Basel, Peter Merian-Weg 6, CH - 4002 Basel. (Email: aleksander.berentsen-at-unibas.ch.)

(Alessandro Marchesiani) Department of Economics and Business, University of Minho, Campus Gualtar, PT - 4710 Braga. (Email: marchesiani-at-gmail.com.)