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On sojourn times in M/GI systems under state-dependent processor sharing

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Abstract We consider a system with Poisson arrivals and i.i.d. service times. The requests are served according to the state-dependent processor-sharing discipline, where each request receives a service capacity which depends on the actual number of requests in the system. The linear systems of PDEs describing the residual and attained sojourn times coincide for this system, which provides time reversibility including sojourn times for this system, and their minimal non-negative solution gives the LST of the sojourn time $V(\tau)$ of a request with required service time τ. For the case that the service time distribution is exponential in a neighborhood of zero, we derive a linear system of ODEs, whose minimal non-negative solution gives the LST of $V(\tau)$, and which yields linear systems of ODEs for the moments of $V(\tau)$ in the considered neighborhood of zero. Numerical results are presented for the variance of $V(\tau)$. In the case of an $M/GI/2$-PS system, the LST of $V(\tau)$ is given in terms of the solution of a convolution equation in the considered neighborhood of zero. For service times bounded from below, surprisingly simple expressions for the LST and variance of $V(\tau)$ in this neighborhood of zero are derived, which yield in particular the LST and variance of $V(\tau)$ in $M/D/2$-PS.

Keywords Poisson arrivals · General service times · Locally exponential service times · Deterministic service times · State-dependent processor sharing · Generalized processor sharing · Many-server · Time reversibility · Insensitivity · $M/GI/r$-PS · $M/D/r$-PS · $M/D/2$-PS · Sojourn time · Laplace–Stieltjes transform · Moments

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1 Introduction

Processor Sharing (PS) systems have been widely used in the last decades for modeling and analyzing computer and communication systems, cf. for example [2, 4, 5, 7, 9, 11, 12, 17, 18], and the references therein. In this paper we deal with sojourn times of requests in a node, where the requests are served according to the following generalized processor-sharing discipline, which we call State-Dependent Processor Sharing (SDPS) discipline, cf. [8, 9]. If there are $n \in \mathbb{N} := \{1, 2, \dots\}$ requests in the node then each of them receives a positive service capacity $\varphi(n)$, i.e., each of the n requests receives during an interval of length $\Delta\tau$ the amount $\varphi(n)\Delta\tau$ of service. In the case of $\varphi_1(n) = 1/n$, $n \in \mathbb{N}$, we obtain the well known single-server PS system, cf. for example [7, 18], in the case of $\varphi_{1,k}(n) = 1/(n+k)$, $n \in \mathbb{N}$, we have a single-server PS system with $k \in \mathbb{N}$ permanent requests in the system, cf. [15, 19], in the case of $\varphi_r(n) = \min(r/n, 1)$, $n \in \mathbb{N}$, an r-server PS system, where all requests are served in a PS mode, but each request receives at most the capacity of one processor, cf. [8, p. 283], [3, 4, 9], in case of $\varphi_{r,k}(n) = \min(r/(n+k), 1)$, $n \in \mathbb{N}$, an r-server PS system with $k \in \mathbb{N}$ permanent requests, in the case of $\varphi_\infty(n) = 1$, $n \in \mathbb{N}$, an infinite-server system.

A system working under the SDPS discipline and where the requests arrive according to a Poisson process of intensity λ, the required service times are i.i.d. with df. $B(x) := P(S \le x)$, where S denotes a generic service time, and finite mean $m_S := ES$ and independent of the arrival process is denoted by $M/GI/$SDPS, the corresponding r-server PS system is denoted by $M/GI/r$-PS.

Networks with nodes working under the SDPS discipline are investigated in [1, 4–6, 8, 10, 20]. In particular, for the $M/GI/$SDPS system some basic results are known, cf. [8], which we will use and therefore shortly review in the following. Let $N(t)$ be the number of requests in the system at time t, $Y^*(t) := (Y_1^*(t), \dots, Y_{N(t)}^*(t))$ the vector of the residual service times of the $N(t)$ requests in the system at time t, ordered randomly, and $\tilde{Y}^*(t) := (\tilde{Y}_1^*(t), \dots, \tilde{Y}_{N(t)}^*(t))$ the vector of the attained service times of the $N(t)$ requests in the system at time t, ordered randomly. The vector processes $(N(t); Y^*(t))$, $t \in \mathbb{R}$, and $(N(t); \tilde{Y}^*(t))$, $t \in \mathbb{R}$, are Markov processes. The $M/GI/$SDPS system is stable, i.e., there exist unique stationary processes $(N(t); Y^*(t))$, $t \in \mathbb{R}$, and $(N(t); \tilde{Y}^*(t))$, $t \in \mathbb{R}$, if and only if

$$\sum_{n=0}^{\infty} \prod_{\ell=1}^{n} \frac{\varrho}{\ell\varphi(\ell)} < \infty, \tag{1.1}$$

where $\varrho := \lambda m_S$ denotes the offered load, cf. [8] (7.18). We assume in the following that the system is stable and in steady state, i.e. that (1.1) is satisfied and that $(N(t); Y^*(t))$, $t \in \mathbb{R}$, and $(N(t); \tilde{Y}^*(t))$, $t \in \mathbb{R}$, are stationary Markov processes. Then the stationary occupancy distribution $p(n) := P(N(t) = n)$, $n \in \mathbb{Z}_+$, as well as the stationary distributions of $(N(t); Y^*(t))$ and $(N(t); \tilde{Y}^*(t))$ on $\{N(t) = n\}$ are given by

$$p(n)=\left(\sum_{m=0}^{\infty}\prod_{\ell=1}^{m}\frac{\varrho}{\ell\varphi(\ell)}\right)^{-1}\prod_{\ell=1}^{n}\frac{\varrho}{\ell\varphi(\ell)},\tag{1.2}$$

$$\begin{aligned}&P\big(N(t)=n;Y_1^*(t)\le y_1,\dots,Y_n^*(t)\le y_n\big)\\&\quad=p(n)\prod_{\ell=1}^{n}B_R(y_\ell)=P\big(N(t)=n;\tilde{Y}_1^*(t)\le y_1,\dots,\tilde{Y}_n^*(t)\le y_n\big),\end{aligned}\tag{1.3}$$

where

$$B_R(x):=\frac{1}{m_S}\int_0^x\big(1-B(\xi)\big)\,\mathrm{d}\xi,\qquad x\in\mathbb{R}_+,\tag{1.4}$$

denotes the stationary residual service time distribution having the density $b_R(x)=(1-B(x))/m_S$, $x\in\mathbb{R}_+$, cf. [8] (7.19) for the case of phase-type distributed service times, [20] for the general case. For the sojourn time V of an arbitrary arriving request with required service time S, from Little's law and (1.2) we find

$$EV=\frac{1}{\lambda}\sum_{n=0}^{\infty}np(n)=m_S\sum_{n=0}^{\infty}\frac{1}{\varphi(n+1)}p(n).\tag{1.5}$$

For the conditional sojourn time $V(\tau)$ of a request with required service time $\tau\in\mathbb{R}_+$

$$EV(\tau)=\frac{\tau}{m_S}EV,\tag{1.6}$$

cf. [8] (7.27). More generally, for $\tau\in\mathbb{R}_+$, $k\in\mathbb{N}$ we have the estimate

$$\tau^k\left(\sum_{n=0}^{\infty}\frac{1}{\varphi(n+1)}p(n)\right)^k\le E\big[V^k(\tau)\big]\le\tau^k\sum_{n=0}^{\infty}\left(\frac{1}{\varphi(n+1)}\right)^k p(n),\tag{1.7}$$

cf. [5] Theorem 1.1. It seems that in the case of the general $M/GI/\mathrm{SDPS}$ system for $V(\tau)$ and V besides (1.5)–(1.7) there are known only asymptotic results for heavy tailed service times, cf. [9]. However, for special cases several results and numerical algorithms are well known. We mention only a few references. For the $M/GI/1$-PS system and special cases, cf. for example [7, 11, 13, 16–18, 21]. The variance of $V(\tau)$ in the $M/M/2$-PS system is given in [14]. The Laplace–Stieltjes transform (LST) and moments of $V(\tau)$ in the general $M/M/r$-PS system are treated in [3] and in the $M/M/\mathrm{SDPS}$ system in [4].

The aim of this paper is to derive analytical results and representations for sojourn times in the $M/GI/\mathrm{SDPS}$ system. The paper is organized as follows. In Sect. 2 we analyze a linear system of partial differential equations (PDEs), which has two different stochastic interpretations, implying time reversibility including sojourn times of the $M/GI/\mathrm{SDPS}$ system. Moreover, its minimal non-negative solution gives the LST of $V(\tau)$, which implies that $V(\tau)$ depends on $B(x)$ for $x>\tau$ only via m_S in distribution. In Sect. 3 we assume that the service time distribution coincides with an exponential distribution in some interval $[0,d)$. We derive a linear system of ordinary differential equations (ODEs) with constant coefficients, whose minimal non-negative solution gives the LST of $V(\tau)$, $\tau \in [0,d)$, and which provides corresponding linear systems of ODEs for the moments of $V(\tau)$, $\tau \in [0,d)$. For the case that the service time is the minimum of an exponential and deterministic time we give the LST and the moments of V in terms of minimal non-negative solutions of linear systems of ODEs. Numerical results are presented for the variance of V in $M/D/r$-PS. In Sect. 4 we assume again that the service time distribution coincides with an exponential distribution in some interval $[0,d)$, but we consider the special case of an $M/GI/r$-PS system. For the $M/GI/2$-PS system a representation for the LST of $V(\tau)$, $\tau \in [0,d]$, in terms of the solution of a convolution equation is given. For the limiting case of service times bounded from below, surprisingly simple expressions for the LST and variance of $V(\tau)$, $\tau \in [0,d]$, are derived, which yield in particular the LST and variance of $V(\tau)$ and V in $M/D/2$-PS.

2 Sojourn times in $M/GI/$SDPS

We assume in the following that the system is stable, i.e., that (1.1) is satisfied, and in steady state. In particular m_S is finite. Moreover, for technical reasons—if not stated otherwise—we make in the following the assumption:

(A1) $B(x)$ has a continuous density $b(x)$ and $B(x) < 1$ for $x \in \mathbb{R}_+$.

For notational convenience let $\bar{B}(x) := 1 - B(x)$, $\bar{B}_R(x) := 1 - B_R(x)$, cf. (1.4), and $\beta(x) := b(x)/\bar{B}(x)$, $\beta_R(x) := b_R(x)/\bar{B}_R(x)$ be the complementary distributions and hazard rates of the service time df. and the stationary residual service time df., respectively. Further we will use several vector notations in this section. If not stated otherwise, let $y := (y_1, \dots, y_\ell) \in \mathbb{R}_+^\ell$ where $\ell = m$ or $\ell = n$, respectively, and

$$\Omega_\ell := \{ y \in \mathbb{R}_+^\ell : 0 < y_1 < \cdots < y_\ell \}.$$

For $y, \tilde{y} \in \mathbb{R}^\ell$ let $y \le \tilde{y}$ if and only if $y_i \le \tilde{y}_i$ for $i = 1, \dots, \ell$.

Besides the randomly ordered residual service times $Y_1^*(t), \dots, Y_{N(t)}^*(t)$ and attained service times $\tilde{Y}_1^*(t), \dots, \tilde{Y}_{N(t)}^*(t)$ we need them ordered increasingly. Let $0 \le Y_1(t) \le \cdots \le Y_{N(t)}(t)$ be the residual service times of the $N(t)$ requests at time t, ordered increasingly, and let $Y(t) := (Y_1(t), \dots, Y_{N(t)}(t))$ be the corresponding vector. In view of the SDPS discipline, this implies that the requests are ordered according to their departure instants in this case. Let $0 \le \tilde{Y}_1(t) \le \cdots \le \tilde{Y}_{N(t)}(t)$ be the attained service times of the $N(t)$ requests at time t, ordered increasingly, and let $\tilde{Y}(t) := (\tilde{Y}_1(t), \dots, \tilde{Y}_{N(t)}(t))$ be the corresponding vector. In view of the SDPS discipline, this implies that the requests are ordered reversely to their arrival instants in this case. For $n \in \mathbb{N}$, $y \in \Omega_n$ let

$$p(n; y) := \frac{\partial^n}{\partial y_1 \dots \partial y_n} P\big(N(t) = n; Y(t) \le y\big)$$

be the density of $Y(t)$ on $\{N(t)=n\}$ and

$$\tilde{p}(n;y) := \frac{\partial^n}{\partial y_1 \dots \partial y_n} P\big(N(t)=n; \tilde{Y}(t) \le y\big)$$

be the density of $\tilde{Y}(t)$ on $\{N(t)=n\}$. On the boundary of Ω_n let $p(n;y)$ and $\tilde{p}(n;y)$ be defined by continuous continuation. The support of $p(n;y)$ and $\tilde{p}(n;y)$ is the closure $\bar{\Omega}_n$ of Ω_n.

Denoting by $\mathcal{S}_n$ the set of all permutations of the set $\{1,\dots,n\}$, from (1.3), (1.4) for $n \in \mathbb{N}$, $y \in \Omega_n$ we obtain

$$\begin{aligned} p(n;y) &= \sum_{\pi \in \mathcal{S}_n} \frac{\partial^n}{\partial y_1 \dots \partial y_n} P\big(N(t)=n; Y_1^*(t) \le y_{\pi(1)}, \dots, Y_n^*(t) \le y_{\pi(n)}\big) \\ &= n! p(n) \prod_{\ell=1}^{n} b_R(y_\ell), \end{aligned} \tag{2.1}$$

and analogously it follows that

$$\tilde{p}(n;y) = n! p(n) \prod_{\ell=1}^{n} b_R(y_\ell). \tag{2.2}$$

By continuous continuation, (2.1) and (2.2) hold for $n \in \mathbb{N}$, $y \in \bar{\Omega}_n$, too.

2.1 PDEs for sojourn times

For the $M/GI/\mathrm{SDPS}$ system let the stability condition (1.1) and (A1) be satisfied. Let $V_\ell(t)$, $\ell = 1,\dots,N(t)$, be the sojourn time of the request with residual service time $Y_\ell(t)$ from time t on until its departure from the system, i.e., its prospective sojourn time from time t on. Since the $Y_\ell(t)$ are ordered increasingly, the SDPS discipline implies that the $V_\ell(t)$ are ordered increasingly, too, i.e., $0 \le V_1(t) \le \cdots \le V_{N(t)}(t)$. Further, $V_1(t)=0$ if and only if $Y_1(t)=0$. In view of (A1) and the distributional and independence assumptions, for $0 < m \le n$, $y \in \Omega_n$, the LSTs

$$v_{n,m}(s;y) := \frac{\partial^n}{\partial y_1 \dots \partial y_n} E\big[e^{-sV_m(t)} \mathbb{I}\big\{N(t)=n, Y(t) \le y\big\}\big] \tag{2.3}$$

of $V_m(t)$ on $\{N(t)=n, Y_1(t) \in \mathrm{d}y_1, \dots, Y_n(t) \in \mathrm{d}y_n\}$ are well defined for $s \in \mathbb{R}_+$. For fixed s and $0 < m \le n$, let $v_{n,m}(s;y)$ be defined on the boundary of Ω_n by continuous continuation. Taking into account that in [5] the residual service times $Y_\ell(t)$ are denoted by $R_\ell(t)$, from [5] (2.2)–(2.6), (1.2), and (1.4) it follows that the $v_{n,m}(s;y)$ satisfy the following linear system of PDEs:

$$\varphi(n) \frac{\partial}{\partial \xi} v_{n,m}(s; y_1+\xi, \dots, y_n+\xi)\Big|_{\xi=0}$$

$$= -\left(\lambda + s + \varphi(n)\sum_{\ell=1}^{n}\beta(y_\ell)\right)v_{n,m}(s; y_1, \dots, y_n)$$
$$+ \varphi(n+1)\sum_{\ell=1}^{n+1}\int_{y_{\ell-1}}^{y_\ell} v_{n+1,m+\mathbb{I}\{\ell\le m\}}(s; y_1, \dots, y_{\ell-1}, \tau, y_\ell, \dots, y_n)\beta(\tau)\,\mathrm{d}\tau \tag{2.4}$$

for $0 < m \le n$, $y \in \Omega_n$, where $y_0 := 0$ and $y_{n+1} := \infty$, with initial condition

$$v_{n,1}(s; 0, y_2, \dots, y_n) = n!p(n)m_S^{-1}\prod_{\ell=2}^{n} b_R(y_\ell), \tag{2.5}$$

$$v_{n,m}(s; 0, y_2, \dots, y_n) = \frac{\lambda}{\varphi(n)}v_{n-1,m-1}(s; y_2, \dots, y_n), \quad 1 < m \le n, \tag{2.6}$$

for $0 \le y_2 \le \dots \le y_n$, and that

$$v_{n,m}(0; y) = n!p(n)\prod_{\ell=1}^{n} b_R(y_\ell), \quad 0 < m \le n, y \in \bar{\Omega}_n. \tag{2.7}$$

Note that these equations are consequences of the Kolmogorov forward equations (Fokker–Planck equations) and of (2.1).

Let $\tilde{V}_\ell(t)$, $\ell = 1, \dots, N(t)$, be the sojourn time of the request with attained service time $\tilde{Y}_\ell(t)$ from its arrival at the system until time t. Since the $\tilde{Y}_\ell(t)$ are ordered increasingly, the SDPS discipline implies that the $\tilde{V}_\ell(t)$ are ordered increasingly, too, i.e. $0 \le \tilde{V}_1(t) \le \dots \le \tilde{V}_{N(t)}(t)$. Further, $\tilde{V}_1(t) = 0$ if and only if $\tilde{Y}_1(t) = 0$. In view of (A1) and the distributional and independence assumptions, for $0 < m \le n$, $y \in \Omega_n$, the LSTs

$$\tilde{v}_{n,m}(s; y) := \frac{\partial^n}{\partial y_1 \dots \partial y_n}E\big[e^{-s\tilde{V}_m(t)}\mathbb{I}\big\{N(t) = n, \tilde{Y}(t) \le y\big\}\big] \tag{2.8}$$

of $\tilde{V}_m(t)$ on $\{N(t) = n, \tilde{Y}_1(t) \in \mathrm{d}y_1, \dots, \tilde{Y}_n(t) \in \mathrm{d}y_n\}$ are well defined for $s \in \mathbb{R}_+$. For fixed s and $0 < m \le n$, let $\tilde{v}_{n,m}(s; y)$ be defined on the boundary of Ω_n by continuous continuation. As $\tilde{v}_{n,m}(0; y)$ corresponds to the density of $\tilde{Y}(t)$ on $\{N(t) = n\}$, from (2.2) we find that

$$\tilde{v}_{n,m}(0; y) = \tilde{p}(n; y) = n!p(n)\prod_{\ell=1}^{n} b_R(y_\ell), \quad 0 < m \le n, y \in \bar{\Omega}_n. \tag{2.9}$$

In the following let $s \in \mathbb{R}_+$ be fixed. The dynamics of the $M/GI/$SDPS system during an interval $[t - h, t]$ of length h provide for $0 < m \le n$, $y \in \Omega_n$ the balance condition

$$
\begin{aligned}
&\tilde{v}_{n,m}(s; y_1, \dots, y_n) \\
&\quad = \left(1 - \lambda h - \varphi(n) h \sum_{\ell=1}^{n} \beta(y_\ell)\right) e^{-sh} \tilde{v}_{n,m}(s; y_1 - \varphi(n)h, \dots, y_n - \varphi(n)h) \\
&\qquad + \varphi(n+1) h \sum_{\ell=1}^{n+1} \int_{y_{\ell-1}}^{y_\ell} \tilde{v}_{n+1, m+\mathbb{I}\{\ell \le m\}}(s; y_1, \dots, y_{\ell-1}, \tau, y_\ell, \dots, y_n) \beta(\tau)\, \mathrm{d}\tau \\
&\qquad + o(h).
\end{aligned}
$$

The first summand on the r.h.s. corresponds to the situation that during $[t-h, t]$ there is no arrival and no departure; the sojourn time increases by h. The second summand corresponds to departures from the system. Subtracting on both sides $\tilde{v}_{n,m}(s; y_1 - \varphi(n)h, \dots, y_n - \varphi(n)h)$, dividing by h and taking $h \downarrow 0$ provides the following linear system of PDEs:

$$
\begin{aligned}
&\varphi(n) \frac{\partial}{\partial \xi} \tilde{v}_{n,m}(s; y_1 + \xi, \dots, y_n + \xi) \Big|_{\xi=0} \\
&\quad = -\left(\lambda + s + \varphi(n) \sum_{\ell=1}^{n} \beta(y_\ell)\right) \tilde{v}_{n,m}(s; y_1, \dots, y_n) \\
&\qquad + \varphi(n+1) \sum_{\ell=1}^{n+1} \int_{y_{\ell-1}}^{y_\ell} \tilde{v}_{n+1, m+\mathbb{I}\{\ell \le m\}}(s; y_1, \dots, y_{\ell-1}, \tau, y_\ell, \dots, y_n) \beta(\tau)\, \mathrm{d}\tau
\end{aligned} \tag{2.10}
$$

for $0 < m \le n$, $y \in \Omega_n$, which correspond to the Kolmogorov forward equations. Taking into consideration arrivals, we find the initial conditions for $0 \le y_2 \le \dots \le y_n$. As $\tilde{Y}_1(t) = 0$ implies $\tilde{V}_1(t) = 0$, from (2.9) and (1.4) we obtain

$$
\tilde{v}_{n,1}(s; 0, y_2, \dots, y_n) = \tilde{v}_{n,1}(0; 0, y_2, \dots, y_n) = n! p(n) m_S^{-1} \prod_{\ell=2}^{n} b_R(y_\ell). \tag{2.11}
$$

In the case of $1 < m \le n$ the dynamics provide

$$
\tilde{v}_{n,m}(s; 0, y_2, \dots, y_n) = \frac{\lambda}{\varphi(n)} \tilde{v}_{n-1,m-1}(s; y_2, \dots, y_n), \quad 1 < m \le n, \tag{2.12}
$$

as the arrival probability for an interval of length $h/\varphi(n)$ is $1 - e^{-h\lambda/\varphi(n)}$.

Note that the $v_{n,m}(s; y)$ and the $\tilde{v}_{n,m}(s; y)$ satisfy the same system of PDEs (2.4)–(2.7).

Lemma 2.1 *For any fixed $s \in (0, \infty)$, the linear system of PDEs (2.4) with the initial conditions (2.5), (2.6) and growth condition*

$$
0 \le v_{n,m}(s; y) \le v_{n,m}(0; y), \quad 0 < m \le n, y \in \bar{\Omega}_n, \tag{2.13}
$$

where $v_{n,m}(0; y)$ is given by (2.7), has at most one solution.

Proof Let $s\in(0,\infty)$ be fixed. Replacing in (2.4) y_i by $y_i+\eta$, $i=1,\dots,n$, integrating with the weight $\exp((\lambda+s)\eta/\varphi(n))\prod_{\ell=1}^n\bar{B}(y_\ell)/\bar{B}(y_\ell+\eta)$ over $[-y_1,0]$ with respect to η and applying (2.5), (2.6) demonstrates that the system of PDEs (2.4) with the initial conditions (2.5), (2.6) is equivalent to the following linear system of integral equations for $0<m\le n$, $y\in\bar{\Omega}_n$:

$$\begin{aligned}
&v_{n,m}(s;y_1,\dots,y_n)\\
&\quad=\mathbb{I}\{m=1\}n!p(n)m_S^{-n}e^{-\frac{\lambda+s}{\varphi(n)}y_1}\left(\prod_{\ell=1}^n\bar{B}(y_\ell)\right)\\
&\qquad+\mathbb{I}\{m>1\}\frac{\lambda}{\varphi(n)}e^{-\frac{\lambda+s}{\varphi(n)}y_1}\left(\prod_{\ell=1}^n\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell-y_1)}\right)v_{n-1,m-1}(s;y_2-y_1,\dots,y_n-y_1)\\
&\qquad+\frac{\varphi(n+1)}{\varphi(n)}\int_{-y_1}^0 e^{\frac{\lambda+s}{\varphi(n)}\xi}\left(\prod_{\ell=1}^n\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell+\xi)}\right)\\
&\qquad\times\sum_{\ell=1}^{n+1}\int_{y_{\ell-1}-\mathbb{I}\{\ell=1\}\xi}^{y_\ell}v_{n+1,m+\mathbb{I}\{\ell\le m\}}(s;y_1+\xi,\dots,y_{\ell-1}+\xi,\tau+\xi,y_\ell+\xi,\\
&\qquad\qquad\dots,y_n+\xi)\beta(\tau+\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi.
\end{aligned}\tag{2.14}$$

Assume that the system of integral equations (2.14) has two different solutions, where both solutions fulfill the growth condition (2.13). Then the difference $\bar{v}_{n,m}(s;y)$ of these solutions satisfies the homogenized version of (2.14). For $0<m\le n$ let $\kappa_{n,m}\in\mathbb{R}_+$ denote the smallest number such that $|\bar{v}_{n,m}(s;y)|\le\kappa_{n,m}v_{n,m}(0;y)$, $y\in\bar{\Omega}_n$. Note that $\kappa_{n,m}\in(0,1]$ due to the growth condition. Let $\kappa_{n-1,0}:=0$ for notational convenience and let $\kappa_0:=\sup_{0<m\le n}\kappa_{n,m}$. The triangle inequality applied to the homogenized version of (2.14), the definitions of $\kappa_{n,m}$ and κ_0, and (2.7), (1.4), (1.2) provide after some algebra for $0<m\le n$, $y\in\bar{\Omega}_n$ that

$$\begin{aligned}
&\left|\bar{v}_{n,m}(s;y_1,\dots,y_n)\right|\\
&\quad\le\kappa_{n-1,m-1}\frac{\lambda}{\varphi(n)}e^{-\frac{\lambda+s}{\varphi(n)}y_1}\left(\prod_{\ell=1}^n\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell-y_1)}\right)v_{n-1,m-1}(0;y_2-y_1,\dots,y_n-y_1)\\
&\qquad+\kappa_0\frac{\varphi(n+1)}{\varphi(n)}\int_{-y_1}^0 e^{\frac{\lambda+s}{\varphi(n)}\xi}\left(\prod_{\ell=1}^n\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell+\xi)}\right)\\
&\qquad\times\sum_{\ell=1}^{n+1}\int_{y_{\ell-1}-\mathbb{I}\{\ell=1\}\xi}^{y_\ell}v_{n+1,m+\mathbb{I}\{\ell\le m\}}(0;y_1+\xi,\dots,y_{\ell-1}+\xi,\tau+\xi,y_\ell+\xi,\\
&\qquad\qquad\dots,y_n+\xi)\beta(\tau+\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\\
&\quad=\left(\kappa_{n-1,m-1}e^{-\frac{\lambda+s}{\varphi(n)}y_1}+\frac{\lambda}{\lambda+s}\kappa_0\left(1-e^{-\frac{\lambda+s}{\varphi(n)}y_1}\right)\right)v_{n,m}(0;y_1,\dots,y_n)
\end{aligned}$$

$$\leq \max\left(\kappa_{n-1,m-1}, \frac{\lambda}{\lambda+s}\kappa_0\right) v_{n,m}(0; y_1, \ldots, y_n).$$

Therefore, by definition of $\kappa_{n,m}$ it follows that

$$\kappa_{n,m} \leq \max\left(\kappa_{n-1,m-1}, \frac{\lambda}{\lambda+s}\kappa_0\right), \quad 0 < m \leq n. \tag{2.15}$$

Let $j \in \mathbb{Z}_+$ be arbitrarily fixed. In view of $\kappa_{j,0} = 0$, from (2.15) we find $\kappa_{j+1,1} \leq \frac{\lambda}{\lambda+s}\kappa_0$. Now induction on $m \in \mathbb{N}$ yields $\kappa_{j+m,m} \leq \frac{\lambda}{\lambda+s}\kappa_0$ in view of (2.15). Thus we obtain $\kappa_0 \leq \frac{\lambda}{\lambda+s}\kappa_0$, which implies the contradiction $\kappa_0 \leq 0$. □

From Lemma 2.1 we conclude that the Kolmogorov forward equations (2.4)–(2.7) and (2.9)–(2.12) provide a complete description of the LSTs $v_{n,m}(s; y)$ and $\tilde{v}_{n,m}(s; y)$, respectively. Moreover, since the $v_{n,m}(s; y)$ and the $\tilde{v}_{n,m}(s; y)$ satisfy the same system of PDEs (2.4)–(2.7), and taking into account (2.1), (2.2), from Lemma 2.1 we obtain the following time reversibility result for $M/GI/$SDPS systems, cf. [10] for the relationship between reversed processes, supplemented by spent and residual lifetimes, and insensitivity in the case of finite state spaces:

Theorem 2.1 *Let the stability condition* (1.1) *for the* $M/GI/$SDPS *system with* (A1) *be satisfied. Then*

$$\tilde{v}_{n,m}(s; y) = v_{n,m}(s; y), \quad 0 < m \leq n, y \in \bar{\Omega}_n, s \in \mathbb{R}_+, \tag{2.16}$$

$$\begin{aligned} &P\big(\tilde{V}_m(t) \leq x | N(t) = n, \tilde{Y}(t) = y\big) = P\big(V_m(t) \leq x | N(t) = n, Y(t) = y\big), \\ &\quad 0 < m \leq n, y \in \bar{\Omega}_n, x \in \mathbb{R}_+. \end{aligned} \tag{2.17}$$

Because of the SDPS discipline, from a probabilistic point of view, for $0 < m \leq n$, $y \in \bar{\Omega}_n$ the sojourn time $V_m(t)$ conditioned on $N(t) = n$, $Y(t) = y$ depends only on $y_1, \ldots, y_m$ and the total number n of requests in the system since the requests with residual service times $y_{m+1}, \ldots, y_n$ have residual service times of an amount greater or equal to y_m and are thus in the system at least as long as the request with service time y_m. However, in the following a rigorous proof will be given. In view of (2.16) and (2.7), (2.9), we try the substitution

$$\begin{aligned} \tilde{v}_{n,m}(s; y_1, \ldots, y_n) &= v_{n,m}(s; y_1, \ldots, y_n) \\ &= u_{n,m}(s; y_1, \ldots, y_m) \prod_{\ell=m+1}^{n} b_R(y_\ell), \end{aligned} \tag{2.18}$$

where the $u_{n,m}(s; y)$, $0 < m \leq n$, $y \in \bar{\Omega}_m$, are continuous functions. The system of PDEs (2.4) is satisfied if the $u_{n,m}(s; y)$ satisfy the following system of PDEs for $0 < m \leq n$, $y \in \Omega_m$:

$$\varphi(n)\frac{\partial}{\partial \xi} u_{n,m}(s; y_1 + \xi, \ldots, y_m + \xi)\bigg|_{\xi=0}$$

$$= -\left(\lambda + s + \varphi(n)\sum_{\ell=1}^{m}\beta(y_\ell)\right)u_{n,m}(s; y_1,\dots,y_m)$$
$$+ \varphi(n+1)\sum_{\ell=1}^{m}\int_{y_{\ell-1}}^{y_\ell} u_{n+1,m+1}(s; y_1,\dots,y_{\ell-1},\tau,y_\ell,\dots,y_m)\beta(\tau)\,\mathrm{d}\tau$$
$$+ \varphi(n+1)b_R(y_m)u_{n+1,m}(s; y_1,\dots,y_m). \tag{2.19}$$

The initial condition (2.5) is satisfied if for $1 = m \le n$

$$u_{n,1}(s;0) = n!p(n)m_S^{-1}, \tag{2.20}$$

and the initial condition (2.6) is satisfied if for $1 < m \le n$, $0 \le y_2 \le \cdots \le y_m$

$$u_{n,m}(s; 0, y_2,\dots,y_m) = \frac{\lambda}{\varphi(n)}u_{n-1,m-1}(s; y_2,\dots,y_m). \tag{2.21}$$

Note that (2.7) is satisfied if for $0 < m \le n$, $y \in \bar{\Omega}_m$

$$u_{n,m}(0; y_1,\dots,y_m) = n!p(n)\prod_{\ell=1}^{m} b_R(y_\ell). \tag{2.22}$$

Lemma 2.2 *For any $s \in \mathbb{R}_+$, the linear system of PDEs (2.19) with the initial conditions (2.20), (2.21) has a minimal non-negative solution, and the minimal non-negative solution is bounded by $u_{n,m}(0; y)$, i.e. by the r.h.s. of (2.22).*

Proof Analogously to the derivation of (2.14) we find that the system of PDEs (2.19) with initial conditions (2.20), (2.21) is equivalent to the following system of integral equations for $0 < m \le n$, $y \in \bar{\Omega}_m$:

$$\begin{aligned}
&u_{n,m}(s; y_1,\dots,y_m)\\
&\quad = e^{-\frac{\lambda+s}{\varphi(n)}y_1}\left(\prod_{\ell=1}^{m}\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell - y_1)}\right)\\
&\qquad \times\left(\mathbb{I}\{m=1\}n!p(n)m_S^{-1} + \mathbb{I}\{m>1\}\frac{\lambda}{\varphi(n)}u_{n-1,m-1}(s; y_2-y_1,\dots,y_m-y_1)\right)\\
&\qquad + \frac{\varphi(n+1)}{\varphi(n)}\int_{-y_1}^{0} e^{\frac{\lambda+s}{\varphi(n)}\xi}\left(\prod_{\ell=1}^{m}\frac{\bar{B}(y_\ell)}{\bar{B}(y_\ell+\xi)}\right)\\
&\qquad \times\left(\sum_{\ell=1}^{m}\int_{y_{\ell-1}-\mathbb{I}\{\ell=1\}\xi}^{y_\ell} u_{n+1,m+1}(s; y_1+\xi,\dots,y_{\ell-1}+\xi,\tau+\xi,y_\ell+\xi,\right.\\
&\qquad\qquad \dots,y_m+\xi)\beta(\tau+\xi)\,\mathrm{d}\tau\\
&\qquad \left. + b_R(y_m+\xi)u_{n+1,m}(s; y_1+\xi,\dots,y_m+\xi)\right)\mathrm{d}\xi.
\end{aligned} \tag{2.23}$$

This system of integral equations can be solved by the method of successive approximation starting from $u_{n,m;0}(s; y_1, \dots, y_m) \equiv 0$ and defining $u_{n,m;i+1}(s; y_1, \dots, y_m)$ recursively with respect to i by the r.h.s. of (2.23), where the $u_{n',m'}(s; \tilde{y}_1, \dots, \tilde{y}_{m'})$ are replaced by $u_{n',m';i}(s; \tilde{y}_1, \dots, \tilde{y}_{m'})$. For this recursion by induction on i it follows that for fixed $0 < m \le n$, $0 \le y_1 \le \cdots \le y_m$ the $u_{n,m;i}(s; y_1, \dots, y_m)$ are monotonically increasing with respect to i and that

$$u_{n,m;i}(s; y_1, \dots, y_m) \le u_{n,m;i}(0; y_1, \dots, y_m). \tag{2.24}$$

Moreover, if $u_{n,m}(s; y_1, \dots, y_m)$ is any non-negative solution of the system of PDEs (2.19) with initial conditions (2.20), (2.21) for fixed $s \in \mathbb{R}_+$, then $u_{n,m}(s; y_1, \dots, y_m)$ is a non-negative solution of (2.23), and thus it follows that $u_{n,m;i}(s; y_1, \dots, y_m) - u_{n,m}(s; y_1, \dots, y_m)$ satisfies the corresponding homogeneous recursion with respect to i. Therefore induction on i yields

$$u_{n,m;i}(s; y_1, \dots, y_m) \le u_{n,m}(s; y_1, \dots, y_m). \tag{2.25}$$

Because of (1.2) and (1.4), the r.h.s. of (2.22) is a solution of (2.19) and (2.20), (2.21) for $s = 0$, thus $u_{n,m;i}(0; y_1, \dots, y_m)$ is bounded by the r.h.s. of (2.22), and in view of (2.24), $u_{n,m;i}(s; y_1, \dots, y_m)$ is bounded by the r.h.s. of (2.22) for any $s \in \mathbb{R}_+$. Thus for any $s \in \mathbb{R}_+$, the limit $\lim_{i\to\infty} u_{n,m;i}(s; y_1, \dots, y_m)$ exists pointwise and is, due to Lebesgue's theorem, a non-negative solution of (2.23). This solution is the minimal non-negative solution of (2.23) because of (2.25). Moreover, in view of (2.24), it is bounded by the minimal non-negative solution of (2.23) for $s = 0$, and thus by the r.h.s. of (2.22) as the r.h.s. of (2.22) is a non-negative solution of (2.19) and (2.20), (2.21) for $s = 0$, i.e. of (2.23) for $s = 0$. □

Summarizing Lemma 2.1 and Lemma 2.2 we have proved the following.

Theorem 2.2 *Let the stability condition* (1.1) *for the* $M/GI/$SDPS *system with* (A1) *be satisfied.*

Then for $s \in (0, \infty)$, *the LSTs* $v_{n,m}(s; y)$ *and* $\tilde{v}_{n,m}(s; y)$ *are given by* (2.18) *for* $0 < m \le n$, $y \in \bar{\Omega}_n$, *where* $u_{n,m}(s; y)$, $y \in \bar{\Omega}_m$, *is the minimal non-negative solution of the linear system of PDEs* (2.19) *with the initial conditions* (2.20), (2.21), *i.e. the non-negative solution of* (2.19)–(2.21) *which is bounded by the r.h.s. of* (2.22).

It seems that for general $\varphi(n)$, $n \in \mathbb{N}$, there is no explicit solution of (2.19)–(2.21). However, for $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, $k \in (-1, \infty)$, the minimal non-negative solution can be given by adopting results of [11, 17], leading to the well known results for $M/GI/1$-PS systems, cf. Example 2.1 in [5].

Example 2.1 Let $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, $k \in (-1, \infty)$, and $s \in \mathbb{R}_+$. We try, cf. [11, 16], the substitution

$$\frac{u_{n,m}(s; y_1, \dots, y_m)}{u_{n,m}(0; y_1, \dots, y_m)} = \delta(s, y_m)^{n+k} \prod_{\ell=1}^{m-1} \frac{1}{\delta(s, y_m - y_\ell)}$$

for $0 < m \le n$, $0 \le y_1 \le \cdots \le y_m$, where $u_{n,m}(0; y_1, \ldots, y_m)$ is given by (2.22) and $\delta(s,\tau)$ is a continuously differentiable function in $\tau \in \mathbb{R}_+$ with initial condition $\delta(s,0) = 1$. The substitution satisfies (2.20) and (2.21). Inserting the substitution into (2.19) and using that $\varphi_{1,k}(n) = 1/(n+k)$, $n \in \mathbb{N}$, one finds that the linear system of PDEs (2.19) is satisfied if $1/\delta(s,\tau)$ satisfies the integro-differential equation

$$\frac{\partial}{\partial \tau}\frac{1}{\delta(s,\tau)} = (s+\lambda)\frac{1}{\delta(s,\tau)} - \lambda \int_0^\tau \frac{\mathrm{d}B(\xi)}{\delta(s,\tau-\xi)} - \lambda \bar{B}(\tau)$$

with initial condition $1/\delta(s,0) = 1$, which has a uniquely determined solution, cf. [16]. Note that $1/\delta(s,\tau)$ is non-decreasing with respect to τ, which implies that $u_{n,m}(s; y_1, \ldots, y_m)$ is bounded by $u_{n,m}(0; y_1, \ldots, y_m)$. The product form solution for $u_{n,m}(s; y_1, \ldots, y_m)/u_{n,m}(0; y_1, \ldots, y_m)$, given above, has been proved for $k = 0$, i.e. for $M/GI/1$-PS systems, for the first time by a decomposition of the sojourn time in [11, 16] and for $k \in \mathbb{N}$, i.e. for the single-server PS model with k permanent requests, in [19].

2.2 LST and moments of $V(\tau)$

For the $M/GI/$SDPS system let the stability condition (1.1) and (A1) be satisfied. For $s \in \mathbb{R}_+$, $0 < m \le n$, $x \in \mathbb{R}_+$ let

$$g_{n,m}(s,x) := \frac{\partial}{\partial x} E\big[e^{-sV_m(t)}\mathbb{I}\big\{N(t) = n, Y_m(t) \le x\big\}\big]. \tag{2.26}$$

Note that in view of Theorem 2.1, also

$$g_{n,m}(s,x) = \frac{\partial}{\partial x} E\big[e^{-s\tilde{V}_m(t)}\mathbb{I}\big\{N(t) = n, \tilde{Y}_m(t) \le x\big\}\big].$$

From Theorem 2.2 by integrating $v_{n,m}(s; y)$ over

$$0 \le y_1 \le \cdots \le y_{m-1} \le x \le y_{m+1} \le \cdots \le y_n$$

with respect to $\mathrm{d}y_1 \ldots \mathrm{d}y_{m-1}\,\mathrm{d}y_{m+1} \ldots \mathrm{d}y_n$ we obtain

$$g_{n,m}(s,x) = e_{n-m}(x) f_{n,m}(s,x), \tag{2.27}$$

where

$$e_\ell(x) := \frac{\bar{B}_R(x)^\ell}{\ell!}, \tag{2.28}$$

$$f_{n,m}(s,x) := \int_{0 \le y_1 \le \cdots \le y_{m-1} \le x} u_{n,m}(s; y_1, \ldots, y_{m-1}, x)\,\mathrm{d}y_1 \ldots \mathrm{d}y_{m-1}. \tag{2.29}$$

For $s \in (0,\infty)$, $0 < m \le n$, $x \in \mathbb{R}_+$, $\ell \in \mathbb{Z}_+$ let

$$g^{(\ell)}_{n,m}(s,x) := (-1)^\ell \frac{\partial^\ell}{\partial s^\ell} g_{n,m}(s,x). \tag{2.30}$$

From (2.26) it follows that

$$g_{n,m}^{(\ell)}(s,x) = \frac{\partial}{\partial x} E\big[V_m^\ell(t) e^{-sV_m(t)} \mathbb{I}\big\{N(t)=n, Y_m(t)\le x\big\}\big], \tag{2.31}$$

which implies

$$0 \le g_{n,m}^{(\ell)}(s,x) \le \ell! s^{-\ell} g_{n,m}(0,x) \tag{2.32}$$

in view of $v^\ell e^{-sv} \le \ell! s^{-\ell}$. Moreover, from (2.31) it follows that $g_{n,m}^{(\ell)}(s,x)$ is monotonically decreasing with respect to $s \in (0,\infty)$. Thus the limit

$$g_{n,m}^{(\ell)}(x) := \lim_{s\downarrow 0} g_{n,m}^{(\ell)}(s,x), \quad 0 < m \le n, x \in \mathbb{R}_+, \ell \in \mathbb{Z}_+, \tag{2.33}$$

exists, but it may be infinite for some $\ell \in \mathbb{N}$.

Let $V(n,\tau)$, $n \in \mathbb{Z}_+$, $\tau \in \mathbb{R}_+$, be the sojourn time of a tagged arriving request with required service time τ (τ-request) finding n requests at its arrival in the system, and let $V(\tau)$, $\tau \in \mathbb{R}_+$, be the sojourn time of a tagged arriving τ-request. For the $M/GI/$SDPS system the following representations for the LSTs and moments of $V(n,\tau)$ and $V(\tau)$ are known, cf. [5] Theorem 1.1, Theorem 2.1, and Theorem 3.1:

Theorem 2.3 *For the $M/GI/$SDPS system let the stability condition (1.1) and (A1) be satisfied. Then for $s \in \mathbb{R}_+$ and $\tau \in \mathbb{R}_+$ the LSTs of $V(n,\tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$ are given by*

$$E\big[e^{-sV(n,\tau)}\big] = \frac{\varphi(n+1)}{\lambda(\tau)p(n)} \sum_{m=1}^{n+1} g_{n+1,m}(s,\tau), \tag{2.34}$$

$$E\big[e^{-sV(\tau)}\big] = \frac{1}{\lambda(\tau)} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} g_{n,m}(s,\tau), \tag{2.35}$$

respectively, where

$$\lambda(x) := \lambda \bar{B}(x), \quad x \in \mathbb{R}_+. \tag{2.36}$$

If additionally

$$\sum_{n=0}^{\infty} \left(\frac{1}{\varphi(n+1)}\right)^k p(n) < \infty \tag{2.37}$$

for some $k \in \mathbb{N}$, then the kth moments of $V(n,\tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$ are finite for $\tau \in \mathbb{R}_+$, and

$$E\big[V^k(n,\tau)\big] = \frac{\varphi(n+1)}{\lambda(\tau)p(n)} \sum_{m=1}^{n+1} g_{n+1,m}^{(k)}(\tau), \tag{2.38}$$

$$E\big[V^k(\tau)\big] = \frac{1}{\lambda(\tau)} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} g_{n,m}^{(k)}(\tau) = k \int_0^\tau \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{g_{n,m}^{(k-1)}(x)}{\lambda(x)} \,\mathrm{d}x. \tag{2.39}$$

Theorem 2.3 yields the following insensitivity property of $V(\tau)$ in $M/GI/\mathrm{SDPS}$.

Theorem 2.4 *Let the stability condition* (1.1) *for the* $M/GI/\mathrm{SDPS}$ *be satisfied. Then the conditional sojourn times* $V(n,\tau)$, $n \in \mathbb{Z}_+$, *and* $V(\tau)$ *depend on the service time distribution* $B(x)$ *for* $x > \tau$ *only via the mean service time* m_S *in distribution.*

Proof Suppose first that (A1) is satisfied. Then $u_{n,m}(s; y_1, \dots, y_m)$ is the minimal non-negative solution of the linear system of PDEs (2.19) with initial conditions (2.20), (2.21). Therefore $u_{n,m}(s; y_1, \dots, y_m)$ depends on $B(x)$ for $x > y_m$ only via m_S, and hence $f_{n,m}(s,\tau)$ depends on $B(x)$ for $x > \tau$ only via m_S because of (2.29). In view of (2.28), (1.4), (2.27), and (2.36), thus the assertion follows from (2.34) and (2.35). The case of a general distribution $B(x)$ of the service time with finite mean m_S is obtained by taking the limit in distribution of a sequence of service time distributions $B_\nu(x)$, $\nu = 1, 2, \dots$, where the service times have the given mean m_S and the $B_\nu(x)$ satisfy (A1) and converge weakly to $B(x)$. □

Example 2.1 (cont.) In the case of $\varphi_{1,k}(n) = 1/(n+k)$, $n \in \mathbb{N}$, $k \in (-1, \infty)$, from the product form solution for $u_{n,m}(s; y_1, \dots, y_m)/u_{n,m}(0; y_1, \dots, y_m)$ and Theorem 2.3 for $s \in \mathbb{R}_+$, $\tau \in \mathbb{R}_+$ it follows that

$$E\big[e^{-sV(\tau)}\big] = \left(\frac{(1-\varrho)\delta(s,\tau)}{1-\varrho\delta(s,\tau)(\int_0^\tau \frac{\mathrm{d}B_R(\xi)}{\delta(s,\tau-\xi)} + \bar{B}_R(\tau))}\right)^{k+1},$$

cf. [11, 16] for $k = 0$ and [19] for $k \in \mathbb{N}$. As $\delta(s,\tau)$ does not depend on $B(x)$ for $x > \tau$, thus $V(\tau)$ depends on $B(x)$ for $x > \tau$ in this case obviously only via the mean service time m_S in distribution.

Note that the outlined approach to the LST of $V(\tau)$ for the $M/GI/1$-PS system is similar to the approach of [13].

3 Locally exponential service times

We assume now that the service time S has finite mean m_S and that its distribution $B(x)$ coincides with an exponential distribution in a neighborhood of zero, i.e.

$$B(x) = 1 - e^{-\mu x}, \quad x \in [0, d), \tag{3.1}$$

where $\mu \in \mathbb{R}_+$, for some $d \in (0, \infty)$. Moreover, for technical reasons we assume (A1) again, i.e., we assume that $B(x)$ has a continuous density $b(x)$ and that $B(x) < 1$ for $x \in \mathbb{R}_+$.

Again we assume in the following that the system is stable, i.e. that (1.1) is satisfied and in steady state. In view of (3.1), for $0 < m \le n$, $0 < y_1 < \cdots < y_m < d$ the system of PDEs (2.19) simplifies to

$$\varphi(n)\frac{\partial}{\partial \xi} u_{n,m}(s; y_1 + \xi, \dots, y_m + \xi)\bigg|_{\xi=0}$$

$$
\begin{aligned}
&= -(\lambda + s + \varphi(n)m\mu)u_{n,m}(s; y_1, \ldots, y_m) \\
&\quad + \varphi(n+1)\mu \sum_{\ell=1}^{m} \int_{y_{\ell-1}}^{y_\ell} u_{n+1,m+1}(s; y_1, \ldots, y_{\ell-1}, \tau, y_\ell, \ldots, y_m)\,\mathrm{d}\tau \\
&\quad + \varphi(n+1)m_S^{-1} e^{-\mu y_m} u_{n+1,m}(s; y_1, \ldots, y_m).
\end{aligned}
$$

Replacing y_i by $y_i + \eta$, $i = 1, \ldots, m$, and integrating over $[-y_1, 0]$ with respect to η yields the system of integral equations

$$
\begin{aligned}
&u_{n,m}(s; y_1, \ldots, y_m) - u_{n,m}(s; 0, y_2 - y_1, \ldots, y_m - y_1) \\
&= \frac{1}{\varphi(n)} \int_{-y_1}^{0} \Biggl(-(\lambda + s + \varphi(n)m\mu)u_{n,m}(s; y_1 + \xi, \ldots, y_m + \xi) \\
&\quad + \varphi(n+1)\mu \sum_{\ell=1}^{m} \int_{y_{\ell-1} - \mathbb{I}\{\ell=1\}\xi}^{y_\ell} u_{n+1,m+1}(s; y_1 + \xi, \ldots, y_{\ell-1} + \xi, \tau + \xi, \\
&\qquad\qquad y_\ell + \xi, \ldots, y_m + \xi)\,\mathrm{d}\tau \\
&\quad + \varphi(n+1)m_S^{-1} e^{-\mu(y_m+\xi)} u_{n+1,m}(s; y_1 + \xi, \ldots, y_m + \xi) \Biggr)\,\mathrm{d}\xi.
\end{aligned}
$$

In view of (2.29), integrating over $0 \le y_1 \le \cdots \le y_m = x < d$ with respect to $\mathrm{d}y_1 \ldots \mathrm{d}y_{m-1}$ and applying Fubini's theorem yields after some algebra

$$
\begin{aligned}
&f_{n,m}(s, x) - \int_{0 \le \xi_2 \le \cdots \le \xi_m \le x} u_{n,m}(s; 0, \xi_2, \ldots, \xi_m)\,\mathrm{d}\xi_2 \ldots \mathrm{d}\xi_m \\
&= \frac{1}{\varphi(n)} \int_0^x \bigl(-(\lambda + s + \varphi(n)m\mu) f_{n,m}(s, \xi_m) + \varphi(n+1)m\mu f_{n+1,m+1}(s, \xi_m) \\
&\quad + \varphi(n+1)m_S^{-1} e^{-\mu\xi_m} f_{n+1,m}(s, \xi_m) \bigr)\,\mathrm{d}\xi_m. \qquad (3.2)
\end{aligned}
$$

Because of (2.20), (2.21) and (2.29), from (3.2) we obtain the linear system of ODEs

$$
\begin{aligned}
&\varphi(n) \frac{\partial}{\partial x} f_{n,m}(s, x) \\
&\quad = -(\lambda + s + \varphi(n)m\mu) f_{n,m}(s, x) \\
&\qquad + \mathbb{I}\{m > 1\}\lambda f_{n-1,m-1}(s, x) + \varphi(n+1)m\mu f_{n+1,m+1}(s, x) \\
&\qquad + \varphi(n+1)m_S^{-1} e^{-\mu x} f_{n+1,m}(s, x), \quad 0 < m \le n, 0 < x < d, \qquad (3.3)
\end{aligned}
$$

with the initial condition

$$
f_{n,m}(s, 0) = \mathbb{I}\{m = 1\} n! p(n) m_S^{-1}, \quad 0 < m \le n. \qquad (3.4)
$$

Note that from (2.29), (2.22), (1.4), and (3.1) it follows that

$$
f_{n,m}(0, x) = n! p(n) \bigl(m_S e^{\mu x}\bigr)^{-m} \frac{F^{m-1}(x)}{(m-1)!}, \quad 0 < m \le n, 0 \le x < d, \qquad (3.5)
$$

where

$$F(x) := \begin{cases} (e^{\mu x} - 1)/\mu, & \mu \neq 0, \\ x, & \mu = 0, \end{cases} \quad x \in \mathbb{R}_+. \tag{3.6}$$

Let

$$h_{n,m}(s,x) := \frac{1}{p(0)} \big(m_S e^{\mu x}\big)^{m-n} f_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d. \tag{3.7}$$

In view of (2.27), (2.28), (1.4), (3.1), (3.6), and (3.7), we obtain

$$g_{n,m}(s,x) = d_{n-m}(x) h_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d, \tag{3.8}$$

where

$$d_\ell(x) := \frac{p(0)}{\ell!} \big(m_S e^{\mu x} - F(x)\big)^\ell, \quad \ell \in \mathbb{Z}_+, x \in \mathbb{R}_+. \tag{3.9}$$

Note that $d_\ell(x) > 0$ for $0 \le x < d$ as

$$m_S = \int_0^\infty \bar{B}(x)\,\mathrm{d}x \ge \int_0^d \bar{B}(x)\,\mathrm{d}x = e^{-\mu d} F(d).$$

Because of (3.7), (3.3), (3.4), and (1.2), the $h_{n,m}$ satisfy the linear system of ODEs with constant coefficients

$$\begin{aligned} &\varphi(n) \frac{\partial}{\partial x} h_{n,m}(s,x) \\ &\quad = -(\lambda + s + \varphi(n) n\mu) h_{n,m}(s,x) \\ &\qquad + \mathbb{I}\{m > 1\} \lambda h_{n-1,m-1}(s,x) + \varphi(n+1) m\mu h_{n+1,m+1}(s,x) \\ &\qquad + \varphi(n+1) h_{n+1,m}(s,x), \quad 0 < m \le n, 0 < x < d, \end{aligned} \tag{3.10}$$

with initial condition

$$h_{n,m}(s,0) = \mathbb{I}\{m = 1\} \prod_{j=1}^n \frac{\lambda}{\varphi(j)}, \quad 0 < m \le n. \tag{3.11}$$

From (3.7), (3.5), and (1.2) it follows that

$$h_{n,m}(0,x) = \left(\prod_{j=1}^n \frac{\lambda}{\varphi(j)} \right) e^{-n\mu x} \frac{F^{m-1}(x)}{(m-1)!}, \quad 0 < m \le n, 0 \le x < d. \tag{3.12}$$

Replacing in (3.10) x by ξ, integrating appropriately weighted over $[0,x]$ with respect to ξ and applying (3.11) demonstrates that the system of ODEs (3.10) with initial condition (3.11) is equivalent to the following system of integral equations:

$$h_{n,m}(s,x) = h^*_{n,m}(s,x) + \frac{1}{\varphi(n)} \int_0^x \big(\mathbb{I}\{m > 1\} \lambda h_{n-1,m-1}(s,\xi)$$

$$+ \varphi(n+1)m\mu h_{n+1,m+1}(s,\xi) + \varphi(n+1)h_{n+1,m}(s,\xi)\big)$$
$$\times e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)(x-\xi)}\,\mathrm{d}\xi, \quad 0 < m \le n, 0 \le x < d, \tag{3.13}$$

where

$$h^*_{n,m}(s,x) = \mathbb{I}\{m=1\}\left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)x}, \quad 0 < m \le n, 0 \le x < d. \tag{3.14}$$

Lemma 3.1 *Let $h^*_{n,m}(s,x)$ be arbitrary continuous functions.*
Then for any fixed $s \in (0,\infty)$, the linear system of integral equations (3.13) with growth condition

$$\left|h_{n,m}(s,x)\right| \le c h_{n,m}(0,x), \quad 0 < m \le n, 0 \le x < d, \tag{3.15}$$

for some $c > 0$, where $h_{n,m}(0,x)$ is given by (3.12), has at most one solution.

Proof The proof runs analogously to the proof of Lemma 2.1. Assume that the system of integral equations (3.13) has two different solutions, where both solutions satisfy the growth condition (3.15). Then the difference $\bar{h}_{n,m}(s,x)$ of these solutions satisfies the homogenized version of (3.13), i.e. (3.13) in the case of $h^*_{n,m}(s,x) = 0$. For $0 < m \le n$ let $\kappa_{n,m} \in \mathbb{R}_+$ denote the smallest number such that $|\bar{h}_{n,m}(s,x)| \le \kappa_{n,m} h_{n,m}(0,x)$, $0 \le x < d$. Note that $\kappa_{n,m} \in (0, 2c]$ due to the growth condition. Let $\kappa_{n-1,0} := 0$ for notational convenience and let $\kappa_0 := \sup_{0<m\le n}\kappa_{n,m}$. The triangle inequality applied to (3.13), the definitions of $\kappa_{n,m}$ and κ_0, and (3.12), (3.6) provide after some algebra for $0 < m \le n$, $0 \le x < d$ that

$$\begin{aligned}
&\left|\bar{h}_{n,m}(s,x)\right| \\
&\quad \le \frac{1}{\varphi(n)}\int_0^x \big(\kappa_{n-1,m-1}\lambda h_{n-1,m-1}(0,\xi) \\
&\qquad + \kappa_0\varphi(n+1)m\mu h_{n+1,m+1}(0,\xi) + \kappa_0\varphi(n+1)h_{n+1,m}(0,\xi)\big)e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)(x-\xi)}\,\mathrm{d}\xi \\
&\quad = \left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)\frac{e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)x}}{(m-1)!} \\
&\qquad \times \int_0^x \left(\kappa_{n-1,m-1}(m-1)F^{m-2}(\xi)e^{\mu\xi} + \kappa_0\frac{\lambda}{\varphi(n)}F^{m-1}(\xi)\right)e^{\frac{\lambda+s}{\varphi(n)}\xi}\,\mathrm{d}\xi \\
&\quad \le \max\left(\kappa_{n-1,m-1}, \frac{\lambda}{\lambda+s}\kappa_0\right)\left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)\frac{e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)x}}{(m-1)!}\big(F^{m-1}(\xi)e^{\frac{\lambda+s}{\varphi(n)}\xi}\big|_{\xi=0}^{x}\big) \\
&\quad \le \max\left(\kappa_{n-1,m-1}, \frac{\lambda}{\lambda+s}\kappa_0\right)h_{n,m}(0,x).
\end{aligned}$$

Therefore, by definition of $\kappa_{n,m}$ it follows that $\kappa_{n,m} \le \max(\kappa_{n-1,m-1}, \frac{\lambda}{\lambda+s}\kappa_0)$, $0 < m \le n$, which provides a contradiction as at the end of the proof of Lemma 2.1. □

Lemma 3.2 *Let $h^*_{n,m}(s,x)$ be arbitrary continuous functions such that*

$$0 \le h^*_{n,m}(s,x) \le h^*_{n,m}(0,x), \quad 0 < m \le n, 0 \le x < d, s \in \mathbb{R}_+. \tag{3.16}$$

Assume that there exists a non-negative solution of the system of integral equations (3.13) *for $s = 0$.*

Then the system of integral equations (3.13) *has a minimal non-negative solution $\tilde{h}_{n,m}(s,x)$ for any $s \in \mathbb{R}_+$, and*

$$\tilde{h}_{n,m}(s,x) \le \tilde{h}_{n,m}(0,x), \quad 0 < m \le n, 0 \le x < d, s \in \mathbb{R}_+. \tag{3.17}$$

Proof For fixed $s \in \mathbb{R}_+$ let $h_{n,m;0}(s,x) := 0$, $0 < m \le n$, $0 \le x < d$, and recursively for $i = 0, 1, \dots$ let

$$\begin{aligned} h_{n,m;i+1}(s,x) := h^*_{n,m}(s,x) + \frac{1}{\varphi(n)} \int_0^x &\big(\mathbb{I}\{m > 1\}\lambda h_{n-1,m-1;i}(s,\xi) \\ &+ \varphi(n+1) m\mu h_{n+1,m+1;i}(s,\xi) + \varphi(n+1) h_{n+1,m;i}(s,\xi)\big) \\ &\times e^{-(\frac{\lambda+s}{\varphi(n)} + n\mu)(x-\xi)}\, d\xi, \quad 0 < m \le n, 0 \le x < d. \end{aligned} \tag{3.18}$$

By induction on $i \in \mathbb{Z}_+$ after some algebra it follows that $h_{n,m;i}(s,x)$ is monotonically increasing with respect to i, that

$$h_{n,m;i}(s,x) \le h_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d, \tag{3.19}$$

for any non-negative solution $h_{n,m}(s,x)$ of (3.13), and that

$$h_{n,m;i}(s,x) \le h_{n,m}(0,x), \quad 0 < m \le n, 0 \le x < d, \tag{3.20}$$

for any non-negative solution $h_{n,m}(0,x)$ of (3.13) for $s = 0$. Thus the limit

$$\tilde{h}_{n,m}(s,x) := \lim_{i\to\infty} h_{n,m;i}(s;x), \quad 0 < m \le n, 0 \le x < d,$$

exists pointwise and represents, due to (3.18) and Lebesgue's theorem, a non-negative solution of (3.13). From (3.19) it follows that $\tilde{h}_{n,m}(s,x)$ is the minimal non-negative solution of (3.13), and (3.20) implies (3.17). □

Now, from Theorem 2.3, and Lemma 3.1, Lemma 3.2 we obtain the following.

Theorem 3.1 *Let the stability condition* (1.1) *for the $M/GI/$SDPS system, where the service time distribution fulfills* (3.1), *be satisfied. Then for $s \in \mathbb{R}_+$ and $\tau \in [0, d)$ the LSTs of $V(n,\tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$ are given by*

$$E\big[e^{-sV(n,\tau)}\big] = \frac{\varphi(n+1)}{\lambda(\tau)p(n)} \sum_{m=1}^{n+1} g_{n+1,m}(s,\tau), \quad \tau \in [0,d), \tag{3.21}$$

$$E\big[e^{-sV(\tau)}\big] = \frac{1}{\lambda(\tau)} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} g_{n,m}(s,\tau), \quad \tau \in [0,d), \tag{3.22}$$

respectively, where

$$\lambda(x) = \lambda e^{-\mu x}, \quad x \in [0, d), \tag{3.23}$$

$g_{n,m}(s,x)$ *is given by* (3.8), (3.9), *and* $h_{n,m}(s,x)$ *is the minimal non-negative solution of the linear system of ODEs* (3.10) *with initial condition* (3.11).

Proof Suppose that (3.1) and (A1) are satisfied. Then Theorem 2.3 yields (3.21), (3.22), (3.23), where $g_{n,m}(s,x)$ is given by (3.8), (3.9), and $h_{n,m}(s,x)$ is a solution of the linear system of ODEs (3.10) with initial condition (3.11). Further, taking into account (3.8), we find that $h_{n,m}(s,x)$ is non-negative and monotonically decreasing with respect to $s \in \mathbb{R}_+$ since $g_{n,m}(s,x)$ is non-negative and monotonically decreasing with respect to $s \in \mathbb{R}_+$ in view of (2.26). The case of a general distribution $B(x)$ of the service time with finite mean m_S and (3.1) is obtained by taking for fixed $\tau \in [0,d)$ the limit in distribution of a sequence of service time distributions $B_\nu(x)$, $\nu = 1, 2, \ldots$, where the service times have the given mean m_S, $B_\nu(x)$ is given by (3.1) for $x \in [0, d')$ for some $d' \in (\tau, d)$, the $B_\nu(x)$ satisfy (A1) and converge weakly to $B(x)$.

As $h_{n,m}(0,x)$ given by (3.12) is a non-negative solution of (3.10), (3.11) for $s = 0$, in view of (3.14) from Lemma 3.2 it follows that there exists a minimal non-negative solution $\tilde{h}_{n,m}(s,x)$ of (3.10), (3.11), and that $0 \le \tilde{h}_{n,m}(s,x) \le \tilde{h}_{n,m}(0,x) \le h_{n,m}(0,x)$. Since $h_{n,m}(s,x)$ is non-negative and monotonically decreasing with respect to $s \in \mathbb{R}_+$, moreover $0 \le h_{n,m}(s,x) \le h_{n,m}(0,x)$. In view of Lemma 3.1 for $c = 1$, thus we have $h_{n,m}(s,x) = \tilde{h}_{n,m}(s,x)$ for $s \in (0,\infty)$.

As $h_{n,m}(s,x)$ is monotonically decreasing with respect to $s \in \mathbb{R}_+$, the limit

$$h_{n,m}(x) := \lim_{s\downarrow 0} h_{n,m}(s,x) = \lim_{s\downarrow 0} \tilde{h}_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d,$$

exists pointwise and represents, due to Lebesgue's theorem, a non-negative solution of (3.13) for $s = 0$, and $\tilde{h}_{n,m}(0,x) \le h_{n,m}(x)$ as $\tilde{h}_{n,m}(0,x)$ is the minimal non-negative solution of (3.13) for $s = 0$. Further, Lemma 3.2 demonstrates that $\tilde{h}_{n,m}(s,x) \le \tilde{h}_{n,m}(0,x)$ for $s \in (0,\infty)$, and by taking the limit $s \downarrow 0$ we find $h_{n,m}(x) \le \tilde{h}_{n,m}(0,x)$. Hence we have $h_{n,m}(x) = \tilde{h}_{n,m}(0,x)$.

Note that $\lim_{s\downarrow 0} E[e^{-sV(n,\tau)}] = 1$ since the distribution of $V(n,\tau)$ is non-defective because of the stability of the system. Taking the limit $s \downarrow 0$ in (3.21) therefore yields

$$\sum_{m=1}^{n} d_{n-m}(x) h_{n,m}(x) = \frac{\lambda(x) p(n-1)}{\varphi(n)}, \quad 0 < n, 0 \le x < d,$$

cf. (3.8). On the other hand, from (3.9), (3.12), (3.23), (1.2) we find

$$\sum_{m=1}^{n} d_{n-m}(x) h_{n,m}(0,x) = \frac{\lambda(x) p(n-1)}{\varphi(n)}, \quad 0 < n, 0 \le x < d.$$

In view of $h_{n,m}(x) = \tilde{h}_{n,m}(0,x) \le h_{n,m}(0,x)$ and $d_{n-m}(x) > 0$, therefore we conclude that

$$\lim_{s \downarrow 0} h_{n,m}(s,x) = \tilde{h}_{n,m}(0,x) = h_{n,m}(0,x). \tag{3.24}$$

Thus $h_{n,m}(0,x)$ given by (3.12) is the minimal non-negative solution of (3.10), (3.11) for $s = 0$. □

Remark 3.1 Note that the data in (3.10)–(3.12) are independent of m_S and thus of $B(\xi)$ for $\xi \ge d$, and therefore $h_{n,m}(s,x)$ as the minimal non-negative solution of (3.10), (3.11) is independent of $B(\xi)$ for $\xi \ge d$, too. Moreover, $h_{n,m}(s,x)$ can be continued for fixed $s \in \mathbb{R}_+$ to the minimal non-negative solution of (3.10), (3.11) for $x \in \mathbb{R}_+$ since the proof of Lemma 3.2 remains valid for any positive d.

For $s \in (0,\infty)$, $\ell \in \mathbb{Z}_+$ let

$$h^{(\ell)}_{n,m}(s,x) := (-1)^\ell \frac{\partial^\ell}{\partial s^\ell} h_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d. \tag{3.25}$$

Note that from (2.30), (3.8) it follows that

$$g^{(\ell)}_{n,m}(s,x) = d_{n-m}(x) h^{(\ell)}_{n,m}(s,x), \quad 0 < m \le n, 0 \le x < d. \tag{3.26}$$

Thus $h^{(\ell)}_{n,m}(s,x)$ is monotonically decreasing with respect to $s \in (0,\infty)$ in view of (2.31), and (2.32) implies

$$0 \le h^{(\ell)}_{n,m}(s,x) \le \ell! s^{-\ell} h_{n,m}(0,x), \quad 0 < m \le n, 0 \le x < d. \tag{3.27}$$

Taking the ℓth derivative with respect to $s \in (0,\infty)$ at both sides of (3.10) yields the linear system of ODEs

$$\begin{aligned}
&\varphi(n) \frac{\partial}{\partial x} h^{(\ell)}_{n,m}(s,x) \\
&\quad = -(\lambda + s + \varphi(n) n\mu) h^{(\ell)}_{n,m}(s,x) \\
&\qquad + \mathbb{I}\{m > 1\} \lambda h^{(\ell)}_{n-1,m-1}(s,x) + \varphi(n+1) m\mu h^{(\ell)}_{n+1,m+1}(s,x) \\
&\qquad + \varphi(n+1) h^{(\ell)}_{n+1,m}(s,x) + \ell h^{(\ell-1)}_{n,m}(s,x), \quad 0 < m \le n,\ 0 < x < d,
\end{aligned} \tag{3.28}$$

where $h^{(-1)}_{n,m}(s,x) := 0$, and from (3.11) we obtain the initial condition

$$h^{(\ell)}_{n,m}(s,0) = \mathbb{I}\{\ell = 0, m = 1\} \prod_{j=1}^{n} \frac{\lambda}{\varphi(j)}, \quad 0 < m \le n. \tag{3.29}$$

Note that the system of ODEs (3.28) with initial condition (3.29) is equivalent to the following system of integral equations:

$$h^{(\ell)}_{n,m}(s,x) = h^*_{n,m}(s,x) + \frac{1}{\varphi(n)} \int_0^x \big(\mathbb{I}\{m > 1\} \lambda h^{(\ell)}_{n-1,m-1}(s,\xi)$$

$$+\varphi(n+1)m\mu h_{n+1,m+1}^{(\ell)}(s,\xi)+\varphi(n+1)h_{n+1,m}^{(\ell)}(s,\xi)\big)$$
$$\times e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)(x-\xi)}\,\mathrm{d}\xi,\quad 0<m\le n, 0\le x<d, \tag{3.30}$$

where

$$h_{n,m}^{*}(s,x)=\mathbb{I}\{\ell=0,m=1\}\left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)x}$$
$$+\frac{\ell}{\varphi(n)}\int_0^x h_{n,m}^{(\ell-1)}(s,\xi)e^{-(\frac{\lambda+s}{\varphi(n)}+n\mu)(x-\xi)}\,\mathrm{d}\xi,\quad 0<m\le n, 0\le x<d. \tag{3.31}$$

Assume now that (2.37) is satisfied for some fixed $k\in\mathbb{N}$. Due to Hölder's inequality and Theorem 2.3, the limits

$$g_{n,m}^{(\ell)}(x)=\lim_{s\downarrow 0}g_{n,m}^{(\ell)}(s,x),\quad 0<m\le n, 0\le x<d,$$

cf. (2.33), exist for $\ell=0,1,\dots,k$. In view of (3.26), thus also the limits

$$h_{n,m}^{(\ell)}(x):=\lim_{s\downarrow 0}h_{n,m}^{(\ell)}(s,x),\quad 0<m\le n, 0\le x<d, \tag{3.32}$$

exist for $\ell=0,1,\dots,k$, and

$$g_{n,m}^{(\ell)}(x)=d_{n-m}(x)h_{n,m}^{(\ell)}(x),\quad 0<m\le n, 0\le x<d, \tag{3.33}$$

for $\ell=0,1,\dots,k$. Taking the limit $s\downarrow 0$ in (3.28) and (3.29) yields the linear systems of ODEs

$$\begin{aligned}
&\varphi(n)\frac{\mathrm{d}}{\mathrm{d}x}h_{n,m}^{(\ell)}(x)\\
&\quad=-(\lambda+\varphi(n)n\mu)h_{n,m}^{(\ell)}(x)+\mathbb{I}\{m>1\}\lambda h_{n-1,m-1}^{(\ell)}(x)\\
&\qquad+\varphi(n+1)m\mu h_{n+1,m+1}^{(\ell)}(x)+\varphi(n+1)h_{n+1,m}^{(\ell)}(x)+\ell h_{n,m}^{(\ell-1)}(x),\\
&\quad 0<m\le n, 0<x<d, \ell\in\{1,\dots,k\},
\end{aligned} \tag{3.34}$$

with the initial conditions

$$h_{n,m}^{(\ell)}(0)=0,\quad 0<m\le n, \ell\in\{1,\dots,k\}. \tag{3.35}$$

In view of (3.32), (3.25), from (3.24) and (3.12) we find

$$h_{n,m}^{(0)}(x)=\left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)e^{-n\mu x}\frac{F^{m-1}(x)}{(m-1)!},\quad 0<m\le n, 0\le x<d. \tag{3.36}$$

Now, from Theorem 2.3, and Lemma 3.1, Lemma 3.2 we obtain the following.

Theorem 3.2 *Let the stability condition* (1.1) *for the* $M/GI/\mathrm{SDPS}$ *system, where the service time distribution fulfills* (3.1), *be satisfied. Further, let* (2.37) *be satisfied for some* $k \in \mathbb{N}$. *Then the kth moments of* $V(n,\tau)$, $n \in \mathbb{Z}_+$, *and* $V(\tau)$ *are finite for* $\tau \in \mathbb{R}_+$, *and for* $\tau \in [0,d)$

$$E\big[V^k(n,\tau)\big] = \frac{\varphi(n+1)}{\lambda(\tau)p(n)} \sum_{m=1}^{n+1} g_{n+1,m}^{(k)}(\tau), \quad \tau \in [0,d), \tag{3.37}$$

$$E\big[V^k(\tau)\big] = k\int_0^\tau \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{g_{n,m}^{(k-1)}(x)}{\lambda(x)}\,\mathrm{d}x, \quad \tau \in [0,d), \tag{3.38}$$

where $\lambda(x)$ *is given by* (3.23) *and* $g_{n,m}^{(\ell)}(x)$ *by* (3.33), (3.9) *for* $\ell = 0,1,\dots,k$, *where* $h_{n,m}^{(0)}(x)$ *is given by* (3.36) *and* $h_{n,m}^{(\ell)}(x)$ *is the minimal non-negative solution of the linear system of ODEs* (3.34), (3.35) *for* $\ell = 1,\dots,k$.

Proof Suppose that (3.1) and (A1) are satisfied. Then Theorem 2.3 yields (3.37), (3.38), where $\lambda(x)$ is given by (3.23), $g_{n,m}^{(\ell)}(x)$ is given by (3.33), (3.9) for $\ell = 0,1,\dots,k$, where $h_{n,m}^{(0)}(x)$ is given by (3.36) and $h_{n,m}^{(\ell)}(x)$ is defined by (3.32) for $\ell = 1,\dots,k$. The case of a general distribution $B(x)$ of the service time with finite mean m_S and (3.1) is obtained by taking for fixed $\tau \in [0,d)$ the limit in distribution of a sequence of service time distributions $B_\nu(x)$, $\nu = 1,2,\dots$, where the service times have the given mean m_S, $B_\nu(x)$ is given by (3.1) for $x \in [0,d')$ for some $d' \in (\tau,d)$, the $B_\nu(x)$ satisfy (A1) and converge weakly to $B(x)$.

Let $\ell \in \{1,\dots,k\}$ be fixed. As $h_{n,m}^{(\ell)}(x)$ is a non-negative solution of (3.28), (3.29) for $s=0$, in view of (3.30), (3.31) from Lemma 3.2 it follows that there exists a minimal non-negative solution $\tilde{h}_{n,m}^{(\ell)}(s,x)$ of (3.28), (3.29) for any $s \in \mathbb{R}_+$, and that $0 \le \tilde{h}_{n,m}^{(\ell)}(s,x) \le \tilde{h}_{n,m}^{(\ell)}(0,x)$. From (3.27) we obtain $0 \le \tilde{h}_{n,m}^{(\ell)}(s,x) \le h_{n,m}^{(\ell)}(s,x) \le \ell! s^{-\ell} h_{n,m}(0,x)$ for $s \in (0,\infty)$, which implies $h_{n,m}^{(\ell)}(s,x) = \tilde{h}_{n,m}^{(\ell)}(s,x)$ for $s \in (0,\infty)$ by applying Lemma 3.1 for $c = \ell! s^{-\ell}$. Since the limit

$$h_{n,m}^{(\ell)}(x) = \lim_{s\downarrow 0} h_{n,m}^{(\ell)}(s,x) = \lim_{s\downarrow 0} \tilde{h}_{n,m}^{(\ell)}(s,x), \quad 0 < m \le n, 0 \le x < d,$$

represents, due to Lebesgue's theorem, a non-negative solution of (3.30) for $s=0$, we obtain $\tilde{h}_{n,m}^{(\ell)}(0,x) \le h_{n,m}^{(\ell)}(x)$ as $\tilde{h}_{n,m}^{(\ell)}(0,x)$ is the minimal non-negative solution of (3.30) for $s=0$. Further, as $\tilde{h}_{n,m}^{(\ell)}(s,x) \le \tilde{h}_{n,m}^{(\ell)}(0,x)$ for $s \in (0,\infty)$, by taking the limit $s \downarrow 0$ we find $h_{n,m}^{(\ell)}(x) \le \tilde{h}_{n,m}^{(\ell)}(0,x)$. Hence $h_{n,m}^{(\ell)}(x) = \tilde{h}_{n,m}^{(\ell)}(0,x)$. Thus $h_{n,m}^{(\ell)}(x)$ is the minimal non-negative solution of (3.28), (3.29) for $s=0$, i.e. of (3.34), (3.35). □

Note that the LSTs and the moments of $V(n,\tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$ at $\tau = d$ are given by continuous continuation from $0 \le \tau < d$.

3.1 Cut exponential service times

Let the service times be distributed according to the minimum of an exponential time with parameter $\mu \in \mathbb{R}_+$ and a deterministic time $d \in (0, \infty)$, i.e.

$$B(x) = 1 - \mathbb{I}\{0 \le x < d\} e^{-\mu x}, \quad x \in \mathbb{R}_+. \tag{3.39}$$

For the mean m_S we find

$$m_S = e^{-\mu d} F(d), \tag{3.40}$$

where $F(d)$ is given by (3.6). Note that the model corresponds to an $M/D/$SDPS system for $\mu = 0$.

Let the stability condition (1.1) for the $M/GI/$SDPS system with service time distribution (3.39) be satisfied. Then the LSTs of $V(n, \tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$, $\tau \in [0, d)$, are given by Theorem 3.1, and the moments of $V(n, \tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$, $\tau \in [0, d)$, are given by Theorem 3.2.

In view of

$$E\big[e^{-sV}\big] = \int_{\mathbb{R}_+} E\big[e^{-sV(\tau)}\big] \, \mathrm{d}B(\tau), \qquad E\big[V^k\big] = \int_{\mathbb{R}_+} E\big[V^k(\tau)\big] \, \mathrm{d}B(\tau),$$

and (3.23), (3.39), (3.40), from Theorem 3.1 and Theorem 3.2 we obtain the following representations for the LST and the moments of the unconditional sojourn time V, respectively, where (1.7) and Fubini's theorem are used for the moments of V.

Theorem 3.3 *Let the stability condition* (1.1) *for the $M/GI/$SDPS system with service time distribution* (3.39) *be satisfied. Then for $s \in (0, \infty)$, the LST of V is given by*

$$E\big[e^{-sV}\big] = \frac{1}{\lambda} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} \left(\mu \int_0^d g_{n,m}(s, x) \, \mathrm{d}x + g_{n,m}(s, d-) \right), \tag{3.41}$$

where $g_{n,m}(s, x)$ is given by (3.8), (3.9), *and $h_{n,m}(s, x)$ is the minimal non-negative solution of the linear system of ODEs* (3.10) *with initial condition* (3.11). *If additionally* (2.37) *is satisfied for some $k \in \mathbb{N}$, then the kth moment of V is finite, and*

$$E\big[V^k\big] = \frac{k}{\lambda} \int_0^d \sum_{n=1}^{\infty} \sum_{m=1}^{n} g_{n,m}^{(k-1)}(x) \, \mathrm{d}x, \tag{3.42}$$

where $g_{n,m}^{(k-1)}(x)$ is given by (3.33), (3.9), *where $h_{n,m}^{(0)}(x)$ is given by* (3.36) *and $h_{n,m}^{(\ell)}(x)$ is the minimal non-negative solution of the linear system of ODEs* (3.34), (3.35) *for $\ell = 1, \dots, k-1$.*

3.2 Numerical results

We apply the results of this section to the variance of the sojourn time V in $M/D/r$-PS, which means $\mu = 0$ in (3.39) and $\varphi(n) = \min(r/n, 1)$, $n \in \mathbb{N}$. Let the stability condition (1.1) for the $M/D/r$-PS system be satisfied, i.e., let $\varrho = \lambda d < r$. Then (2.37) is satisfied for all $k \in \mathbb{N}$. Thus from Theorem 3.3 it follows that $E[V^2]$ in $M/D/r$-PS is given by (3.42) for $k = 2$, where $g^{(1)}_{n,m}(x)$ is given by (3.33), (3.9) for $\ell = 1$, where $h^{(1)}_{n,m}(x)$ is the minimal non-negative solution of the linear system of ODEs (3.34) with the initial condition (3.35) for $\ell = 1$, and $h^{(0)}_{n,m}(x)$ is given by (3.36). In view of $\mu = 0$, from (3.33), (3.9), (3.40), (3.6), (3.34), (3.36), and (3.35) we obtain

$$g^{(1)}_{n,m}(x) = p(0)\frac{(d-x)^{n-m}}{(n-m)!}h^{(1)}_{n,m}(x), \quad 0 < m \le n, 0 \le x < d, \tag{3.43}$$

where $h^{(1)}_{n,m}(x)$ is the minimal non-negative solution of the linear system of ODEs

$$\begin{aligned}\varphi(n)\frac{\mathrm{d}}{\mathrm{d}x}h^{(1)}_{n,m}(x) &= -\lambda h^{(1)}_{n,m}(x) + \mathbb{I}\{m > 1\}\lambda h^{(1)}_{n-1,m-1}(x) + \varphi(n+1)h^{(1)}_{n+1,m}(x)\\ &\quad + \left(\prod_{j=1}^{n}\frac{\lambda}{\varphi(j)}\right)\frac{x^{m-1}}{(m-1)!}, \quad 0 < m \le n, 0 < x < d,\end{aligned} \tag{3.44}$$

with the initial condition

$$h^{(1)}_{n,m}(0) = 0, \quad 0 < m \le n. \tag{3.45}$$

The minimal non-negative solution $h^{(1)}_{n,m}(x)$, $0 < m \le n$, of (3.44), (3.45) can be approximated by solving a suitable finite version $n \le n'$ of the linear system of ODEs (3.44), (3.45), where $h^{(1)}_{n,m}(x)$ is replaced by 0 on the r.h.s. of (3.44) if $n > n'$. The second moment $E[V^2]$ can be computed subsequently via (3.42), (3.43) by numerical integration. The first moment $E[V]$ can be computed via Little's law (1.5).

Note that the sequence of the solutions of the finite versions of (3.44), (3.45) indexed by n' converges pointwise monotonically increasing to $h^{(1)}_{n,m}(x)$ for $n' \to \infty$: Applying the construction of the minimal non-negative solution given in the proof of Lemma 3.2 to the corresponding finite versions of (3.30), (3.31) for $s = 0$ and $\ell = 1$, by induction on $i \in \mathbb{Z}_+$ and taking then the limit $i \to \infty$ we obtain the monotonicity of the solutions of the finite versions of (3.30), (3.31) with respect to n' and that they are bounded by $h^{(1)}_{n,m}(x)$. Thus the sequence of these solutions converges pointwise monotonically increasing to a non-negative solution of (3.30), (3.31), which is bounded by $h^{(1)}_{n,m}(x)$. Since $h^{(1)}_{n,m}(x)$ is the minimal non-negative solution of (3.30), (3.31), the assertion is proved. Therefore, due to Lebesgue's theorem, $E[V^2]$ can be computed with arbitrary accuracy by choosing n' sufficiently large.

In Table 1 there are given EV and $\mathrm{var}(V)$ for $r = 2, 4, 8, 16$. Note that in the case of $r = 2$ the simple expression (4.54) for $\mathrm{var}(V) = \mathrm{var}(V(d))$ has been used. Without loss of generality we have chosen $d := 1$ (unit of time).

Table 1 The mean and variance of the sojourn time V in $M/D/r$-PS for $r = 2, 4, 8, 16$ in case of $d = 1$

ϱ/r	$r=2$		$r=4$		$r=8$		$r=16$	
	EV	$\mathrm{var}(V)$	EV	$\mathrm{var}(V)$	EV	$\mathrm{var}(V)$	EV	$\mathrm{var}(V)$
0.30	1.0989	0.0559	1.0132	0.0034	1.0006	0.0001	1.0000	0.0000
0.35	1.1396	0.0885	1.0232	0.0070	1.0017	0.0002	1.0000	0.0000
0.40	1.1905	0.1354	1.0378	0.0133	1.0039	0.0006	1.0001	0.0000
0.45	1.2539	0.2031	1.0584	0.0239	1.0079	0.0014	1.0004	0.0000
0.50	1.3333	0.3011	1.0870	0.0411	1.0148	0.0033	1.0011	0.0001
0.55	1.4337	0.4454	1.1260	0.0689	1.0260	0.0069	1.0029	0.0003
0.60	1.5625	0.6628	1.1794	0.1142	1.0436	0.0140	1.0065	0.0009
0.65	1.7316	1.0018	1.2532	0.1889	1.0708	0.0274	1.0137	0.0025
0.70	1.9608	1.5563	1.3572	0.3166	1.1128	0.0529	1.0270	0.0063
0.75	2.2857	2.5273	1.5094	0.5476	1.1785	0.1030	1.0511	0.0151
0.80	2.7778	4.4060	1.7455	1.0045	1.2860	0.2083	1.0953	0.0366
0.85	3.6036	8.6560	2.1489	2.0534	1.4771	0.4604	1.1805	0.0933
0.90	5.2632	21.3348	2.9694	5.2085	1.8769	1.2376	1.3696	0.2797
0.95	10.2564	92.7299	5.4571	23.0418	3.1104	5.6830	1.9752	1.3814

4 $M/GI/r$-PS with locally exponential service times

We consider now an $M/GI/r$-PS system, i.e.

$$\varphi(n) = \min(r/n, 1), \quad n \in \mathbb{N}, \tag{4.1}$$

where the service time S has finite mean m_S and its distribution $B(x)$ coincides with an exponential distribution in a neighborhood of zero, i.e.

$$B(x) = 1 - e^{-\mu x}, \quad x \in [0, d), \tag{4.2}$$

where $\mu \in (0, \infty)$, for some $d \in (0, \infty)$. Moreover, we assume that the stability condition (1.1) is satisfied, i.e. that $\varrho < r$, and that the system is in steady state. From Theorem 3.1 it follows that the LST of $V(\tau)$ for $s \in (0, \infty)$ is given by

$$E\big[e^{-sV(\tau)}\big] = \frac{e^{\mu\tau}}{\lambda} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} \frac{p(0)}{(n-m)!} \big(m_S e^{\mu\tau} - F(\tau)\big)^{n-m} h_{n,m}(s, \tau),$$
$$\tau \in [0, d), \tag{4.3}$$

where $h_{n,m}(s, x)$ is the minimal non-negative solution of the linear system of ODEs (3.10) with initial condition (3.11). Applying the substitution

$$h_{n,m}(s, x) = \lambda^{n+1-m} \left(\prod_{j=1}^{n} \frac{1}{\varphi(j)} \right) e^{-\mu x} \vartheta_{n,m}(s/\lambda, \lambda x),$$
$$0 < m \le n, 0 \le x < d, \tag{4.4}$$

and the notation $\kappa := \mu/\lambda$, it follows that $\vartheta_{n,m}(\sigma,\xi)$ is the minimal non-negative solution of

$$\begin{aligned}\frac{\partial}{\partial\xi}\vartheta_{n,m}(\sigma,\xi) = -\left(\frac{1+\sigma}{\varphi(n)} + (n-1)\kappa\right)\vartheta_{n,m}(\sigma,\xi)\\ + \mathbb{I}\{m>1\}\vartheta_{n-1,m-1}(\sigma,\xi) + \frac{m\kappa}{\varphi(n)}\vartheta_{n+1,m+1}(\sigma,\xi)\\ + \frac{1}{\varphi(n)}\vartheta_{n+1,m}(\sigma,\xi), \quad 0<m\le n, \xi\in(0,\infty),\end{aligned} \tag{4.5}$$

with initial condition

$$\vartheta_{n,m}(\sigma,0) = \mathbb{I}\{m=1\}, \quad 0<m\le n, \tag{4.6}$$

restricted to $\xi \in [0,\lambda d)$, cf. Remark 3.1. The linear system of ODEs (4.5), (4.6) is equivalent to the system of integral equations

$$\begin{aligned}\vartheta_{n,m}(\sigma,\xi) &= \mathbb{I}\{m=1\}e^{-(\frac{1+\sigma}{\varphi(n)}+(n-1)\kappa)\xi}\\ &\quad + \int_0^\xi \biggl(\mathbb{I}\{m>1\}\vartheta_{n-1,m-1}(\sigma,\eta) + \frac{m\kappa}{\varphi(n)}\vartheta_{n+1,m+1}(\sigma,\eta)\\ &\quad + \frac{1}{\varphi(n)}\vartheta_{n+1,m}(\sigma,\eta)\biggr)e^{-(\frac{1+\sigma}{\varphi(n)}+(n-1)\kappa)(\xi-\eta)}\,\mathrm{d}\eta, \quad 0<m\le n, \xi\in\mathbb{R}_+.\end{aligned} \tag{4.7}$$

As $(e^{-(n-1)\kappa\xi}/(m-1)!)((e^{\kappa\xi}-1)/\kappa)^{m-1}$, $0<m\le n$, $\xi\in\mathbb{R}_+$, is a non-negative solution of (4.5), (4.6) for $\sigma=0$, the minimal non-negative solution $\vartheta_{n,m}(\sigma,\xi)$, $0<m\le n$, $\xi\in\mathbb{R}_+$, of (4.7) for $\sigma>0$ can be constructed by the method of successive approximation starting from $\vartheta_{n,m;0}(\sigma,\xi)\equiv 0$, defining $\vartheta_{n,m;i+1}(\sigma,\xi)$ recursively with respect to i by the r.h.s. of (4.7), where the $\vartheta_{n',m'}(\sigma,\eta)$ are replaced by $\vartheta_{n',m';i}(\sigma,\eta)$, and taking the limit $i\to\infty$. This construction provides the estimate

$$0\le\vartheta_{n,m}(\sigma,\xi)\le\frac{e^{-(n-1)\kappa\xi}}{(m-1)!}\left(\frac{e^{\kappa\xi}-1}{\kappa}\right)^{m-1}, \quad 0<m\le n, \xi\in\mathbb{R}_+, \tag{4.8}$$

and it follows that $\vartheta_{n,m}(\sigma,\xi)$ for $0<m\le n-(r-1)$ does not depend on $\varphi(\ell)$ for $\ell<r$. By continuous continuation from (2.34), (2.36), (4.2), (2.27), (2.28), (3.7), and (4.4) we find that

$$E\bigl[e^{-\lambda\sigma V(0,\xi/\lambda)}\bigr] = \vartheta_{1,1}(\sigma,\xi), \quad \xi\in[0,\lambda d], \tag{4.9}$$

where $V(0,\tau)$ denotes the sojourn time of a τ-request finding at its arrival the $M/GI/r$-PS system empty. Further, by continuous continuation from (4.3), (4.4), and (3.6) we find that

$$E\bigl[e^{-\lambda\sigma V(\xi/\lambda)}\bigr] = p(0)\sum_{m=1}^\infty\sum_{n=m}^\infty\left(\prod_{j=1}^{n-1}\frac{1}{\varphi(j)}\right)\left(\frac{(\kappa\varrho-1)e^{\kappa\xi}+1}{\kappa}\right)^{n-m}$$

$$\times \frac{\vartheta_{n,m}(\sigma,\xi)}{(n-m)!}, \quad \xi \in [0, \lambda d]. \tag{4.10}$$

Note that $\varrho = 1/\kappa$ for the $M/M/r$-PS system.

As $\vartheta_{n,m}(\sigma,\xi)$ for $0 < m \leq n-(r-1)$ does not depend on $\varphi(\ell)$ for $\ell < r$, as well as the $M/GI/r$-PS system we consider the corresponding $M/GI/$SDPS system where the service capacity $\varphi(n) = \min(r/n, 1)$, $n \in \mathbb{N}$, is replaced by

$$\tilde{\varphi}(n) = r/n, \quad n \in \mathbb{N}. \tag{4.11}$$

Note that this $M/GI/$SDPS system is equivalent to an $M/GI/1$-PS system by replacing the service time S by S/r. We denote by $\tilde{p}(n)$ the stationary occupancy distribution of the $M/GI/$SDPS system with service capacity (4.11), by $\tilde{V}(\tau)$ the sojourn time of a τ-request in this system and by $\tilde{\vartheta}_{n,m}(\sigma,\xi)$ the minimal non-negative solution of the corresponding system of ODEs

$$\begin{aligned}\frac{\partial}{\partial\xi}\tilde{\vartheta}_{n,m}(\sigma,\xi) = -\left(\frac{1+\sigma}{\tilde{\varphi}(n)} + (n-1)\kappa\right)\tilde{\vartheta}_{n,m}(\sigma,\xi)\\ + \mathbb{I}\{m>1\}\tilde{\vartheta}_{n-1,m-1}(\sigma,\xi) + \frac{m\kappa}{\tilde{\varphi}(n)}\tilde{\vartheta}_{n+1,m+1}(\sigma,\xi)\\ + \frac{1}{\tilde{\varphi}(n)}\tilde{\vartheta}_{n+1,m}(\sigma,\xi), \quad 0 < m \leq n, \xi \in (0,\infty),\end{aligned} \tag{4.12}$$

with initial condition

$$\tilde{\vartheta}_{n,m}(\sigma,0) = \mathbb{I}\{m=1\}, \quad 0 < m \leq n, \tag{4.13}$$

which is equivalent to the system of integral equations

$$\begin{aligned}\tilde{\vartheta}_{n,m}(\sigma,\xi) &= \mathbb{I}\{m=1\}e^{-(\frac{1+\sigma}{\tilde{\varphi}(n)}+(n-1)\kappa)\xi}\\ &\quad + \int_0^\xi \Bigg(\mathbb{I}\{m>1\}\tilde{\vartheta}_{n-1,m-1}(\sigma,\eta) + \frac{m\kappa}{\tilde{\varphi}(n)}\tilde{\vartheta}_{n+1,m+1}(\sigma,\eta)\\ &\quad + \frac{1}{\tilde{\varphi}(n)}\tilde{\vartheta}_{n+1,m}(\sigma,\eta)\Bigg)e^{-(\frac{1+\sigma}{\tilde{\varphi}(n)}+(n-1)\kappa)(\xi-\eta)}\,\mathrm{d}\eta, \quad 0 < m \leq n, \xi \in \mathbb{R}_+,\end{aligned} \tag{4.14}$$

cf. (4.5)–(4.7). Analogously to (4.8) we obtain the estimate

$$0 \leq \tilde{\vartheta}_{n,m}(\sigma,\xi) \leq \frac{e^{-(n-1)\kappa\xi}}{(m-1)!}\left(\frac{e^{\kappa\xi}-1}{\kappa}\right)^{m-1}, \quad 0 < m \leq n, \xi \in \mathbb{R}_+. \tag{4.15}$$

Further, analogously to (4.9) it follows that

$$E\big[e^{-\lambda\sigma\tilde{V}(0,\xi/\lambda)}\big] = \tilde{\vartheta}_{1,1}(\sigma,\xi), \quad \xi \in [0,\lambda d], \tag{4.16}$$

where $\tilde{V}(0,\tau)$ denotes the sojourn time of a τ-request finding at its arrival the $M/GI/$SDPS system with service capacity (4.11) empty. Using the well known expression for the LST of the sojourn time of a τ-request finding at its arrival the corresponding $M/GI/1$-PS system with scaled service time empty, cf. [18] (2.24) and (2.26), where, because of (4.2), $\psi(s,u)$ for $u\in[0,d)$ can be determined directly from [18] (2.22) and the initial condition $\psi(s,0)=1$, in view of (4.16), after some algebra we find that

$$\begin{aligned}\tilde{\vartheta}_{1,1}(\sigma,\xi) &= E\big[e^{-\lambda\sigma\tilde{V}(0,\xi/\lambda)}\big]\\ &= \frac{r_2-r_1}{(r_2-1)e^{(r_2-r\kappa)\xi/r}+(1-r_1)e^{(r_1-r\kappa)\xi/r}}, \quad \xi\in[0,\lambda d], \end{aligned}\tag{4.17}$$

where

$$r_{1,2} := \big(1+\sigma+r\kappa \mp \sqrt{(1+\sigma+r\kappa)^2-4r\kappa}\big)/2 \tag{4.18}$$

are the zeroes of $z^2-(1+\sigma+r\kappa)z+r\kappa$. Note that (4.18) implies

$$0<r_1<\min(r\kappa,1), \qquad r_2>\max(r\kappa,1) \tag{4.19}$$

for $\sigma>0$. Moreover, analogously to (4.10) we find that

$$\begin{aligned}E\big[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}\big] &= \tilde{p}(0)\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\prod_{j=1}^{n-1}\frac{1}{\tilde{\varphi}(j)}\right)\left(\frac{(\kappa\varrho-1)e^{\kappa\xi}+1}{\kappa}\right)^{n-m}\\ &\quad\times\frac{\tilde{\vartheta}_{n,m}(\sigma,\xi)}{(n-m)!}, \quad \xi\in[0,\lambda d]. \end{aligned}\tag{4.20}$$

On the other hand, by using the well known expression for the LST of the sojourn time of a τ-request in the corresponding $M/GI/1$-PS system with scaled service time, cf. [18] (2.29), in view of (4.2), after some algebra we obtain

$$\begin{aligned}E\big[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}\big] &= \frac{1-\frac{\varrho}{r}}{\frac{(1-r_2)^2}{r_2(r_2-r_1)}e^{(r_2-r\kappa)\xi/r}-\frac{(1-r_1)^2}{r_1(r_2-r_1)}e^{(r_1-r\kappa)\xi/r}+\frac{1-\kappa\varrho}{r\kappa}},\\ &\xi\in[0,\lambda d]. \end{aligned}\tag{4.21}$$

Note that (4.21) for $\varrho=1/\kappa$ also follows from the expression for the LST of the sojourn time of a τ-request in the $M/M/1$-PS system, cf. [18] (2.43).

As $\vartheta_{n,m}(\sigma,\xi)$ for $0<m\le n-(r-1)$ does not depend on $\varphi(\ell)$ for $\ell<r$ and since $\varphi(\ell)=\tilde{\varphi}(\ell)$ for $\ell\ge r$, we find that

$$\vartheta_{n,m}(\sigma,\xi)=\tilde{\vartheta}_{n,m}(\sigma,\xi), \quad 0<m\le n-(r-1), \xi\in\mathbb{R}_+. \tag{4.22}$$

Because of (4.22), (4.1), and (4.11), from (4.10) and (4.20) we obtain

$$
\begin{aligned}
&E\big[e^{-\lambda\sigma V(\xi/\lambda)}\big] \\
&\quad = p(0)\sum_{m=1}^{\infty}\sum_{n=m}^{m+r-2}\left(\prod_{j=1}^{n-1}\frac{1}{\varphi(j)}\right)\left(\frac{(\kappa\varrho-1)e^{\kappa\xi}+1}{\kappa}\right)^{n-m}\frac{\bar{\vartheta}_{n,m}(\sigma,\xi)}{(n-m)!} \\
&\qquad - p(0)\sum_{m=1}^{r-1}\sum_{n=m}^{r-1}\left(\left(\prod_{j=n}^{r-1}\frac{r}{j}\right)-1\right)\left(\frac{(\kappa\varrho-1)e^{\kappa\xi}+1}{\kappa}\right)^{n-m}\frac{\tilde{\vartheta}_{n,m}(\sigma,\xi)}{(n-m)!} \\
&\qquad + \frac{r^r p(0)}{r!\tilde{p}(0)}E\big[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}\big], \quad \xi\in[0,\lambda d], \qquad (4.23)
\end{aligned}
$$

where

$$
\bar{\vartheta}_{n,m}(\sigma,\xi) := \vartheta_{n,m}(\sigma,\xi) - \tilde{\vartheta}_{n,m}(\sigma,\xi), \quad 0<m\le n, \xi\in\mathbb{R}_+. \qquad (4.24)
$$

From (4.6) and (4.13) it follows that

$$
\bar{\vartheta}_{n,m}(\sigma,0) = 0, \quad 0<m\le n, \qquad (4.25)
$$

and (4.22) yields

$$
\bar{\vartheta}_{n,m}(\sigma,\xi) = 0, \quad 0<m\le n-(r-1), \xi\in\mathbb{R}_+. \qquad (4.26)
$$

Moreover, from (4.8) and (4.15) we find the estimate

$$
\big|\bar{\vartheta}_{n,m}(\sigma,\xi)\big| \le \frac{e^{-(n-1)\kappa\xi}}{(m-1)!}\left(\frac{e^{\kappa\xi}-1}{\kappa}\right)^{m-1}, \quad 0<m\le n, \xi\in\mathbb{R}_+. \qquad (4.27)
$$

4.1 $M/GI/2$-PS with locally exponential service times

Let us consider the $M/GI/2$-PS system, i.e. the case of $r=2$, in more detail. We assume that the stability condition (1.1) for the $M/GI/2$-PS system is satisfied, i.e. that $\varrho<2$, and that the system is in steady state. In view of (1.2), (4.1), (4.11), in the case of $r=2$ the representation (4.23) for the LST of the sojourn time of a τ-request simplifies to

$$
\begin{aligned}
E\big[e^{-\lambda\sigma V(\xi/\lambda)}\big] &= \frac{1-\varrho/2}{1+\varrho/2}\big(2G(\xi,1)-G(\xi,0)-\tilde{\vartheta}_{1,1}(\sigma,\xi)\big) \\
&\quad + \frac{2}{1+\varrho/2}E\big[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}\big], \quad \xi\in[0,\lambda d], \qquad (4.28)
\end{aligned}
$$

where $\tilde{\vartheta}_{1,1}(\sigma,\xi)$ and $E[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}]$ are given by (4.17) and (4.21) for $r=2$, respectively, and where for fixed positive σ

$$
G(\xi,z) := \sum_{m=1}^{\infty}\frac{(m-1)!}{2^{m-1}}\bar{\vartheta}_{m,m}(\sigma,\xi)z^{m-1}. \qquad (4.29)
$$

The series on the r.h.s. of (4.29) converges for $\xi \in [0, \lambda d]$, $|z| < 2/\varrho$ as well as for $\xi \in \mathbb{R}_+$, $|z| < 2\kappa$ due to (4.27), (4.2), and it follows that

$$\big|G(\xi, z)\big| < \frac{2\kappa}{2\kappa - |z|}, \quad \xi \in \mathbb{R}_+, |z| < 2\kappa. \tag{4.30}$$

Further, (4.25) yields that

$$G(0, z) = 0. \tag{4.31}$$

Because of (4.30), the Laplace transform

$$G^*(w, z) := \int_{\mathbb{R}_+} e^{-w\xi} G(\xi, z)\, \mathrm{d}\xi \tag{4.32}$$

of $G(\xi, z)$ with respect to ξ exists for $\Re w > 0$, $|z| < 2\kappa$. From (4.1), (4.5), (4.11), (4.12), (4.24), and (4.26) we find for $\xi \in (0, \infty)$ that

$$\frac{\partial}{\partial \xi}\bar{\vartheta}_{1,1}(\sigma, \xi) = -(1+\sigma)\bar{\vartheta}_{1,1}(\sigma, \xi) + \kappa\bar{\vartheta}_{2,2}(\sigma, \xi) + \frac{\partial}{\partial \xi}\tilde{\vartheta}_{1,1}(\sigma, \xi), \tag{4.33}$$

$$\begin{aligned}\frac{\partial}{\partial \xi}\bar{\vartheta}_{m,m}(\sigma, \xi) = {}&-\left(\frac{m(1+\sigma)}{2} + (m-1)\kappa\right)\bar{\vartheta}_{m,m}(\sigma, \xi) + \bar{\vartheta}_{m-1,m-1}(\sigma, \xi)\\ &+ \frac{m^2\kappa}{2}\bar{\vartheta}_{m+1,m+1}(\sigma, \xi), \quad m \geq 2.\end{aligned} \tag{4.34}$$

In view of (4.15) and (4.27), the Laplace transform $\tilde{\vartheta}^*_{1,1}(\sigma, w)$ of $\tilde{\vartheta}_{1,1}(\sigma, \xi)$ and the Laplace transforms $\bar{\vartheta}^*_{m,m}(\sigma, w)$ of $\bar{\vartheta}_{m,m}(\sigma, \xi)$, $m \geq 1$, exist for $\Re w > 0$. Because of (4.13) and (4.25), from (4.33) and (4.34) it follows for $\Re w > 0$ that

$$(1+\sigma+w)\bar{\vartheta}^*_{1,1}(\sigma, w) = \kappa\bar{\vartheta}^*_{2,2}(\sigma, w) + w\tilde{\vartheta}^*_{1,1}(\sigma, w) - 1, \tag{4.35}$$

$$\begin{aligned}&\left(\frac{m(1+\sigma)}{2} + (m-1)\kappa + w\right)\bar{\vartheta}^*_{m,m}(\sigma, w)\\ &\quad = \bar{\vartheta}^*_{m-1,m-1}(\sigma, w) + \frac{m^2\kappa}{2}\bar{\vartheta}^*_{m+1,m+1}(\sigma, w), \quad m \geq 2,\end{aligned} \tag{4.36}$$

respectively. In view of (4.29), (4.32), and (4.35), multiplying both sides of (4.36) by $(m-1)!(z/2)^{m-1}$ and summing up for $m \geq 2$ yields after some algebra that

$$\begin{aligned}&\big(z^2 - (1+\sigma+2\kappa)z + 2\kappa\big)\frac{\partial}{\partial z}G^*(w, z) + \big(z - (1+\sigma+2w)\big)G^*(w, z)\\ &\quad = 1 - wG^*(w, 0) - w\tilde{\vartheta}^*_{1,1}(\sigma, w), \quad \Re w > 0, |z| < 2\kappa.\end{aligned} \tag{4.37}$$

As the zeroes $r_{1,2}$ of $z^2 - (1+\sigma+2\kappa)z + 2\kappa$ satisfy (4.19), $G^*(w, z)$ is a solution of the ODE (4.37) for fixed w which is continuous at $z = r_1$. For real $z \in (r_1, 2\kappa)$ thus from (4.37) it follows that

$$G^*(w, z) = \big(wG^*(w, 0) + w\tilde{\vartheta}^*_{1,1}(\sigma, w) - 1\big)$$

$$\times \int_{r_1}^{z} \left(\frac{(x-r_1)(r_2-z)}{(z-r_1)(r_2-x)} \right)^{\frac{r_2-2\kappa+2w}{r_2-r_1}} \frac{\mathrm{d}x}{(x-r_1)(r_2-z)},$$
$$\Re w > 0,\ z \in (r_1, 2\kappa). \tag{4.38}$$

The substitution $((x-r_1)(r_2-z))/((z-r_1)(r_2-x)) = e^{-\frac{r_2-r_1}{2}\xi}$ yields

$$G^*(w,z) = \frac{r_2-r_1}{2} \big(wG^*(w,0) + w\tilde{\vartheta}_{1,1}^*(\sigma,w) - 1 \big)$$
$$\times \int_0^\infty \frac{e^{-w\xi}\,\mathrm{d}\xi}{(r_2-z)e^{(r_2-2\kappa)\xi/2} + (z-r_1)e^{(r_1-2\kappa)\xi/2}},$$
$$\Re w > 0, z \in (r_1, 2\kappa). \tag{4.39}$$

By means of analytic continuation to $z=0$ and applying the geometric series with common ratio $(r_1/r_2)e^{(r_1-r_2)\xi/2}$ to the integrand, for $\Re w > 0$ from (4.39) we obtain the representation

$$G^*(w,0) = \frac{\sum_{k=0}^\infty \frac{(r_1/r_2)^k}{(r_2-2\kappa+2w)/(r_2-r_1)+k}}{r_2 - w\sum_{k=0}^\infty \frac{(r_1/r_2)^k}{(r_2-2\kappa+2w)/(r_2-r_1)+k}} \big(w\tilde{\vartheta}_{1,1}^*(\sigma,w) - 1 \big) \tag{4.40}$$

of the Laplace transform of $G(\xi,0)$ by a hypergeometric function. On the other hand, in view of (4.13), (4.31), and of the properties of the Laplace transform, from (4.39) we find that

$$G(\xi,z) = \frac{r_2-r_1}{2} \int_0^\xi \left(\frac{\partial}{\partial\eta} \big(G(\eta,0) + \tilde{\vartheta}_{1,1}(\sigma,\eta) \big) \right)$$
$$\times \frac{\mathrm{d}\eta}{(r_2-z)e^{(r_2-2\kappa)(\xi-\eta)/2} + (z-r_1)e^{(r_1-2\kappa)(\xi-\eta)/2}},$$
$$\xi \in \mathbb{R}_+, z \in (r_1, 2\kappa), \tag{4.41}$$

where (4.41) even holds for $\xi \in \mathbb{R}_+$, $|z| < r_2$ due to the uniqueness theorem for analytic functions and Taylor's theorem, and where $r_2 > 1$ because of (4.19). In view of (4.13) and (4.31), integration by parts provides us with

$$G(\xi,z) = \frac{1}{2} \big(G(\xi,0) + \tilde{\vartheta}_{1,1}(\sigma,\xi) \big)$$
$$- \frac{1}{2} \frac{r_2-r_1}{(r_2-z)e^{(r_2-2\kappa)\xi/2} + (z-r_1)e^{(r_1-2\kappa)\xi/2}}$$
$$- \frac{r_2-r_1}{4} \int_0^\xi \big(G(\eta,0) + \tilde{\vartheta}_{1,1}(\sigma,\eta) \big)$$
$$\times \frac{(r_2-z)(r_2-2\kappa)e^{(r_2-2\kappa)(\xi-\eta)/2} + (z-r_1)(r_1-2\kappa)e^{(r_1-2\kappa)(\xi-\eta)/2}}{((r_2-z)e^{(r_2-2\kappa)(\xi-\eta)/2} + (z-r_1)e^{(r_1-2\kappa)(\xi-\eta)/2})^2}\,\mathrm{d}\eta,$$
$$\xi \in \mathbb{R}_+, |z| < r_2. \tag{4.42}$$

In particular, choosing $z=0$ in (4.42) yields

$$\begin{aligned}G(\xi,0)&=\tilde{\vartheta}_{1,1}(\sigma,\xi)-\frac{r_2-r_1}{r_2e^{(r_2-2\kappa)\xi/2}-r_1e^{(r_1-2\kappa)\xi/2}}\\&\quad-\frac{r_2-r_1}{2}\int_0^\xi\big(G(\eta,0)+\tilde{\vartheta}_{1,1}(\sigma,\eta)\big)\\&\quad\times\frac{r_2(r_2-2\kappa)e^{(r_2-2\kappa)(\xi-\eta)/2}-r_1(r_1-2\kappa)e^{(r_1-2\kappa)(\xi-\eta)/2}}{(r_2e^{(r_2-2\kappa)(\xi-\eta)/2}-r_1e^{(r_1-2\kappa)(\xi-\eta)/2})^2}\,\mathrm{d}\eta,\quad \xi\in\mathbb{R}_+.\end{aligned}\tag{4.43}$$

Due to (4.43), (4.29), (4.24), (4.9), and (4.17), moreover, we obtain

$$\begin{aligned}E\big[e^{-\lambda\sigma V(0,\xi/\lambda)}\big]&=\frac{2(r_2-r_1)}{(r_2-1)e^{(r_2-2\kappa)\xi/2}+(1-r_1)e^{(r_1-2\kappa)\xi/2}}\\&\quad-\frac{r_2-r_1}{r_2e^{(r_2-2\kappa)\xi/2}-r_1e^{(r_1-2\kappa)\xi/2}}-\frac{r_2-r_1}{2}\int_0^\xi E\big[e^{-\lambda\sigma V(0,\eta/\lambda)}\big]\\&\quad\times\frac{r_2(r_2-2\kappa)e^{(r_2-2\kappa)(\xi-\eta)/2}-r_1(r_1-2\kappa)e^{(r_1-2\kappa)(\xi-\eta)/2}}{(r_2e^{(r_2-2\kappa)(\xi-\eta)/2}-r_1e^{(r_1-2\kappa)(\xi-\eta)/2})^2}\,\mathrm{d}\eta,\\&\quad \xi\in[0,\lambda d],\end{aligned}\tag{4.44}$$

cf. [3] Theorem 4 for the case of an $M/M/2$-PS system.

The Volterra integral equation (convolution equation) (4.43) can be used for computing $G(\xi,0)$, $\xi\in[0,\lambda d]$, for example by means of the Neumann series. Alternatively, (4.40) can be used for computing $G(\xi,0)$, $\xi\in[0,\lambda d]$, by inverting both factors on the r.h.s. and applying the convolution formula. Subsequently, $G(\xi,1)$, $\xi\in[0,\lambda d]$, can be computed via (4.41) or (4.42) for $z=1$.

4.2 $M/GI/2$-PS with bounded from below service times

In this section we consider the $M/GI/2$-PS system with bounded from below service times, i.e., we consider the case of $r=2$ where

$$B(x)=0,\quad x\in[0,d),\tag{4.45}$$

for some $d\in(0,\infty)$. Moreover, we assume that the stability condition (1.1) for the $M/GI/2$-PS system is satisfied, i.e. that $\varrho<2$, and that the system is in steady state. The LST of $V(\tau)$ is given by the limit $\mu\downarrow 0$ of the LST of $V(\tau)$ in the corresponding system with locally exponential service times, cf. Sect. 4.1. Thus from (4.28) we obtain

$$\begin{aligned}E\big[e^{-\lambda\sigma V(\xi/\lambda)}\big]&=\lim_{\kappa\downarrow 0}\bigg(\frac{1-\varrho/2}{1+\varrho/2}\big(2G(\xi,1)-G(\xi,0)-\tilde{\vartheta}_{1,1}(\sigma,\xi)\big)\\&\quad+\frac{2}{1+\varrho/2}E\big[e^{-\lambda\sigma\tilde{V}(\xi/\lambda)}\big]\bigg),\quad \xi\in[0,\lambda d].\end{aligned}\tag{4.46}$$

In view of

$$\lim_{\kappa\downarrow 0} r_1 = 0, \qquad \lim_{\kappa\downarrow 0} r_2 = 1+\sigma, \tag{4.47}$$

cf. (4.18), from (4.17) we find that

$$\lim_{\kappa\downarrow 0} \tilde{\vartheta}_{1,1}(\sigma,\xi) = \frac{1+\sigma}{1+\sigma e^{(1+\sigma)\xi/2}}, \quad \xi\in[0,\lambda d]. \tag{4.48}$$

Moreover, after tedious algebra from (4.21) and (4.18) it follows that

$$\lim_{\kappa\downarrow 0} E\big[e^{-\lambda\sigma \tilde{V}(\xi/\lambda)}\big] = \frac{1-\varrho/2}{(1-\varrho/2)+\frac{\sigma}{1+\sigma}\xi/2+(\frac{\sigma}{1+\sigma})^2(e^{(1+\sigma)\xi/2}-1)},$$
$$\xi\in[0,\lambda d]. \tag{4.49}$$

In the limiting case of $\kappa\downarrow 0$, in view of (4.47) and (4.48), the Volterra integral equation (4.43) for $G(\xi,0)$ simplifies to

$$\begin{aligned} G(\xi,0) &= \frac{1+\sigma}{1+\sigma e^{(1+\sigma)\xi/2}} - e^{-(1+\sigma)\xi/2} \\ &\quad - \frac{1+\sigma}{2}\int_0^\xi \left(G(\eta,0)+\frac{1+\sigma}{1+\sigma e^{(1+\sigma)\eta/2}}\right)e^{-(1+\sigma)(\xi-\eta)/2}\,\mathrm{d}\eta, \\ &\xi\in[0,\lambda d]. \end{aligned} \tag{4.50}$$

Multiplying by $e^{(1+\sigma)\xi/2}$, differentiating with respect to ξ, and multiplying again by $e^{(1+\sigma)\xi/2}$ demonstrates that

$$\frac{\partial}{\partial\xi}\big(e^{(1+\sigma)\xi}G(\xi,0)\big) = -\frac{\sigma(1+\sigma)^2}{2}\frac{e^{(1+\sigma)3\xi/2}}{(1+\sigma e^{(1+\sigma)\xi/2})^2}, \quad \xi\in(0,\lambda d).$$

In view of $G(0,0)=0$, cf. (4.50), thus we find that

$$G(\xi,0) = -\frac{\sigma(1+\sigma)}{\zeta^2}\int_1^\zeta \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\bigg|_{\zeta=e^{(1+\sigma)\xi/2}}, \quad \xi\in[0,\lambda d], \tag{4.51}$$

where

$$\int_1^\zeta \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta = \frac{1}{\sigma^3}\left(\sigma(\zeta-1) - 2\log\left(\frac{1+\sigma\zeta}{1+\sigma}\right) + \frac{\sigma(\zeta-1)}{(1+\sigma)(1+\sigma\zeta)}\right)$$

for $\sigma>0$. Further, in the limiting case of $\kappa\downarrow 0$, taking into account (4.47), (4.48), (4.51), and using integration by parts, from (4.41) for $z=1$ after some algebra we obtain

$$G(\xi,1) = \frac{\sigma(1+\sigma)^2}{2}\int_1^\zeta \frac{\sigma\zeta}{(\omega+\sigma\zeta)^2\omega^2}\int_1^\omega \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\,\mathrm{d}\omega$$

$$
\begin{aligned}
&\quad - \frac{\sigma(1+\sigma)^2}{2}\int_1^\zeta \frac{\zeta}{(1+\sigma\omega)(\omega+\sigma\zeta)^2}\,\mathrm{d}\omega \\
&\quad - \frac{\sigma(1+\sigma)}{2\zeta^2}\int_1^\zeta \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\bigg|_{\zeta=e^{(1+\sigma)\xi/2}} \\
&= -\sigma(1+\sigma)^2\int_1^\zeta \frac{1}{(\omega+\sigma\zeta)^2\omega}\int_1^\omega \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\,\mathrm{d}\omega \\
&\quad - \frac{\sigma(1+\sigma)}{\zeta^2}\int_1^\zeta \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\bigg|_{\zeta=e^{(1+\sigma)\xi/2}}, \quad \xi\in[0,\lambda d], \quad (4.52)
\end{aligned}
$$

where for the last equation again integration by parts has been used. Note that the r.h.s. of (4.52) can be evaluated by using the dilogarithm function

$$
\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-\omega)}{\omega}\,\mathrm{d}\omega.
$$

Summarizing, from (4.46), (4.52), (4.51), (4.48), and (4.49) we find the following representation for the LST of $V(\tau)$.

Theorem 4.1 *Let the stability condition $\varrho < 2$ for the $M/GI/2$-PS system, where the service time fulfills* (4.45), *be satisfied. Then for $s \in \mathbb{R}_+$ and $\tau \in [0,d]$ the LST of $V(\tau)$ is given by*

$$
\begin{aligned}
E\big[e^{-sV(\tau)}\big] &= \frac{1-\varrho/2}{1+\varrho/2}\Bigg(-2\sigma(1+\sigma)^2\int_1^\zeta \frac{1}{(\omega+\sigma\zeta)^2\omega}\int_1^\omega \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta\,\mathrm{d}\omega \\
&\quad - \frac{\sigma(1+\sigma)}{\zeta^2}\int_1^\zeta \left(\frac{\eta}{1+\sigma\eta}\right)^2 \mathrm{d}\eta - \frac{1+\sigma}{1+\sigma\zeta} \\
&\quad + \frac{2(1+\sigma)^2}{(1-\varrho/2)(1+\sigma)^2+\sigma\log(\zeta)+\sigma^2(\zeta-1)}\Bigg)\bigg|_{\sigma=s/\lambda,\,\zeta=e^{(\lambda+s)\tau/2}}, \\
&\tau\in[0,d]. \qquad (4.53)
\end{aligned}
$$

Note that Theorem 4.1 also provides the LST of $V(\tau)$, $\tau\in[0,m_S]$, and in particular the LST of $V = V(m_S)$ in $M/D/2$-PS for $d = m_S$.

Taking the kth derivative with respect to s at $s=0$ on both sides of (4.53) yields the kth moment of $V(\tau)$ for $\tau\in[0,d]$. In particular, we find the following simple expression for the variance of $V(\tau)$ after tedious algebra.

Corollary 4.1 *Let the stability condition $\varrho < 2$ for the $M/GI/2$-PS system, where the service time fulfills* (4.45), *be satisfied. Then for $\tau\in[0,d]$ the variance of $V(\tau)$ is given by*

$$
\mathrm{var}\big(V(\tau)\big) = \frac{1}{\lambda^2}\Bigg(\frac{2\varrho\xi^2}{(1+\varrho/2)^2(1-\varrho/2)^2} - \frac{4(e^\xi-1-\xi)}{(1+\varrho/2)(1-\varrho/2)}
$$

$$+\frac{2}{9}\frac{1-\varrho/2}{1+\varrho/2}\left((12\xi-10)e^{\xi}+9+e^{-2\xi}\right)\bigg)\bigg|_{\xi=\lambda\tau/2}, \quad \tau\in[0,d]. \tag{4.54}$$

Note that Corollary 4.1 also provides the variances of $V(\tau)$, $\tau\in[0,m_S]$, and of $V=V(m_S)$ in $M/D/2$-PS for $d=m_S$, cf. Table 1 for $r=2$.

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