Risk of Bitcoin Market: Volatility, Jumps, and Forecasts

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Crytocurrency Market

- Inherent risk of BTC, technologic nature
- Controversial asset. Yermack (2015), Hafner (2018)
- Econometric Investigation. Conrad et al. (2018), Scaillet et al. (2018) More Literature Review

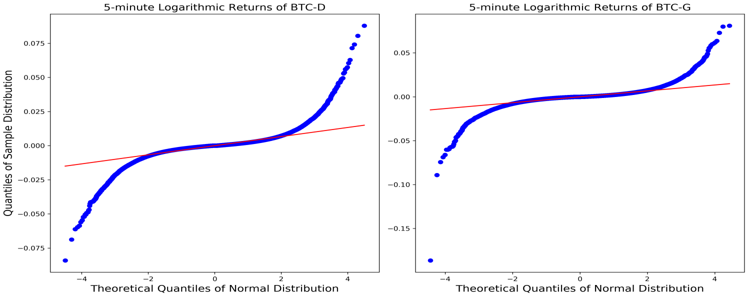


Figure 1: Q-Q plot on 5-minute logreturns of BTC

Annualized Realized Volatility Comparison

	AEX	DJI	FTSE	HSI	SPX	SSEC	BTC-D	BTC-G
count	4842	4704	4769	4645	4709	4508	864	883
mean	0.16	0.12	0.14	0.15	0.13	0.23	1.16	0.93
std	0.38	0.30	0.32	0.41	0.32	0.46	2.05	1.76
min	0.10%	0.08%	0.16%	0.35%	0.04%	0.23%	2.00%	0.76%
25%	0.02	0.02	0.02	0.03	0.02	0.03	0.08	0.04
50%	0.05	0.04	0.05	0.06	0.05	0.09	0.56	0.40
75%	0.14	0.11	0.13	0.14	0.12	0.23	3.70	3.17
max	7.04	5.55	7.74	16.46	7.18	7.71	26.07	21.99

- Jan. 2017 - Jul. 2019. Sum of Square logreturns using 5-minute freq data.
- Overnight bias from trading hours corrected (Bollerslev et al. 2018).
- Extraordinary high risk level, mean and std, against indices globally.

Research Questions

- Risk dynamics of BTC, (upside/downside) Realized volatility and $(+/-)$jumps

  Anything special?

  Jumps contribution to risk
- Predictability of *RV*
- Role of jumps in *RV* forecasting

Main Results

- Decomposition of *RV* of BTC robust to consecutive jumps (ABDL(2001,2003), Corsi et al.(2010)); Upside/downside risk. Positive/Negative Jumps (BNKS(2010), Patton, Sheppard (2015))
- Up to 68% of the sample days with jump(s), however, little contribution from jumps to risk.
- Symmetric on risk and jumps. Jump risk reduction by simple "portfolio".
- $RSV_t^- \uparrow \Rightarrow RV_{t+1} \uparrow$, and $(T)J_t^+ \uparrow \Rightarrow RV_{t+1} \uparrow$ (In-sample)
- Necessary to Model jumps? Forecasting horizon matters on choosing models.

Outline

1. Introduction
2. Theoretical Framework
3. Data and Preliminary Analysis
4. Accounting Separated Jumps with *RV* Forecasting
5. Conclusion

Continuous-Time Framework

- A general continous-time jump diffussion process:

$$dp(t) = \mu(t)\,\mathrm{d}\,t + \sigma(t)\,\mathrm{d}\,W(t) + \kappa(t)\,\mathrm{d}\,q(t), 0 \leq t \leq T \quad (1)$$

p: Logarithmic price
μ: Continous and locally bounded variation process
σ: Strictly positive stochastic volatility process with right continuous sample path
W: Brownian motion
κ: Size of a jumps
q: Counting measure for jumps

Realized Variance

- Realized Variance RV_{t+1} during $[t : t+1]$. $r_{t+j\Delta}$: j_{th} logreturn in day t; Δ: Sampling interval

$$RV_{t+1}(\Delta) \stackrel{\text{def}}{=} \sum_{j=1}^{1/\Delta} r^2_{t+j\Delta} \tag{2}$$

- Convergence of RV as Δ goes to 0 (Quadratic variance theory and BNS(2004, 2006))

$$RV_{t+1}(\Delta) \xrightarrow{p} \underbrace{\int_t^{t+1} \sigma^2(s)ds}_{IV_{t+1}} + \underbrace{\sum_{t<s\leq t+1} \kappa^2(s)}_{J_{t+1}} \tag{3}$$

Jump J Separation

- Convergence of Realized BiPower Variance BPV_{t+1} during $[t, t+1]$ to IV as Δ goes to 0

$$BPV_{t+1}(\Delta) \stackrel{\text{def}}{=} \frac{\pi}{2} \sum_{j=2}^{1/\Delta} |r_{t+j\Delta}||r_{t+(j-1)\Delta}| \qquad (4)$$

$$BPV_{t+1}(\Delta) \rightarrow \int_t^{t+1} \sigma^2(s) ds \qquad (5)$$

- Jump estimator J_{t+1}:

$$RV_{t+1}(\Delta) - BPV_{t+1}(\Delta) \xrightarrow{\text{p}} \sum_{t<s\leq t+1} \kappa^2(s)$$
$$J_{t+1} \stackrel{\text{def}}{=} \mathbf{I}(z > \Phi_\alpha) \max\{RV_{t+1}(\Delta) - BPV_{t+1}(\Delta), 0\} \qquad (6)$$

Jump test z-test

Realized Variance Separation, BTCD

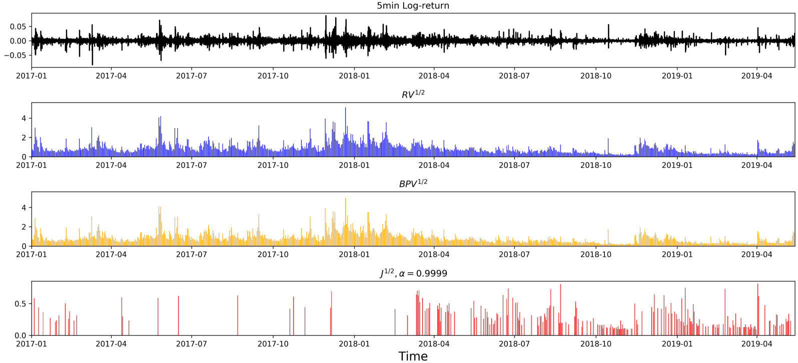


Figure 2: 5-minute Logreturn, annualized daily Realized Volatility $RV^{1/2}$, Bipower Volatility $BPV^{1/2}$ and significant Jump $J^{1/2}$ Process

▸ BTCG

BPV Biased to Consecutive Jumps

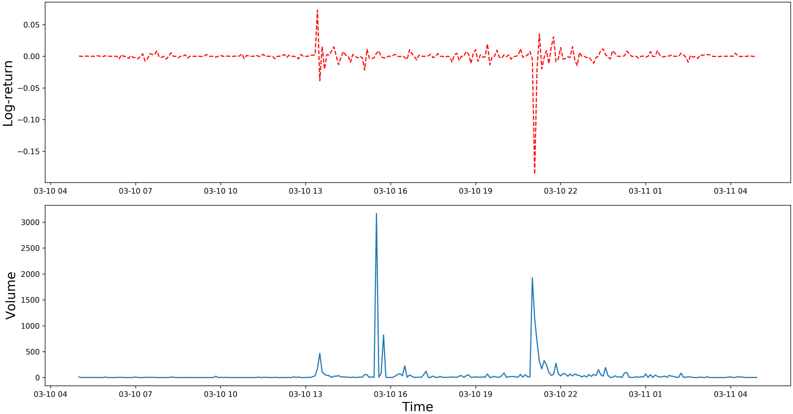


Figure 3: Price process of BTCG on 10th, Mar 2017, Log-return; Trading Volume, at 5-minute frequency

Threshold Bipower Variation Estimator

- Threshold Bipower Variance $TBPV_{t+1}$ during $[t, t+1]$ (Mancini(2009), Corsi et al.(2010))

$$TBPV_{t+1}(\Delta) = \mu_1^{-2} \cdot \sum_{j=2}^{1/\Delta} C_1(r_{t+j\Delta}, \theta_{t+j\Delta}) C_1(r_{t+(j-1)\Delta}, \theta_{t+(j-1)\Delta}) \quad (7)$$

- Where $\theta_\tau = c_\theta^2 \cdot \widehat{V}_\tau$, and $\mu_1 = \sqrt{2/\pi}$. Let $c_\theta = 3$

$$C_\eta(r_\tau, \theta_\tau) = \begin{cases} |r_\tau|^\eta & , r_\tau^2 \leq \theta_\tau \\ r_\tau^e(\theta_\tau, \eta) & , r_\tau^2 > \theta_\tau \end{cases} \quad (8)$$

- Thresholded Jump TJ_{t+1} during [t,t+1]

$$TJ_{t+1}(\alpha) \stackrel{\text{def}}{=} \mathbf{I}(c\text{-}z > \Phi_\alpha) \cdot \max(RV_{t+1} - TBPV_{t+1}, 0) \quad (9)$$

▸ Conditional Expected Returns ▸ Local Variance Estimation ▸ *TJ c-z*-test

Threshold Estimator Jump Separation, BTCD

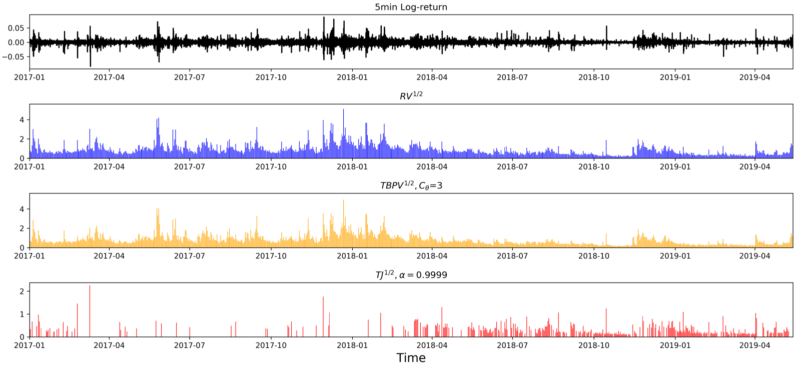


Figure 4: $TJ^{1/2}$(99.99%) separation. Significant thresholded jumps separation using TBPV.

▸ BTCG

Realized Semivariance

- Positive and Negative realized Semivariance

$$RS_{t+1}^{+(-)} = \sum_{j=1}^{1/\Delta} r_{t+j\Delta}^2 \cdot \mathrm{I}\{r_{t_j\Delta} > (<) 0\} \tag{10}$$

- Convergence in probability (BNKS 2010), $\Delta p_s = p_s - p_{s_-}$

$$RS_{t+1}^{+} \xrightarrow{p} \underbrace{\frac{1}{2}\int_t^{t+1} \sigma^2(s)ds}_{\frac{1}{2}IV_{t+1}} + \underbrace{\sum_{t<s\leq t+1} (\Delta p_s)^2 \cdot \mathrm{I}\{\Delta p_s > 0\}}_{J_{t+1}^{+}} \tag{11}$$

- Positive and negative jumps

$$(T)J_{t+1}^{+(-)} = \max\left\{RS_{t+1}^{+(-)} - \frac{1}{2}(T)BPV_{t+1}, 0\right\} \tag{12}$$

Data Source

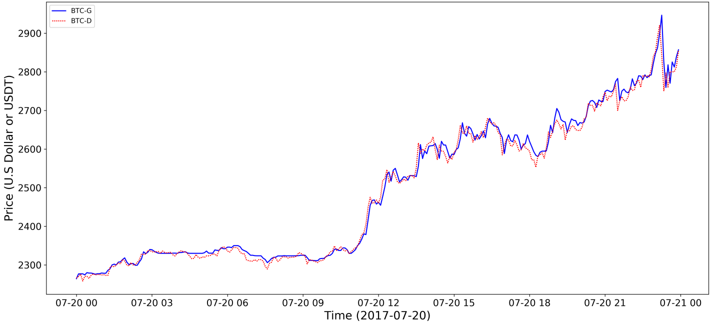


Figure 5: Price trajectories on 20th July, 2017.

- BTCD: DYOS solutions GmbH. Equal-weighted prices from 3 actively trading exchanges. Jan. 2017 - May 2019.
- BTCG: Gemini exchanges, price from one exchange. Jan. 2017 - May(BTCD) and Jul.(BTCG) 2019. ▸ Date Selection

Summary Statistics for RV Estimators, BTCD

	RV	RSV^+	RSV^-	$\log(RV)$	$\log(RSV^+)$	$\log(RSV^-)$	C	TC
mean	1.16	0.57	0.59	−0.60	−1.32	−1.30	1.12	1.08
std	2.05	1.02	1.05	1.20	1.22	1.23	2.06	2.01
min	0.02	0.01	0.01	−3.98	−4.88	−4.51	0.01	0.01
5%	0.08	0.04	0.04	−2.54	−3.29	−3.27	0.06	0.05
50%	0.56	0.27	0.26	−0.58	−1.31	−1.33	0.51	0.47
95%	3.70	1.81	2.06	1.31	0.59	0.72	3.70	3.69
max	26.07	13.28	12.80	3.26	2.59	2.55	26.07	26.07
skewness	5.59	5.78	5.41	0.06	0.04	0.10	5.60	5.73
kurtosis	43.38	46.24	40.12	−0.04	−0.03	−0.15	43.40	45.86
acf(1)	0.53	0.51	0.52	0.79	0.78	0.77	0.53	0.56
acf(7)	0.17	0.16	0.17	0.58	0.58	0.57	0.17	0.18
acf(30)	0.11	0.10	0.11	0.38	0.37	0.38	0.12	0.13
acf(100)	0.06	0.06	0.06	0.23	0.23	0.23	0.06	0.07
ADF	−3.16	−3.22	−3.12	−4.36	−4.34	−4.42	−3.12	−3.10

- Similar intensity, upside risk and downside risk. ACF
- Long-memory effect, Lognormal distributed. Lognormal
- Discontinuities do not contribute much to risk.

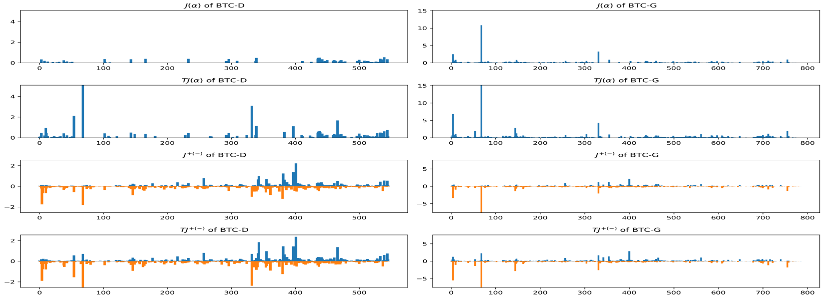
BTC-G

Summary Statistics for Jump Estimators, BTCD

	$J(\alpha)$	$TJ(\alpha)$	J^+	J^-	TJ^+	TJ^-
prop.†	0.25	0.39	0.68	0.73	0.76	0.79
mean	0.12	0.20	0.08	0.09	0.09	0.11
std	0.14	0.40	0.17	0.17	0.21	0.25
min(%)	0.83	0.92	0.01	0.01	0.02	0.03
5%(%)	1.00	1.00	0.30	0.35	0.36	0.46
50%	0.07	0.09	0.03	0.03	0.03	0.04
95%	0.40	0.64	0.3	0.32	0.34	0.39
max	0.66	5.09	2.22	1.81	2.36	4.39
skewness	1.69	7.28	6.37	5.28	5.80	9.46
kurtosis	2.55	73.22	56.20	40.47	43.31	131.43

- Idiosyncratic jump risk significantly by BTCD (simple weigted "portfolio"). BTC-G
- Almost equal between positive and negative jump, quantity and size. Jump Evolving
- †: Report only non-zero values.

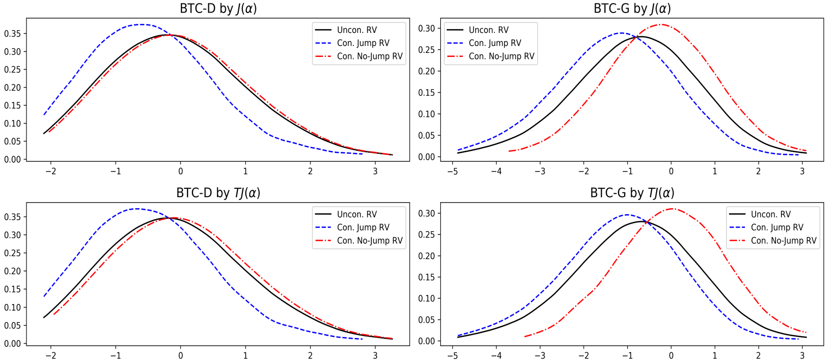
Jump Dynamics



- Top to down: $J(\alpha)$, $TJ(\alpha)$, $J^{+(-)}$, $TJ^{+(-)}$; Left to right: BTCD, BTCG

▸ Back

RV Distr. Conditioning on (Non-)Jump



- Negative shifting RV_t distr. conditioning on jump$_{t-1}$
- Consistent results from 2 estimators in 2 processes

HAR Models

- HAR with Jump Components

$$RV_{t,t+h} = \alpha + X_{RV}^{\top}\beta_{RV} + X_{j}^{\top}\beta_{j} + \varepsilon_{t,t+h}, \quad t = 1, \ldots, T \tag{13}$$

▸ Regressors Details

- 8 forecasting models ▸ Details of Models in **square form** and **logarithmic form**
- Serial correlation and heteroskedasticity corrected by Newey-West covariance matrix estimator

Regression Results

- Inconsistent results in BTCD and BTCG (Full-sample fitting)
- Adaptive Method:

  A fixed 90-day rolling window

  Parameters estimated daily

  Parameters changing over time systematically ▸ Details
- Out-of-sample forecasting:

  "Insanity filter". ▸ Details

  4 metrics, D-M test for significance ▸ Details

  Economic Values ▸ Details

▸ Parameters Evolving

Out-of-Sample, BTCD, Square Form

	HAR	RVJ	RVTJ	RVSJ	RVSTJ	RSV	RSVSJ	RSVSTJ
	h=1							
MZ-R^2	0.267	0.227	0.248	0.228	0.219	0.285	0.215	0.165
MSE	3.636	4.023†	3.850	4.032	3.984	3.676	4.414†	4.812†
HRMSE	1.569	2.031†	1.576	1.792†	1.737†	1.767†	1.841†	1.756†
QLIKE	0.846	1.151†	1.100†	1.000†	1.172†	0.988†	1.176†	1.545†
	h=7							
MZ-R^2	0.365	0.360	0.416	0.440	0.470	0.431	0.508	0.458
MSE	1.643	1.789†	1.437	1.552	1.285*	1.573	1.328*	1.272*
HRMSE	1.359	1.646†	1.278	1.358	1.303	1.455	1.243	1.279
QLIKE	0.984	1.146†	0.997	1.026	0.996	1.037	0.880*	1.043
	h=30							
MZ-R^2	0.543	0.583	0.682	0.717	0.623	0.628	0.743	0.677
MSE	0.673	0.649	0.462*	0.398*	0.558*	0.514*	0.357*	0.440*
HRMSE	0.779	0.637*	0.614*	0.525*	0.528*	0.698*	0.483*	0.444*
QLIKE	0.937	0.943	0.898*	0.847*	0.883*	0.877*	0.842*	0.874*

Table 1: Adaptive Out-of-Sample Forecasts Performance Evaluation. $\dagger$ ($*$): HAR out(under)performs other models. Log Form

Economic Values, BTCD

	HAR	RVJ	RVTJ	RVSJ	RVSTJ	RSV	RSVSJ	RSVSTJ
Square form								
h=1	2.938	2.403	2.496	2.613	2.268	2.626	2.356	1.421
h=7	3.071	2.614	3.048	2.935	3.024	2.916	3.339	2.874
h=30	3.569	3.522	3.642	3.760	3.667	3.708	3.769	3.678
Logarithmic form								
h=1	2.619	2.486	2.404	2.660	2.486	2.583	2.403	2.146
h=7	3.133	2.969	3.128	3.196	3.234	3.140	3.204	3.143
h=30	3.479	3.527	3.591	3.619	3.641	3.671	3.676	3.665

Table 2: Economic Values Evaluation of Out-of-Sample Forecasts (%)

- Higher accuracy & utility in the long-horizon forecasting
- Forecasting horizon matters:
  Short-horizon: Roughest model
  Long-horizon: Finely calibrated models
- Consistent and robust conclusion

▸ BTCG

Conclusions

- Risk characteristics of BTC

  Significant larger scale and frequent jumps (up to 68%), Consecutive jumps (Threshold method)

  Small contribution from jumps. Jump risk reduction of BTCD

  Log-noraml distributed and long-memory of *RV*

- Managing risk

  Forecasting horizon is important on choosing models

  Longer forecasting horizon, higher utility

  Short-horizon setting: Roughest model outperforms

  Long-horizon setting: Modelling jumps is profitable

Summary

- Realized Volatility Processes
  Differences: Significant larger scale and frequent jumps
  Similarities: Log-noraml distributed and long-memory
  Maybe useful for cryptocurrencies options pricing
- Statistics Findings
  Threshold jump method overcomes consecutive jumps and provides more information
  Significant positive impact from TJ on RV. No mean-reversion
  Non-linear models perform better
- Economic Perspective
  Investors gain higher economic value by modeling jumps
  Longer investment horizon, higher utility
  Non-linear modeling is necessary to capture more market changes

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Literature Review

- TBA

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Jump Test z-test

- Distribution results from BNS (2004) and other extensions in which the statistic goes to normal distribution given a series of conditions and in the absence of jump. The z-test is the ratio-statistic provided by Huang and Tauchen (2005).

$$z_{t+1} = \frac{\{RV_{t+1}(\Delta) - BPV_{t+1}(\Delta)\}\, RV_{t+1}^{-1}(\Delta)}{\sqrt{\Delta \cdot \zeta \cdot \max\left\{1, \frac{TPV_{t+1}(\Delta)}{\{BPV_{t+1}(\Delta)\}^2}\right\}}} \tag{14}$$

Where $\zeta = \frac{\pi^2}{4} + \pi - 5$

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Conditional Expected Returns

- The expected value of η-power returns conditioning on the square returns larger than threshold

$$
\begin{aligned}
r^e(\theta,\eta) &= \mathsf{E}\left\{ |r|^\eta \middle| r^2 > \theta \right\} \\
&= \frac{(2\sigma^2)^{\frac{\eta}{2}}}{2\sqrt{\pi}\Phi\left(-\frac{\sqrt{\theta}}{\sigma}\right)} \cdot \Gamma\left(\frac{\eta+1}{2}, \frac{\theta}{2\sigma^2}\right) \\
&= \frac{1}{2\sqrt{\pi}\Phi\left(-c_\theta\right)} \cdot \left(\frac{2\theta_t}{c_\theta^2}\right)^{\frac{\eta}{2}} \cdot \Gamma\left(\frac{\eta+1}{2}, \frac{c_\theta^2}{2}\right)
\end{aligned} \tag{15}
$$

- Where Φ and Γ

$$
\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{2}} ds, \Gamma(\alpha, x) = \int_{x}^{+\infty} s^{\alpha-1} e^{-s} ds \tag{16}
$$

Realized Variance Separation, BTCG

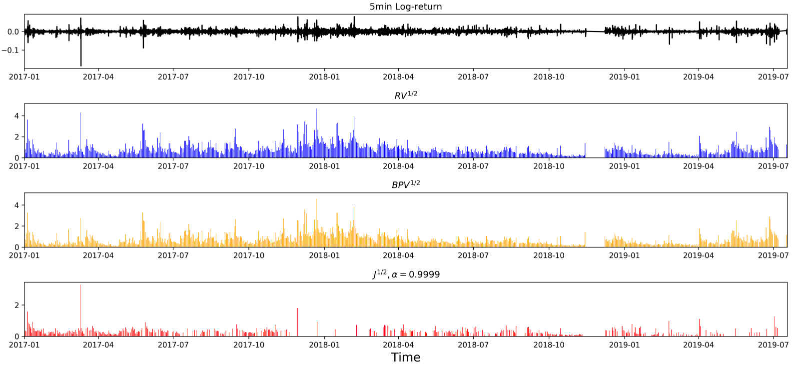

Figure 6: 5-minute Logreturn, annualized daily Realized Volatility $RV^{1/2}$, Bipower Volatility $BPV^{1/2}$ and Jump $J^{1/2}$(99.99%) Process

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Local Variance Estimation $\widehat{V}_t$

- a non-parametric filter of length $2L+1$ (Fan and Yao 2003)

$$\widehat{V}_t^Z = \frac{\sum_{i=-L, i\neq -1,0,1}^{L} K(\frac{i}{L}) \cdot r_{t+i}^2 \cdot \mathrm{I}\{r_{t+i}^2 \leq c_\theta^2 \cdot \widehat{V}_{t+i}^{Z-1}\}}{\sum_{i=-L, i\neq -1,0,1}^{L} K(\frac{i}{L}) \cdot \mathrm{I}\{r_{t+i}^2 \leq c_\theta^2 \cdot \widehat{V}_{t+i}^{Z-1}\}}, Z = 1, 2, 3... \tag{17}$$

- $K(x)$: Gaussian Kernel. c_θ: immaterial coefficient, trade-off between effciency and bias.
- Initial value $\widehat{V}_t^0 = \inf$; Recursive calculation until converged.
- Evaluation within each day, avoid using furture information.

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Threshold Jump

- Test statistics $c\text{-}z$ for TJ_t (BNS 2004, Corsi et al. 2010)

$$c\text{-}z = \Delta^{-\frac{1}{2}} \frac{\{RV_t - TBPV_t\} RV_t^{-1}}{\sqrt{\left(\frac{\pi^2}{4} + \pi - 5\right) \max\{1, \frac{TTriPV_t}{TBPV_t^2}\}}} \tag{18}$$

- Threshold continuous process, hereafter TC_{t+1} for period $[t, t+1]$

$$TC_{t+1} \stackrel{\text{def}}{=} RV_{t+1} - TJ_{t+1} \tag{19}$$

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Threshold Estimator Jump Separation, BTCG

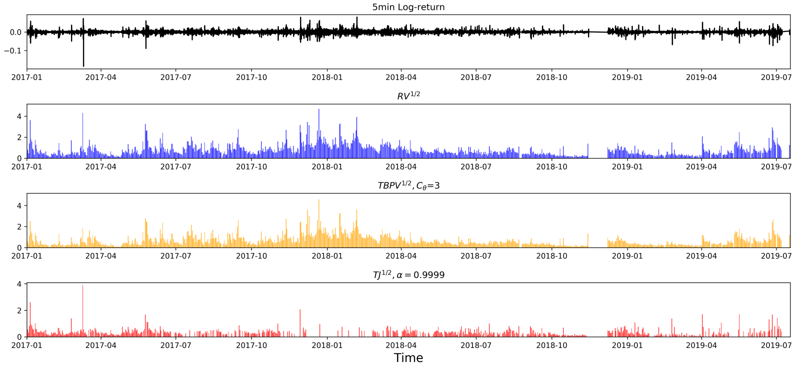

Figure 7: $TJ^{1/2}$(99.99%) separation. Significant thresholded jumps separation using TBPV.

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Liquidity of Bitcoin Market

- Defining a measure on relative trading intensity. $Minvol_{t+j\Delta,K} = \min_{k=-K\ldots,0,\ldots K} vol_{t+(j+k)\Delta}$, likewise for $Maxvol_{t+j\Delta,K}$.

$$
\begin{aligned}
RTI_{t+j\Delta} &\stackrel{def}{=} \frac{vol_{t+j\Delta} - Minvol_{t+j\Delta,K}}{Maxvol_{t+j\Delta,K} - Minvol_{t+j\Delta,K}} \\
ARTI_{j\Delta} &\stackrel{def}{=} \frac{1}{T}\sum_{t=1}^{T} RTI_{t+j\Delta}
\end{aligned}
\tag{20}
$$

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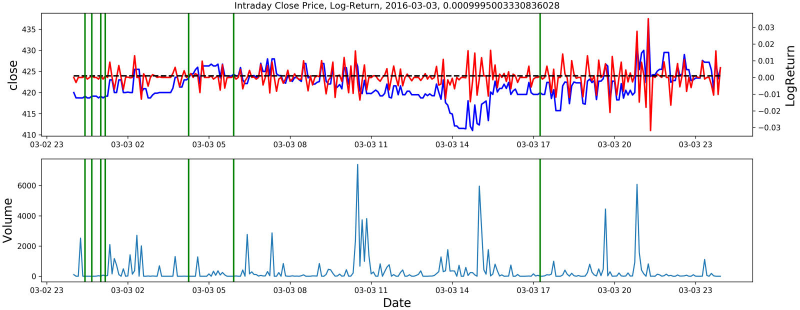
Algo-Trading



- Same Logreturn (rounded to 19 decimals) appears 7 times on March 3rd, 2016 and 100 times in 2016.
- Not removing weekends or holidays

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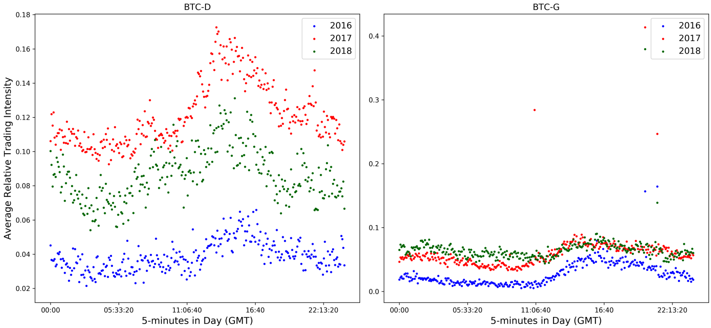
Liquidity of Bitcoin Market



- A rolling-window measure on relative trading intensity $RTI_{t_j\Delta}$.
- ARTI of BTC-D (left) and BTC-G (right), 2016, 2017, 2018.
- Algo-trading ‣ Possible Algorithmic Trading

‣ Def: Relative Trading Intensity

Summary Statistics for *RV* Estimators, BTCG

	RV	RSV^+	RSV^-	$\log(RV)$	$\log(RSV^+)$	$\log(RSV^-)$	C	TC
mean	0.93	0.45	0.49	−0.92	−1.66	−1.64	0.85	0.77
std	1.76	0.84	0.98	1.32	1.32	1.37	1.64	1.55
min	0.01	0.00	0.00	−4.88	−5.51	−5.63	0.00	0.00
5%	0.04	0.02	0.02	−3.15	−3.90	−4.00	0.03	0.02
50%	0.40	0.19	0.19	−0.91	−1.65	−1.64	0.33	0.28
95%	3.17	1.53	1.73	1.15	0.42	0.55	3.09	2.95
max	21.99	11.78	14.52	3.09	2.47	2.68	21.99	21.99
skewness	5.86	6.32	6.85	−0.04	−0.04	−0.01	5.73	6.21
kurtosis	47.28	59.42	68.92	−0.10	−0.14	−0.19	48.03	57.88
acf(1)	0.42	0.45	0.35	0.74	0.73	0.71	0.47	0.50
acf(7)	0.13	0.14	0.11	0.49	0.51	0.47	0.16	0.19
acf(30)	0.09	0.09	0.07	0.26	0.26	0.25	0.12	0.14
acf(100)	0.03	0.04	0.02	0.11	0.11	0.11	0.03	0.04
ADF	−3.03	−3.12	−4.89	−5.06	−3.61	−5.27	−2.95	−2.77

Table 3: Summary statistics for volatility related measures (annualized) from 5-min freq BTC, $ac(n)$: n-days autocorrelation. Confidence level $\alpha = 0.9999$

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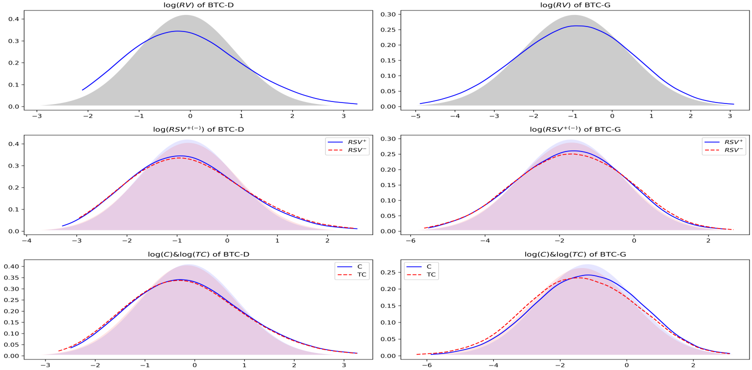
Distribution of *RV*



- Lognormal distribution.
- Shadows of normal distribution pdf in the background.

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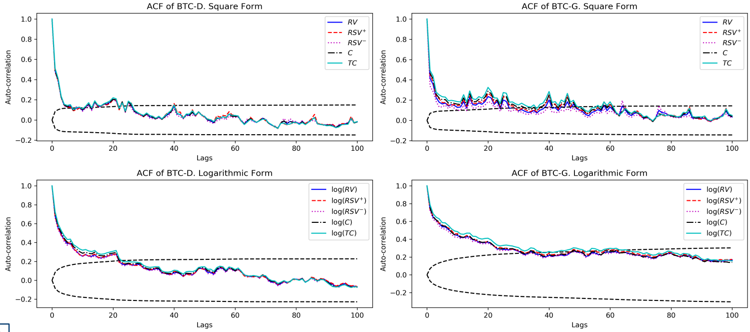
Autocorrelation Decay



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Summary Statistics for Jump Estimators, BTCG

	$J(\alpha)$	$TJ(\alpha)$	J^+	J^-	TJ^+	TJ^-
prop.[†]	0.51	0.68	0.75	0.81	0.83	0.87
mean	0.17	0.24	0.08	0.12	0.11	0.15
std	0.56	0.76	0.16	0.45	0.23	0.56
min	0.30%	0.34%	0.01%	0.00%	0.00%	0.02%
5%	0.90%	1.00%	0.24%	0.33%	0.32%	0.38%
50%	0.08	0.11	0.04	0.04	0.04	0.05
95%	0.44	0.65	0.34	0.37	0.38	0.44
max	10.82	15.14	2.16	10.71	2.86	12.87
skewness	15.90	14.40	6.45	19.27	6.77	17.33
kurtosis	290.20	259.24	61.08	437.7	62.06	368.50

- †: Report only non-zero values.
- Slightly larger negative jump, quantity and size.

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Estimators Construction for Forecasting Models

- Given a daily estimator $\hat{Y}_\tau$

$$\hat{Y}_{\tau_1,\tau_2} = \frac{1}{\tau_2 - \tau_1} \sum_{\tau=\tau_1+1}^{\tau_2} \hat{Y}_\tau \tag{21}$$

- Thress stepwise estimators

$$\begin{aligned} \text{Daily estimator, } \hat{Y}_D &\stackrel{def}{=} \hat{Y}_{t-1,t} \\ \text{Weekly estimator, } \hat{Y}_W &\stackrel{def}{=} \hat{Y}_{t-7,t-1} \\ \text{monthly estimator, } \hat{Y}_M &\stackrel{def}{=} \hat{Y}_{t-30,t-7} \end{aligned} \tag{22}$$

- Regressors vector

$$X_{\dot{RV}}^\top = \left(\dot{RV}_D, \dot{RV}_W, \dot{RV}_M\right) X_{\dot{J}}^\top = \left(\dot{J}_D, \dot{J}_W, \dot{J}_M\right) \tag{23}$$

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Metrics

- 3 loss function

$$L^{\text{MSE}} = \left(RV_{t,t+h} - \widehat{RV}_{t,t+h}\right)^2 \quad (24)$$

$$L^{\text{HRMSE}} = \left(\frac{RV_{t,t+h} - \widehat{RV}_{t,t+h}}{RV_{t,t+h}}\right)^2 \quad (25)$$

$$L^{\text{QLIKE}} = \log \widehat{RV}_{t,t+h} + \frac{RV_{t,t+h}}{\widehat{RV}_{t,t+h}} \quad (26)$$

- Mincer-Zarnowitz R^2, and D-M test with 3 loss functions.

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Insanity Filter

- Any forecast is no smaller (larger) than the minimum (maximum) realization of the past. (Patteon and Sheppard (2015), Swanson and White (1997), Bollerslev et al (2018))

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Parameters Evolving

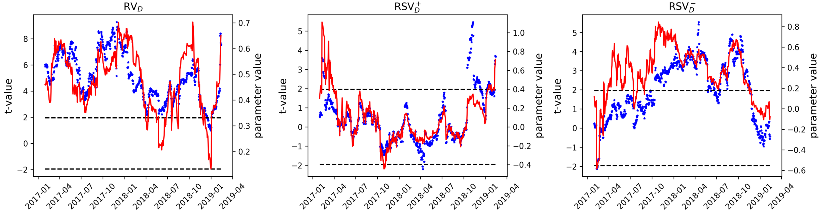

Figure 8: BTCD, parameters and *t*-values

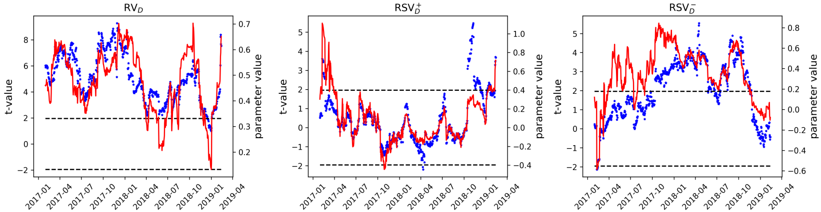

Figure 9: BTCG, parameters and *t*-values

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HAR Models

- HAR with jumps, **RVJ** and **RVTJ**

$$RV_{t,t+h} = \alpha + X_{RV}^{\top}\beta_{RV} + X_{J}^{\top}\beta_{J} + \varepsilon_{t,t+h} \quad (27)$$

$$RV_{t,t+h} = \alpha + X_{RV}^{\top}\beta_{RV} + X_{TJ}^{\top}\beta_{TJ} + \varepsilon_{t,t+h} \quad (28)$$

- HAR with signed jumps, **RVSJ** and **RVSTJ**

$$RV_{t,t+h} = \alpha + X_{RV}^{\top}\beta_{RV} + X_{J^+}^{\top}\beta_{J^+} + X_{J^-}^{\top}\beta_{J^-} + \varepsilon_{t,t+h} \quad (29)$$

$$RV_{t,t+h} = \alpha + X_{RV}^{\top}\beta_{RV} + X_{TJ^+}^{\top}\beta_{TJ^+} + X_{TJ^-}^{\top}\beta_{TJ^-} + \varepsilon_{t,t+h} \quad (30)$$

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HAR Models

- Realized Semi-Variance model, **RSV**

$$RV_{t,t+h} = \alpha + X_{RSV^+}^\top \beta_{RSV^+} + X_{RSV^-}^\top \beta_{RSV^-} + \varepsilon_{t,t+h} \quad (31)$$

- RSV with signed jumps, **RSVSJ** and **RSVSTJ**

$$\begin{aligned} RV_{t,t+h} &= \alpha + X_{RSV^+}^\top \beta_{RSV^+} + X_{RSV^-}^\top \beta_{RSV^-} \\ &\quad + X_{J^+}^\top \beta_{J^+} + X_{J^-}^\top \beta_{J^-} + \varepsilon_{t,t+h} \end{aligned} \quad (32)$$

$$\begin{aligned} RV_{t,t+h} &= \alpha + X_{RSV^+}^\top \beta_{RSV^+} + X_{RSV^-}^\top \beta_{RSV^-} \\ &\quad + X_{TJ^+}^\top \beta_{TJ^+} + X_{TJ^-}^\top \beta_{TJ^-} + \varepsilon_{t,t+h} \end{aligned} \quad (33)$$

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Out-of-Sample, BTCD, Log Form

	HAR	RVJ	RVTJ	RVSJ	RVSTJ	RSV	RSVSJ	RSVSTJ
	h=1							
MZ-R^2	0.261	0.221	0.218	0.238	0.249	0.266	0.243	0.235
MSE	2.217	2.329	2.353	2.284	2.260	2.209	2.262	2.298
HRMSE	0.979	0.986	1.004	1.167	1.118	0.985	1.161	1.143
QLIKE	0.906	$0.998^{\dagger}$	1.017	$1.056^{\dagger}$	$1.074^{\dagger}$	0.945	$1.112^{\dagger}$	$1.170^{\dagger}$
		h=7						
MZ-R^2	0.329	0.281	0.277	0.344	0.377	0.352	0.352	0.406
MSE	0.949	1.014	1.016	0.968	0.912	0.955	1.082	0.931
HRMSE	0.908	0.906	$0.989^{\dagger}$	0.865	0.862	0.888	1.022	0.971
QLIKE	0.906	0.909	$1.001^{\dagger}$	1.000	$1.009^{\dagger}$	0.929	0.945	0.978
		h=30						
MZ-R^2	0.591	0.626	0.635	0.651	0.671	0.632	0.678	0.708
MSE	0.348	0.303*	0.305*	0.288*	0.276*	0.307*	0.267*	0.247*
HRMSE	0.582	0.514*	0.549*	0.481*	0.461*	0.509*	0.397*	0.384*
QLIKE	0.773	0.731*	0.746*	0.726*	0.736*	0.744*	0.715*	0.725*

Table 4: Consistent results from BTCG in square and log forms Back

Realized Utility Framework

- Investor: Mean-variance preference with constant sharp ratio on time-varying volatility asset(Bollerslev et al (2018))
- Expected utility function approximation with assuming $W_{t+1} \sim \mathrm{N}(\mu_t, \sigma_t^2)$, $\gamma^A = -\frac{u''}{u'}$ as Pratt-Arrow absolute risk aversion function

$$\mathrm{E}[u(W_{t+1})] = \mu_t - \frac{1}{2}\gamma^A \sigma_t^2 \tag{34}$$

- Investment: ω_t on Cryptocurrencies and $1 - \omega_t$ on risk-free asset at time t

$$W_{t+1} = W_t(1 + r_f + \omega_t r_{t+1}) \tag{35}$$

Where $r_{t+1} - r_f$ as excess return at time $t + 1$ and risk free return

Volatility Timing Strategy

- Under assumption of the known constant Sharp ratio $SR = \frac{\mathsf{E}(r_{t+1})}{\sqrt{\mathsf{E}(RV_{t+1})}}$, rewriting expected utility $\mathsf{EU}(\omega_t)$ by replacing $\mathsf{V}(r_{t+1})$ with RV_{t+1}

$$\mathsf{EU}(\omega_t) = W_t \left[\omega_t \mathsf{E}(r_{t+1}) + \frac{\gamma}{2}\omega_t^2 \mathsf{V}(r_{t+1})\right] \tag{36}$$

$$= W_t \left[\omega_t \mathsf{E}(r_{t+1}) + \frac{\gamma}{2}\omega_t^2 RV_{t+1}\right] \tag{37}$$

$$= W_t \left[\omega_t SR \cdot \sqrt{RV_{t+1}} + \frac{\gamma}{2}\omega_t^2 RV_{t+1}\right] \tag{38}$$

Where $\gamma = \gamma^A W_t$ as relative risk aversion

- Optimal weight ω_t^* targeting SR/γ: Volatility timing strategy

$$\omega_t^* = \frac{SR/\gamma}{\sqrt{RV_{t+1}}} \tag{39}$$

Evaluating RV Forecasting

- Optimal expected utility function

$$\mathrm{EU}(\omega_t^*) = \frac{SR^2}{2\gamma} W_t \tag{40}$$

- For estimated $\widehat{RV}_{t+1}$ and corresponding optimal $\hat{\omega}_t$, the expected utility per wealth

$$\frac{\mathrm{EU}(\hat{\omega}_t)}{W_t} = \frac{SR^2}{\gamma}\left(\sqrt{\frac{RV_{t+1}}{\widehat{RV}_{t+1}}} - \frac{1}{2}\frac{RV_{t+1}}{\widehat{RV}_{t+1}}\right) \tag{41}$$

- Realized utility (RU): Averaging the realized expression by out-of-sample forecast $\widehat{RV}$

$$RU\left(\widehat{RV}_{t+1}\right) = \frac{SR^2}{\gamma}\frac{1}{T}\sum_{t=1}^{T}\left(\sqrt{\frac{RV_{t+1}}{\widehat{RV}_{t+1}}} - \frac{1}{2}\frac{RV_{t+1}}{\widehat{RV}_{t+1}}\right) \tag{42}$$

Economic Values, BTCG

	HAR	RVJ	RVTJ	RVSJ	RVSTJ	RSV	RSVSJ	RSVSTJ
Square form								
h=1	2.650	1.364	1.330	-0.244	0.560	2.041	-0.421	-1.843
h=7	3.164	2.271	1.862	0.569	1.294	2.906	1.090	1.100
h=30	3.585	3.687	3.637	3.669	3.597	3.689	3.697	3.621
Logarithmic form								
h=1	2.098	1.871	1.728	1.663	1.584	2.019	1.513	1.393
h=7	2.793	2.805	2.549	2.496	2.466	2.710	2.699	2.631
h=30	3.576	3.668	3.633	3.672	3.647	3.635	3.690	3.662

Table 5: Economic Values Evaluation of Out-of-Sample Forecasts (%)

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Out-of-Sample Forecast, *RV* form

	HAR	HAR-CJ	HAR-TCJs	HAR-Exp-CJs	HAR-Exp-TCJs
α	0.122	0.266	0.221	0.237	0.203
	(1.341)	(3.040)	(2.083)	(2.590)	(1.899)
β_D	0.242	0.283	0.268	0.528	0.579
	(3.329)	(3.125)	(2.528)	(4.286)	(3.634)
β_W	0.104	0.079	-0.097	0.759	1.229
	(1.127)	(0.675)	(-0.43)1	(1.971)	(1.999)
β_M	0.533	0.604	1.023	-0.302	-0.614
	(2.054)	(1.822)	(1.983)	(-1.153)	(-1.352)
β_{JD}		0.002	0.184	-0.038	0.167
		(0.010)	(1.282)	(-0.232)	(1.226)
β_{JW}		-0.004	0.326	-0.052	0.212
		(-0.026)	(1.502)	(-0.276)	(1.252)
β_{JM}		-0.704	-0.372	-0.471	-0.183
		(-1.749)	(-1.788)	(-1.588)	(-1.119)
R^2	0.389	0.394	0.368	0.404	0.390
MSE	1.597	1.644	1.739	1.623	1.671
QLIKE	1.459	1.673	1.686	1.571	1.591
HRMSE	1.288	1.502	1.468	1.474	1.459

Table 6: *RV*, t-values in the parantheses