Localized Conditional Autoregressive Expectile Model

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Motivation

- Time-varying conditional autoregressive expectile (CARE) parameters ▸ Parameter Dynamics
- Practice: time-homogeneous parameters
- Regime shifts, structural breaks

Objectives

(i) Localizing CARE Models

- Local parametric approach (LPA)
- Balance between modelling bias and parameter variability

(ii) Tail Risk Dynamics

- Estimation windows with potentially varying lengths
- Time-varying expectile parameters

Statistics and Risk Management

Statistics

- Modelling bias vs. parameter variability
- Flexible framework

Risk Management

- Parameter dynamics and structural changes
- Measuring tail risk

Example

Risk Management

An investor observes 250 index returns and estimates the underlying risk exposure. Tasks:

(a) Statistical properties of the CARE parameter estimates

(b) Estimate the 1% expectile level

Outline

1. Motivation ✓
2. Conditional Autoregressive Expectile (CARE)
3. Local Parametric Approach (LPA)
4. Empirical Results
5. Conclusions

Conditional Autoregressive Expectile

- Taylor (2008), Kuan et al. (2009)
- CARE specification, $\mathcal{F}_t$ - information set up to t

$$y_t = e_{t,\tau} + \varepsilon_{t,\tau} \qquad \mathsf{E}\left[\varepsilon_{t,\tau} \,|\mathcal{F}_{t-1}\right] = 0, \mathsf{Var}\left(\varepsilon_{t,\tau} \,|\mathcal{F}_{t-1}\right) = \sigma^2_{\varepsilon,\tau}$$

AND

$$e_{t,\tau} = \alpha_{0,\tau} + \alpha_{1,\tau} y_{t-1} + \alpha_{2,\tau}\left(y^+_{t-1}\right)^2 + \alpha_{3,\tau}\left(y^-_{t-1}\right)^2$$

- Expectile $e_{t,\tau}$ at $\tau \in (0,1)$, $\theta_\tau = \left\{\alpha_{0,\tau}, \alpha_{1,\tau}, \alpha_{2,\tau}, \alpha_{3,\tau}, \sigma^2_{\varepsilon,\tau}\right\}^\top$
- Returns: $y^+_{t-1} = \max\{y_{t-1}, 0\}$, $y^-_{t-1} = \min\{y_{t-1}, 0\}$

Parameter Estimation

- Data calibration with time-varying intervals
- Observed returns $\mathcal{Y} = \{y_1, \dots, y_n\}$
- Quasi maximum likelihood estimate (QMLE)

$$\widetilde{\theta}_{I,\tau} = \arg \max_{\theta_\tau \in \Theta} \ell_I (\mathcal{Y}; \theta_\tau)$$

  - $I = [t_0 - m, t_0]$ - interval of $(m+1)$ observations at t_0
  - $\ell_I (\cdot)$ - quasi log likelihood Details

Estimation Quality

- Mercurio and Spokoiny (2004), Spokoiny (2009)
- Quality of estimating *true* parameter vector θ^*_τ by QMLE $\widetilde{\theta}_{I,\tau}$ in terms of Kullback-Leibler divergence; $\mathcal{R}_r(\theta^*_\tau)$ - risk bound

$$\mathsf{E}_{\theta^*_\tau} \left| \ell_I(\mathcal{Y}; \widetilde{\theta}_{I,\tau}) - \ell_I(\mathcal{Y}; \theta^*_\tau) \right|^r \leq \mathcal{R}_r(\theta^*_\tau)$$

▸ Gaussian Regression

- 'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)
- 'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

Solomon Kullback and Richard A. Leibler on BBI: 

Local Parametric Approach (LPA)

- LPA, Spokoiny (1998, 2009)
  - Time series parameters can be locally approximated
  - Finding the (longest) *interval of homogeneity*
  - Balance between modelling bias and parameter variability

- Time series literature
  - GARCH$(1, 1)$ models - Čížek et al. (2009)
  - Realized volatility - Chen et al. (2010)
  - Multiplicative Error Models - Härdle et al. (2014)

Interval Selection

- $(K+1)$ nested intervals with length $n_k = |I_k|$

$$\begin{matrix} I_0 & \subset & I_1 & \subset \cdots \subset & I_k & \subset \cdots \subset & I_K \\ \widetilde{\boldsymbol{\theta}}_0 & & \widetilde{\boldsymbol{\theta}}_1 & & \widetilde{\boldsymbol{\theta}}_k & & \widetilde{\boldsymbol{\theta}}_K \end{matrix}$$

Example: Daily index returns

Fix i_0, $I_k = [i_0 - n_k, i_0]$, $n_k = \left[n_0 c^k\right]$, $c > 1$

$\{n_k\}_{k=0}^{11} = \{20 \text{ days}, 25 \text{ days}, \ldots, 250 \text{ days}\}$, $c = 1.25$

Local Change Point Detection

- Fix t_0, sequential test $(k = 1, \ldots, K)$

H_0 : parameter homogeneity within I_k vs. $H_1 : \exists$ change point within J_k

$$T_{k,\tau} = \sup_{s \in J_k} \left\{ \ell_{A_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{A_{k,s},\tau} \right) + \ell_{B_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{B_{k,s},\tau} \right) - \ell_{I_{k+1}} \left(\mathcal{Y}, \widetilde{\theta}_{I_{k+1},\tau} \right) \right\},$$

with $J_k = I_k \setminus I_{k-1}$, $A_{k,s} = [t_0 - n_{k+1}, s]$ and $B_{k,s} = (s, t_0]$

Data

- Series
  - DAX and FTSE100 returns, 20040102-20141231 (2869 days)
  - Research Data Center (RDC) - Datastream
- Setup
  - Expectile levels: $\tau = 0.05$ and $\tau = 0.01$
  - Interval lengths: 20, 60, 125 and 250 trading days

Parameter Dynamics ▸ Motivation

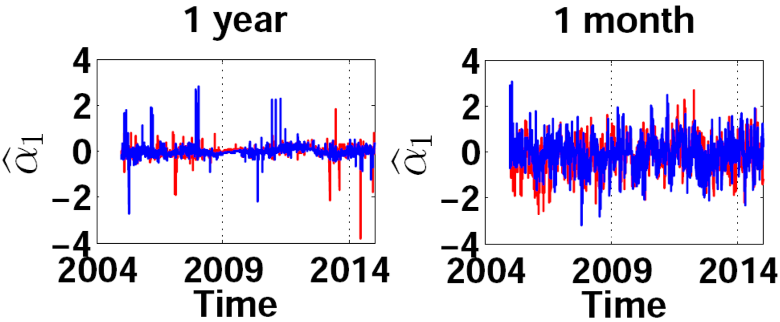

Figure 1: Estimated $\alpha_{1,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters

Parameter Dynamics ▸ Motivation

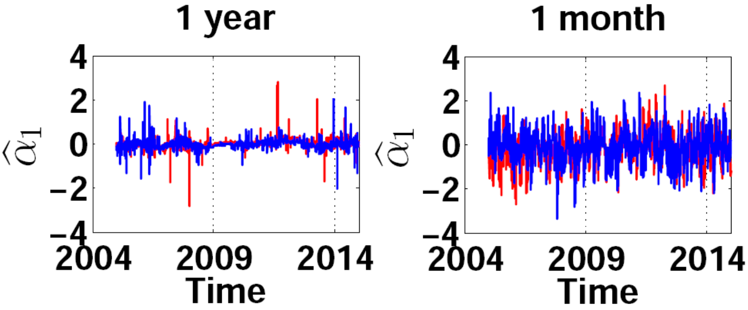

Figure 2: Estimated $\alpha_{1,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters

Parameter Distributions

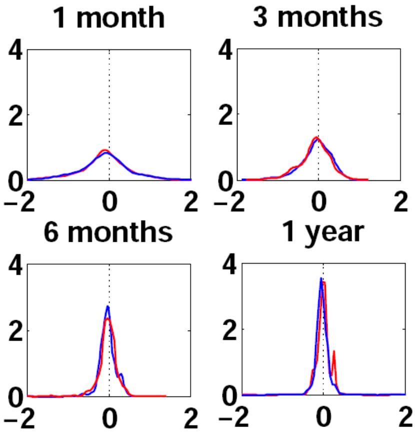

Figure 3: Kernel density estimates of $\alpha_{1,0.05}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Parameter Distributions

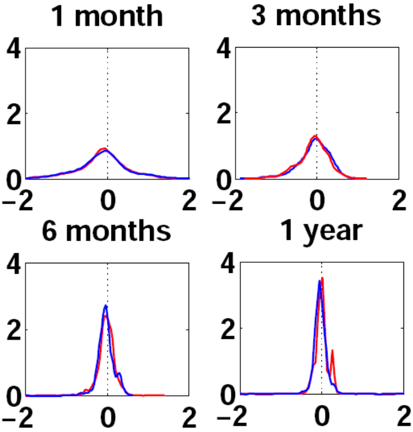

Figure 4: Kernel density estimates of $\alpha_{1,0.01}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Expectile Dynamics

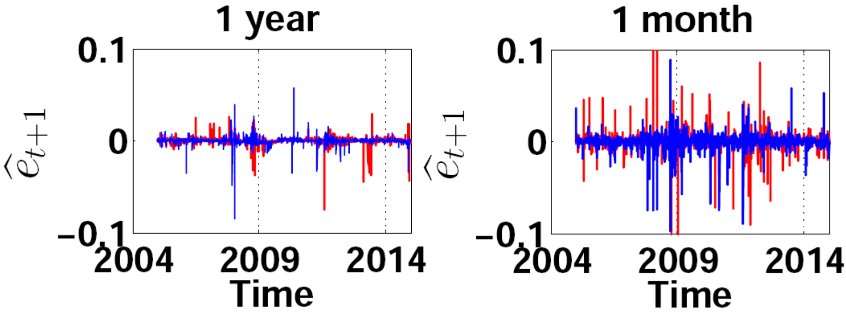

Figure 5: Estimated $\alpha_{1,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Expectile Dynamics

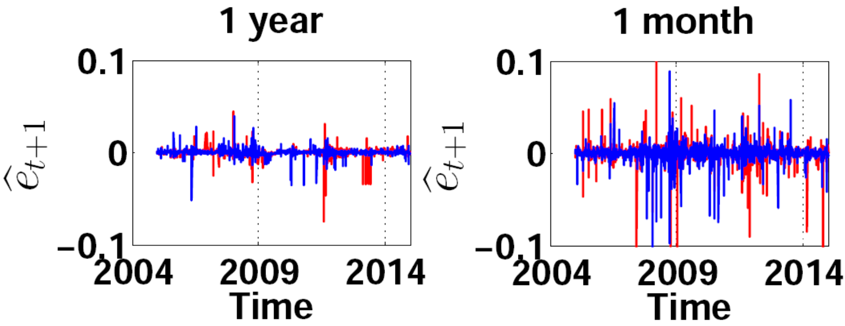

Figure 6: Estimated $\alpha_{1,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Conclusions

(i) Localizing CARE Models
- Balance between modelling bias and parameter variability
- Parameter dynamics

(ii) Tail Risk Dynamics
- Varying distributional characteristics
- Expectile levels $\tau = 0.05$ and $\tau = 0.01$

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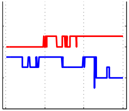

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The Annals of Statistics **37**(3): 1405–1436, 2009

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Journal of Financial Econometrics **6**(2): 231–252, 2008

Asymmetric Normal Distribution ▸ CARE

- If $\varepsilon_\tau \sim \text{AND}\left(\mu, \omega_\varepsilon^2, \tau\right)$ with pdf

$$f_\varepsilon(u) = \frac{1}{\omega_\varepsilon}\sqrt{\frac{2}{\pi}} \exp\left\{-\frac{(u-\mu)^2}{2\omega_\varepsilon^2}\right\} \Phi\left(\tau \frac{u-\mu}{\omega_\varepsilon}\right)$$

  - $\mathsf{E}\left[\varepsilon_\tau\right] = 0$ when $\mu = -\frac{\tau\omega_\varepsilon}{\sqrt{1+\tau^2}}\sqrt{\frac{2}{\pi}}$
  - $\mathsf{Var}\left(\varepsilon_\tau\right) = \sigma_{\varepsilon,\tau}^2 = \omega_\varepsilon^2\left\{1 - \frac{2}{\pi}\left(\frac{\tau}{\sqrt{1+\tau^2}}\right)^2\right\}$

Quasi Log Likelihood Function

▸ Parameter Estimation

- If $\varepsilon_\tau \sim \text{AND}\left(\mu, \omega_\varepsilon^2, \tau\right)$ with pdf $f_\varepsilon(\cdot)$ then $y \sim \text{AND}\left(e_\tau + \mu, \omega_\varepsilon^2, \tau\right)$
- Quasi log likelihood function for observed data $\mathcal{Y} = \{y_1, \ldots, y_n\}$ over a fixed interval I

$$\ell_I(\mathcal{Y}; \theta_\tau) = \sum_{t \in I} \log f_\varepsilon(y_t - e_{t,\tau})$$

Gaussian Regression

▸ Estimation Quality

$$Y_i = f(X_i) + \varepsilon_i, i = 1, \ldots, n, \text{ weights } W = \{w_i\}_{i=1}^{n}$$

$$L(W, \theta) = \sum_{i=1}^{n} \ell\{Y_i, f_\theta(X_i)\} w_i, \text{ log-density } \ell(\cdot), \widetilde{\theta} = \arg\max_{\theta \in \Theta} L(W, \theta)$$

1. Local constant, $f(X_i) \approx \theta^*$, $\varepsilon_i \sim \mathrm{N}\left(0, \sigma^2\right)$

$$\mathrm{E}_{\theta^*} \left| L(W, \widetilde{\theta}) - L(W, \theta^*) \right|^r \leq \mathrm{E}\, |\xi|^{2r}, \quad \xi \sim \mathrm{N}(0, 1)$$

2. Local linear, $f(X_i) \approx \theta^{*\top} \Psi_i$, $\varepsilon_i \sim \mathrm{N}\left(0, \sigma^2\right)$, basis functions $\Psi = \{\psi_1(X_1), \ldots, \psi_p(X_p)\}$, multivariate ξ

$$\mathrm{E}_{\theta^*} \left| L(W, \widetilde{\theta}) - L(W, \theta^*) \right|^r \leq \mathrm{E}\, |\xi|^{2r}, \quad \xi \sim \mathrm{N}(0, \mathcal{I}_p)$$

Parameter Dynamics

▸ Parameter Dynamics

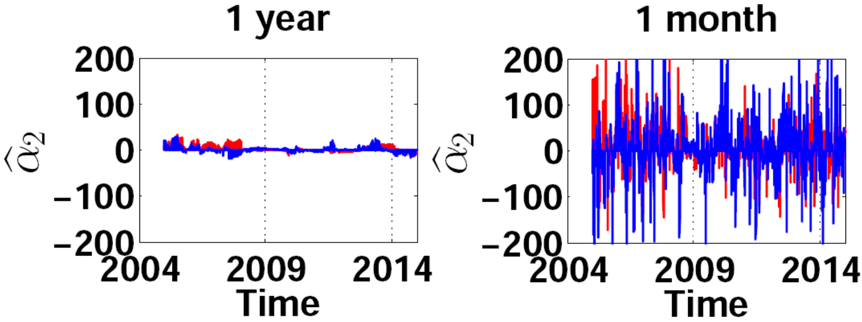

Figure 7: Estimated $\alpha_{2,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

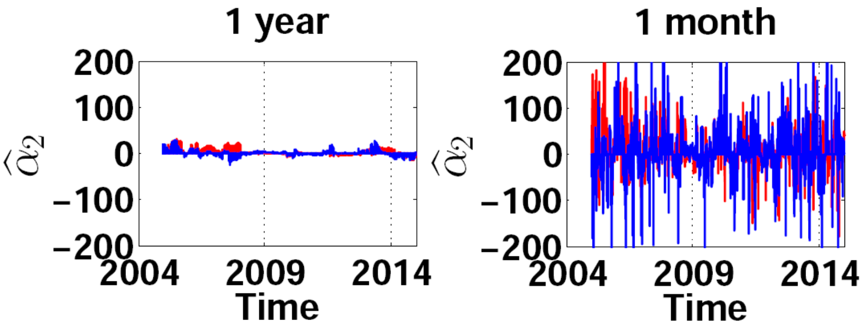

Figure 8: Estimated $\alpha_{2,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

Parameter Dynamics

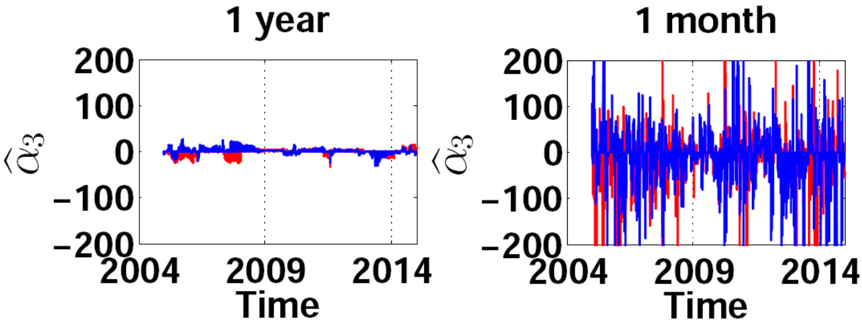

Figure 9: Estimated $\alpha_{3,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

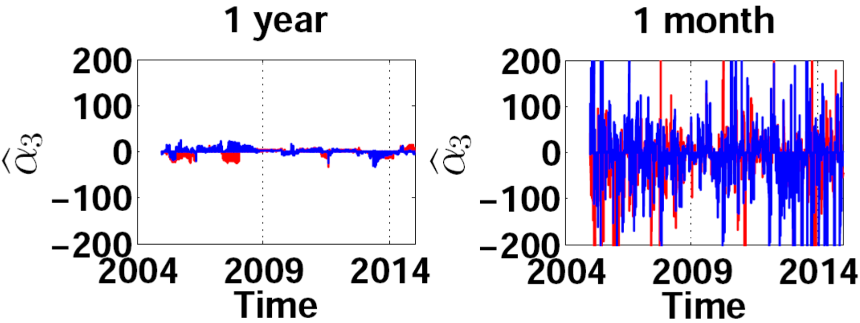

Figure 10: Estimated $\alpha_{3,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations