Adaptive Order Flow Forecasting with MEMs

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Trading Volume

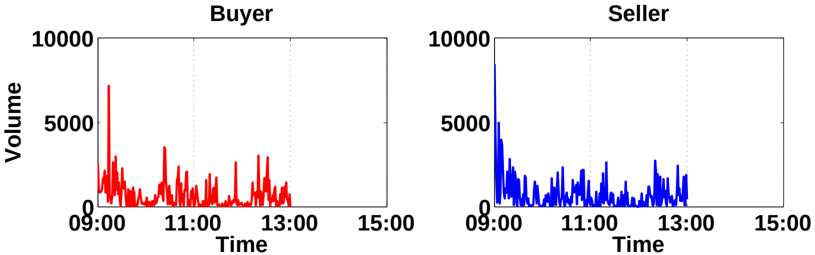


Figure 1: One-minute aggregated size of buyer-initiated and seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305 ▸ Volume (Description)

Trading Volume

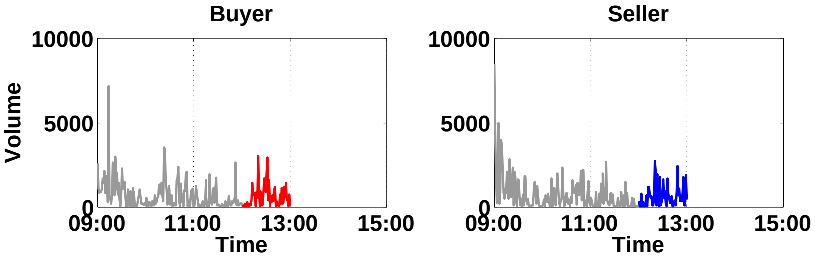


Figure 2: One-minute aggregated size of buyer-initiated and seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305 with an estimation window length of 60

Trading Volume

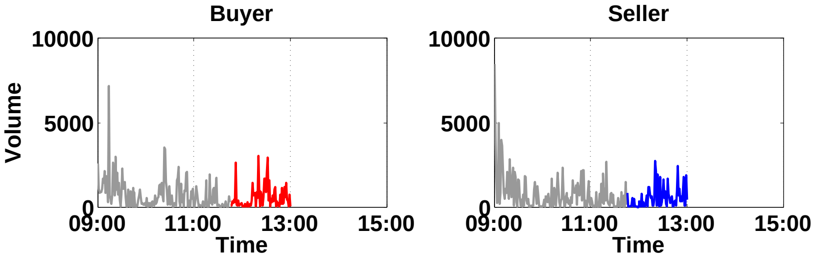


Figure 3: One-minute aggregated size of buyer-initiated and seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305 with an estimation window length of 75

Trading Volume

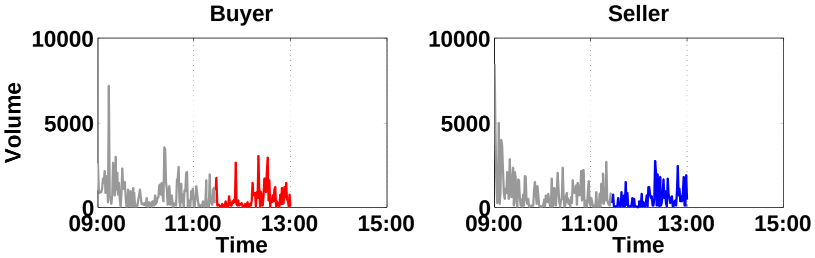


Figure 4: One-minute aggregated size of buyer-initiated and seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305 with an estimation window length of 94

Trading Volume

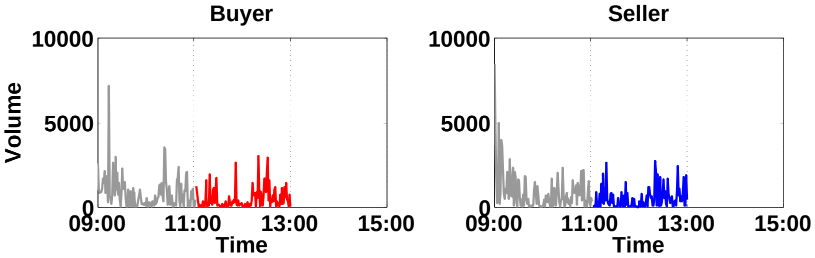


Figure 5: One-minute aggregated size of buyer-initiated and seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305 with an estimation window length of 118

Order Flow

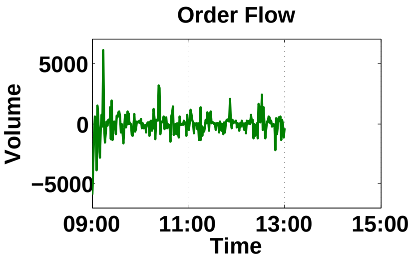


Figure 6: Order flow (difference between the buyer-initiated and the seller-initiated volume) for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange on 20130305

Example

A brokerage decides to trade contracts over a forecasted period (e.g., 5min.) during March 2013. A position (buy/sell) is entered ahead the forecasted period and closed afterwards.

Trading Strategy

- Calibration (February 2013)
  - Predict the order flow and execution costs at each period
  - Strategy (i): 'buy if predicted order flow is positive'
  - Strategy (ii): 'sell if predicted order flow is positive'
- Evaluation (March 2013): employ the best strategy found in February 2013

Objectives

- Short-Term Order Flow Forecasting
  - Local adaptive Multiplicative Error Model (MEM)
  - Balance between modelling bias and parameter variability
  - Estimation windows with potentially varying lengths
- Intra-Day Trading
  - Buyer- and seller-initiated trades and order flow dynamics
  - Performance based trading strategy (signal)

Statistics and Quantitative Finance

Statistics

- Modelling bias vs. parameter variability
- Flexible framework and predictive accuracy

Quantitative Finance Practice

- Trading strategies and equity curves
- Hardware: field-programmable gate array (FPGA)

Outline

1. Motivation ✓
2. Local Adaptive Multiplicative Error Model (MEM)
3. Modelling Trading Volumes
4. Order Flow Forecasting
5. Intra-Day Trading
6. Conclusions

Multiplicative Error Model (MEM)

- Engle (2002), MEM(p,q), $\mathcal{F}_i$ - information set up to i

$$y_i = \mu_i \varepsilon_i, \qquad \mathsf{E}\left[\varepsilon_i \,|\mathcal{F}_{i-1}\right] = 1$$

$$\mu_i = \omega + \sum_{j=1}^{p} \alpha_j y_{i-j} + \sum_{j=1}^{q} \beta_j \mu_{i-j}, \qquad \omega > 0, \alpha_j, \beta_j \geq 0$$

- Hautsch (2012) - comprehensive MEM literature overview
  - y_i - squared (de-meaned) log return: GARCH(p,q)
  - y_i - volume, bid-ask spread, duration: ACD(p,q)

Engle, Robert F. on BBI:

Autoregressive Conditional Duration (ACD)

1. Exponential-ACD, Engle and Russel (1998) EACD

   $\varepsilon_i \sim Exp(1)$, $\boldsymbol{\theta}_E = (\omega, \boldsymbol{\alpha}, \boldsymbol{\beta})^\top$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)$

2. Weibull-ACD, Engle and Russel (1998) WACD

   $\varepsilon_i \sim \mathcal{G}(s, 1)$, $\boldsymbol{\theta}_W = (\omega, \boldsymbol{\alpha}, \boldsymbol{\beta}, s)^\top$

Weibull, E. H. Waloddi on BBI: 

Parameter Estimation

- Consistent parameter estimation
- Data calibration with time-varying intervals
- Quasi maximum likelihood estimates (QMLEs) of $\boldsymbol{\theta}_E$ and $\boldsymbol{\theta}_W$

$$\widetilde{\boldsymbol{\theta}}_I = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L_I\left(y;\boldsymbol{\theta}\right) \tag{1}$$

  - $I = [i_0 - n, i_0]$ - interval of $(n+1)$ observations at i_0
  - $L_I\left(\cdot\right)$ - log likelihood, ‣ EACD and ‣ WACD

Estimation Quality

- Mercurio and Spokoiny (2004), Spokoiny (2009)
- Quality of estimating *true* parameter vector $\boldsymbol{\theta}^*$ by QMLE $\widetilde{\boldsymbol{\theta}}_I$ in terms of Kullback-Leibler divergence; $\mathcal{R}_r(\boldsymbol{\theta}^*)$ - risk bound

$$\mathsf{E}_{\boldsymbol{\theta}^*}\left|L_I(\widetilde{\boldsymbol{\theta}}_I) - L_I(\boldsymbol{\theta}^*)\right|^r \leq \mathcal{R}_r(\boldsymbol{\theta}^*)$$

Gaussian Regression

- Likelihood based confidence sets

- 'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)
- 'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

Kullback, Solomon and Leibler, Richard A. on BBI:

Local Parametric Approach (LPA)

- LPA, Spokoiny (1998, 2009)
  - Time series parameters can be locally approximated
  - Finding the (longest) *interval of homogeneity*
  - Balance between modelling bias and parameter variability

- Time series literature
  - GARCH$(1,1)$ models - Čížek et al. (2009)
  - Realized volatility - Chen et al. (2010)
  - Multiplicative Error Models - Härdle et al. (2014)

Interval Selection

- $(K+1)$ nested intervals with length $n_k = |I_k|$

$$
\begin{array}{ccccccc}
I_0 & \subset & I_1 & \subset \cdots \subset & I_k & \subset \cdots \subset & I_K \\
\widetilde{\boldsymbol{\theta}}_0 & & \widetilde{\boldsymbol{\theta}}_1 & & \widetilde{\boldsymbol{\theta}}_k & & \widetilde{\boldsymbol{\theta}}_K
\end{array}
$$

Example: Trading volumes aggregated over 1-min periods, Härdle et al. (2013)

Fix i_0, $I_k = [i_0 - n_k, i_0]$, $n_k = \left[n_0 c^k\right]$, $c > 1$

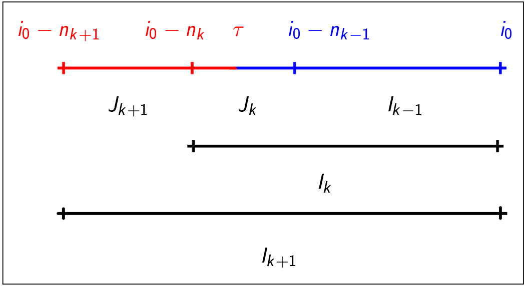
$\{n_k\}_{k=0}^{13} = \{60 \text{ min.}, 75 \text{ min.}, \dots, 1 \text{ week}\}$, $c = 1.25$

Local Change Point Detection

▸ Example

- Fix i_0, sequential test $(k = 1, \ldots, K)$

H_0 : parameter homogeneity within I_k vs. H_1 : $\exists$ change point within J_k



$$T_k = \sup_{\tau \in J_k} \left\{ L_{A_{k,\tau}}\left(\widetilde{\theta}_{A_{k,\tau}}\right) + L_{B_{k,\tau}}\left(\widetilde{\theta}_{B_{k,\tau}}\right) - L_{I_{k+1}}\left(\widetilde{\theta}_{I_{k+1}}\right) \right\},$$

with $J_k = I_k \setminus I_{k-1}$, $A_{k,\tau} = [i_0 - n_{k+1}, \tau]$ and $B_{k,\tau} = (\tau, i_0]$

Critical Values, $\mathfrak{z}_k$

Critical Values

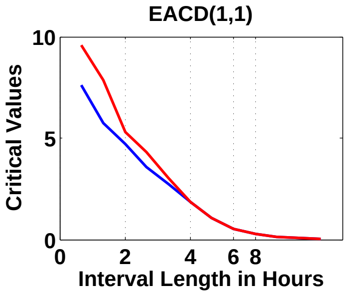


Figure 7: Critical values for low $(\widetilde{\alpha}+\widetilde{\beta}=0.84)$ and high $(\widetilde{\alpha}+\widetilde{\beta}=0.93)$ weekly persistence and 'modest' risk $(r=0.5)$ with $\rho=0.25$, Härdle et al. (2013)

Adaptive Estimation

- Compare T_k at every step k with $\mathfrak{z}_k$
- Data window index of the *interval of homogeneity* - $\widehat{k}$
- Adaptive estimate

$$\widehat{\boldsymbol{\theta}} = \widetilde{\boldsymbol{\theta}}_{\widehat{k}}, \quad \widehat{k} = \max_{k \le K} \{k : T_\ell \le \mathfrak{z}_\ell, \ell \le k\}$$

- Note: rejecting the null at $k = 1$, $\widehat{\theta}$ equals QMLE at I_0
  If the algorithm goes until K, $\widehat{\theta}$ equals QMLE at I_K

Data

- Osaka Securities Exchange
  - Series: mini Nikkei 225 index futures
  - NO12U 20120628-20120913, NO12Z 20120914-20121213, NO13H 20121214-20130307, NO13M 20130308-20130524

- Span: 20120628 - 20130524
  - Data on 20130304 removed as trading stopped between 11:06-14:10
  - 218 trading days (78480 minutes), 09:01-15:00

Notation

- One-minute cumulated volumes at day d and minute i
  - Buyer-initiated volume $\breve{y}^b_{i,d}$
  - Seller-initiated volume $\breve{y}^s_{i,d}$
- Prices: ask $S^a_{d,i}$, bid $S^b_{d,i}$, transaction $S_{d,i}$ at beginning of minute i
- Seasonally adjusted volume ▸ Periodicity component $s_{i,d-1}$
  - Buyer-initiated volume, $y^b_{i,d} = \breve{y}^b_{i,d} / s^b_{i,d-1}$
  - Seller-initiated volume, $y^s_{i,d} = \breve{y}^s_{i,d} / s^s_{i,d-1}$
  - $d = 31, \ldots, 218$ $(20120813 - 20130524)$, $i = 1, \ldots, 360$

Intraday Periodicity

Figure 8: Estimated intraday periodicity factors for the buyer-initiated and the seller-initiated trades for the 'mini Nikkei 225 index futures' traded at the Osaka Securities Exchange from 20120813-20130524

Adaptive Estimation

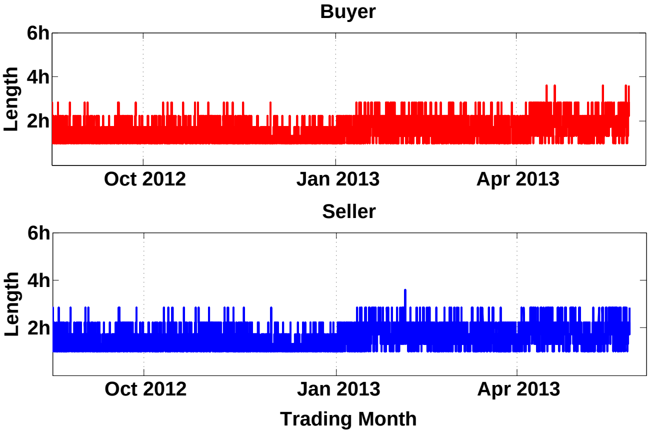


Figure 9: Estimated length of *intervals of homogeneity* for the buyer-initiated and the seller-initiated trades from 20120820-20130524 using the EACD(1,1) model with $r = 0.5$ and $\rho = 0.25$

Adaptive Estimation

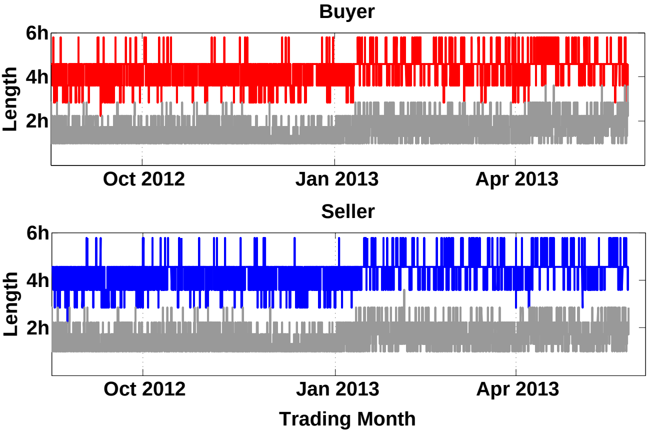


Figure 10: Estimated length of *intervals of homogeneity* for the buyer-initiated and the seller-initiated trades from 20120820-20130524 using the EACD(1,1) model with $r = 1$ and $\rho = 0.25$

Forecasting Setup

- Forecasting period: 20120820 - 20130523 (65520 minutes)
- Forecasts at each minute, horizon $h = 1, \ldots, 60$ min.
  - Computed recursively
  - Multiplied by the seasonality component associated with the previous 30 days
- Local MEM: EACD$(1,1)$, $r \in \{0.5, 1\}$, $\rho \in \{0.25, 0.5\}$

Forecasting Performance

	RMSE	RMSPE
Buyer-initiated		
$r = 0.50$	1220	1787
$r = 1.00$	871	1065
Seller-initiated		
$r = 0.50$	1195	1766
$r = 1.00$	878	1103

Table 1: Overall RMSEs and RMSPEs for the buyer-initiated and the seller-initiated volume from 20120820-20130523 using the EACD$(1,1)$ model with $\rho = 0.25$

Forecasting Performance

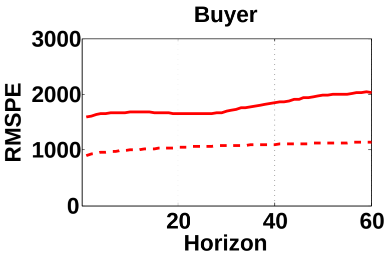


Figure 11: RMSPEs for the buyer-initiated and the seller-initiated from 20120820-20130523 across all forecasting horizons $h = 1, \ldots, 60$ using the EACD(1,1) model with $r = 0.5$ (solid), $r = 1$ (dashed). We set $\rho = 0.25$.

Forecasting Performance

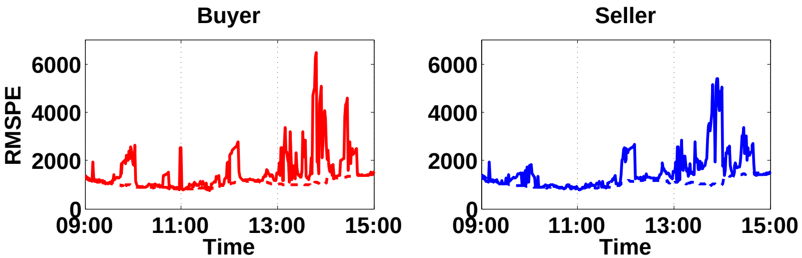


Figure 12: RMSPEs for the buyer-initiated and the seller-initiated from 20120820-20130523 across all trading minutes $i = 1, \ldots, 360$ using the EACD(1,1) model with $r = 0.5$ (solid), $r = 1$ (dashed). We set $\rho = 0.25$.

Intra-Day Trading

- Setup
  - Forecasted period: 5 minutes; from 20120903-20130430
  - Predict the order flow using the local MEM: EACD(1, 1), $r = 1$ and $\rho = 0.25$
  - 'Execute' orders and evaluate trading costs
- Trading strategies
  - Strategy (i): 'buy if predicted order flow is positive'
  - Strategy (ii): 'sell if predicted order flow is positive'
  - In the current month employ the strategy with the lowest costs during the previous month

Trading Costs

	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
9:01, 9:06, ...	0.96	1.02*	0.93	0.94	1.28*	1.13*	1.07*	0.89
9:02, 9:07, ...	0.95	1.05*	0.92	0.86	1.21*	0.98	0.96	0.94
9:03, 9:08, ...	0.90	0.99	0.97	0.80	0.99	1.10*	1.05*	1.00
9:04, 9:09, ...	0.88	1.02*	0.94	0.76	1.14*	1.29*	1.01*	1.03*
9:05, 9:10, ...	0.89	0.98	1.06*	0.90	1.24*	1.04*	1.00*	0.91

Table 2: Costs of executing strategy (i) relative to costs of strategy (ii) from 20120903 to 20130430. Superiority of strategy (i) is denoted by an asterisk (*). The costs are calculated based on trades at bid and ask prices.

Trading Profit

	Oct	Nov	Dec	Jan	Feb	Mar	Apr
9:01, 9:06, ...	-41	39	-22	55	-9	-17	64
9:02, 9:07, ...	7	48	-72	47	47	-33	-27
9:03, 9:08, ...	-46	-49	-89	-24	13	10	27
9:04, 9:09, ...	-16	35	-114	18	-70	14	13
9:05, 9:10, ...	-31	4	70	65	6	29	39

Table 3: Monthly profit from executing previous month's best strategy in thousands of JPY per one contract from 20121001 to 20130430. The profits are calculated based on trades at transaction prices.

Trading Profit

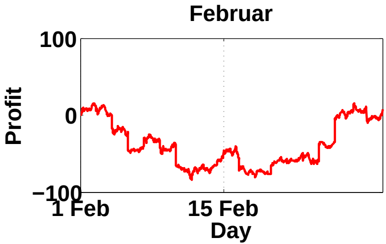


Figure 13: Equity curves in thousands of JPY per one contract from 20130201-20130228 and 20130301-20130329 using the strategy (i). The profits are calculated based on trades at transaction prices at 5 minute intervals (09:01, 09:06, . . .).

Trading Profit

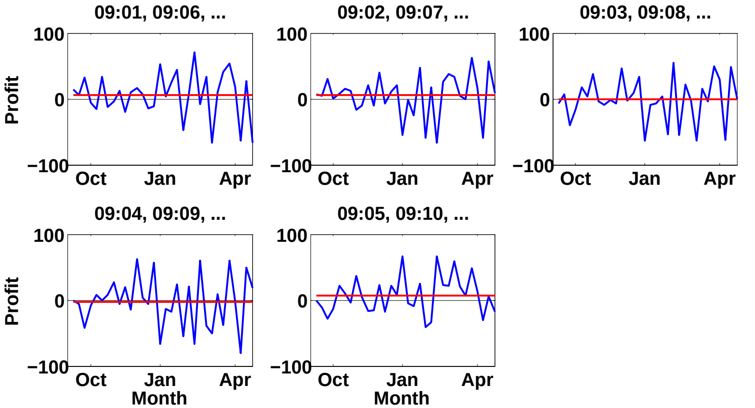


Figure 14: Equity curves in thousands of JPY per one contract from 20120903-20130430 employing the strategy with lowest costs during the past week; based on trades at transaction prices on a weekly basis.

Trading Profit

	Monthly			Weekly		
	Mean	St.Dev	t	Mean	St.Dev	t
9:01, 9:06, ...	10.0	41.7	0.63	5.9	33.9	0.98
9:02, 9:07, ...	2.4	47.6	0.13	5.4	32.1	0.95
9:03, 9:08, ...	-22.6	41.6	-1.44	-0.9	33.4	-0.16
9:04, 9:09, ...	-17.2	54.7	-0.83	-2.6	38.5	-0.38
9:05, 9:10, ...	25.9	35.9	1.90	7.4	28.0	1.50

Table 4: Mean, standard deviation (St.Dev) and the test statistics t (null hypothesis: average profit equals zero) of the profit in thousands of JPY per one contract from 20120903-20130430 given a monthly and a weekly evaluation scheme relative to the daily execution time points

Conclusions

Short-Term Order Flow Forecasting

- Order flow predicted successfully using the local adaptive MEM
- 'Conservative' adaptive estimation ($r = 1$) outperforms modest risk approach ($r = 0.5$) and requires 3-6 hours of data

Intra-Day Trading

- Introduced strategies share similar performance
- Trading profit achieved

Adaptive Order Flow Forecasting with MEMs

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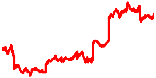

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Volume Description

▸ Trading Volume

- Buyer-initiated volume at minute i
  - Consider market orders only
  - Number of contracts that have been bought during minute i (at the ask price)
- Seller-initiated volume at minute i
  - Consider market orders only
  - Number of contracts that have been sold during minute i (at the bid price)

Exponential-ACD (EACD)

ACD Parameter Estimation

- Engle and Russel (1998), $\varepsilon_i \sim Exp(1)$

$$L_I(y;\boldsymbol{\theta_E}) = \sum_{i=max(p,q)+1}^{n} \left(-\log \mu_i - \frac{y_i}{\mu_i}\right) \mathbf{I}\{i \in I\} \qquad (2)$$

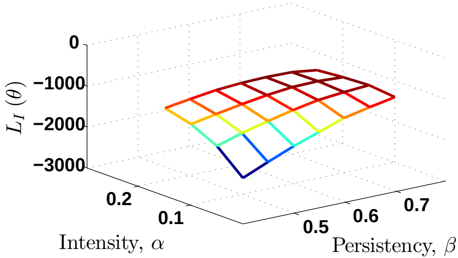

Figure 15: Log likelihood - EACD(1,1), $\boldsymbol{\theta_E^*} = (0.10, 0.20, 0.65)^\top$

Weibull-ACD (WACD)

▸ ACD ▸ Parameter Estimation

- Engle and Russel (1998), $\varepsilon_i \sim \mathcal{G}(s, 1)$

$$L_I(y; \boldsymbol{\theta}_W) = \sum_{i \in I} \left[\log \frac{s}{y_i} + s \log \frac{\Gamma(1 + 1/s)\, y_i}{\mu_i} - \left\{ \frac{\Gamma(1 + 1/s)\, y_i}{\mu_i} \right\}^s \right] \mathbf{I}\{i \in I\} \tag{3}$$

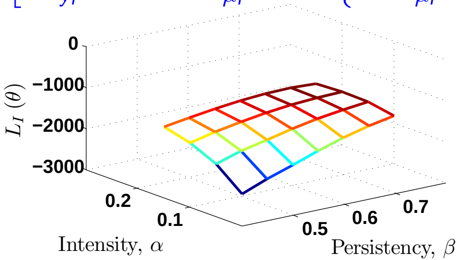

Figure 16: Log likelihood - WACD(1,1), $\boldsymbol{\theta}^*_W = (0.10, 0.20, 0.65, 0.85)^\top$

Gaussian Regression

Estimation Quality

$Y_i = f(X_i) + \varepsilon_i, i = 1, \ldots, n$, weights $W = \{w_i\}_{i=1}^n$

$L(W, \theta) = \sum_{i=1}^{n} \ell\{Y_i, f_\theta(X_i)\} w_i$, log-density $\ell(\cdot)$, $\widetilde{\theta} = \arg\max_{\theta \in \Theta} L(W, \theta)$

1. Local constant, $f(X_i) \approx \theta^*$, $\varepsilon_i \sim \mathrm{N}(\theta^*, \sigma^2)$

$$\mathrm{E}_{\theta^*} \left| L(W, \widetilde{\theta}) - L(W, \theta^*) \right|^r \leq \mathrm{E} |\xi|^{2r}, \quad \xi \sim \mathrm{N}(0, 1)$$

2. Local linear, $f(X_i) \approx \theta^{*\top} \Psi_i$, $\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$, basis functions $\Psi = \{\psi_1(X_1), \ldots, \psi_p(X_p)\}$, multivariate ξ

$$\mathrm{E}_{\theta^*} \left| L(W, \widetilde{\theta}) - L(W, \theta^*) \right|^r \leq \mathrm{E} |\xi|^{2r}, \quad \xi \sim \mathrm{N}(0, \mathcal{I}_p)$$

Local Change Point Detection

▸ LCP

Example: Trading volumes aggregated over 1-min periods

- Scheme with $(K+1) = 14$ intervals and fix i_0
- Assume $I_0 = 60$min. is homogeneous

- H_0 : parameter homogeneity within $I_1 = 75$min.
  - Define $J_1 = I_1 \setminus I_0$ - observations from y_{i_0-75} up to y_{i_0-60}
  - For each $\tau \in J_1$ fit log likelihoods over $A_{1,\tau}$, $B_{1,\tau}$ and I_2
  - Find the largest likelihood ratio - T_{I_1,J_1}

Critical Values, $\mathfrak{z}_k$ ▸ Critical Values

- Simulate $\mathfrak{z}_k$ - homogeneity of the interval sequence $I_0, \ldots, I_k$
- 'Propagation' condition (under H_0)

$$\mathrm{E}_{\theta^*} \left| L_{I_k}(\widetilde{\theta}_k) - L_{I_k}(\widehat{\theta}_k) \right|^r \leq \rho_k \mathcal{R}_r(\theta^*), \quad k = 1, \ldots, K \qquad (4)$$

$\rho_k = \rho k / K$ for given significance level ρ, ▸ $\widehat{\theta}_k$ - adaptive estimate

- Check $\mathfrak{z}_k$ for (nine) different θ^* ▸ Parameter Dynamics - Quartiles
  - EACD and WACD, $K \in \{8, 13\}$, $r \in \{0.5, 1\}$, $\rho \in \{0.25, 0.50\}$
  - Findings: $\mathfrak{z}_k$ are virtually invariable w.r.t. θ^* given a scenario
    Largest differences at first two or three steps

Parameter Dynamics

Estimation window	EACD(1,1)			WACD(1,1)		
	Q25	Q50	Q75	Q25	Q50	Q75
1 week	0.85	0.89	0.93	0.82	0.88	0.92
2 days	0.77	0.86	0.92	0.74	0.84	0.91
1 day	0.68	0.82	0.90	0.63	0.79	0.89
3 hours	0.54	0.75	0.88	0.50	0.72	0.87
2 hours	0.45	0.70	0.86	0.42	0.67	0.85
1 hour	0.33	0.58	0.80	0.31	0.57	0.80

Table 5: Quartiles of estimated persistence levels $\left(\widetilde{\alpha}+\widetilde{\beta}\right)$ for all five stocks at each minute from 20080222-20081231 (215 trading days). Calibration period: 20080102 - 20080221, Härdle et al. (2013)

Parameter Dynamics

Critical Values

Model	Low Persistence			Moderate Persistence			High Persistence		
	Q25	Q50	Q75	Q25	Q50	Q75	Q25	Q50	Q75
EACD, $\widetilde{\alpha}$	0.28	0.22	0.18	0.30	0.23	0.19	0.31	0.24	0.20
EACD, $\widetilde{\beta}$	0.56	0.62	0.67	0.59	0.66	0.71	0.62	0.68	0.73
WACD, $\widetilde{\alpha}$	0.28	0.21	0.17	0.30	0.23	0.18	0.32	0.24	0.19
WACD, $\widetilde{\beta}$	0.54	0.60	0.65	0.58	0.65	0.70	0.60	0.68	0.74

Table 6: Quartiles of 774,000 estimated ratios $\widetilde{\beta}/\left(\widetilde{\alpha}+\widetilde{\beta}\right)$ (estimation windows covering 1800 observations) from 20080222-20081231 conditional on the persistence level: low {EACD (0.85), WACD (0.82)}, moderate {EACD (0.89), WACD (0.88)} or high {EACD (0.93), WACD (0.92)}. 774,000 ratios = 215 days $\times$ 360 minutes/day $\times$ 5 stocks $\times$ 2 models, Härdle et al. (2013)

Intra-day Periodicity

▸ Notation

- Flexible Fourier Series (FFS) approximation, Gallant (1981)

  Intraday periodicity components $\left(s_{1,d-30}, \dots, s_{360,d-1}\right)^\top$

- Estimation: 30-day rolling window with $s_{i,d-1} = \dots = s_{i,d-30}$, Engle and Rangel (2008)

$$s_{i,d-1} = \delta \bar{\imath}_i + \sum_{m=1}^{M} \{\delta_{c,m} \cos(\bar{\imath}_i \cdot 2\pi m) + \delta_{s,m} \sin(\bar{\imath}_i \cdot 2\pi m)\}$$

$\bar{\imath} = (\bar{\imath}_1, \dots, \bar{\imath}_{360})^\top = (1/360, \dots, 360/360)^\top$ - intraday time trend