Confidence Corridors for Multivariate Generalized Quantile Regression

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Does treatment help?

(a) Before

(b) After

Treatment effect

- $Y_1 (Y_0)$: the treatment (control) group; D: dummy variable,
  - Mean: the average treatment effect $\Delta_m = \mathsf{E}[Y_1 - Y_0]$
  - Quantile: $\Delta_\tau = F_1^{-1}(\tau) - F_0^{-1}(\tau)$
- Randomized experiment:
  $\mathsf{E}[Y_1 - Y_0] = \mathsf{E}[Y_1|D = 1] - \mathsf{E}[Y_0|D = 0]$
- Measure Δ_m through a dummy-variable regression:

$$Y_i = \alpha + D_i\gamma + \boldsymbol{X}_i^\top \beta + e_i, \qquad \text{(Location shift)}$$
$$Y_i = \alpha + \boldsymbol{X}_i^\top (\beta + D_i\gamma) + e_i, \qquad \text{(scaling)}$$

Quantile treatment effect (QTE)

Doksum (1974) "Horizontal distance" $\Delta(y)$:

$$F_1(y) = F_0\{y + \Delta(y)\},$$

express as:

$$\Delta(y) = F_0^{-1}\{F_1(y)\} - y,$$

set $\tau = F_1(y)$, one gets the quantile treatment effect:

$$\Delta_\tau = \Delta\{F_1^{-1}(\tau)\} \stackrel{\text{def}}{=} F_0^{-1}(\tau) - F_1^{-1}(\tau).$$

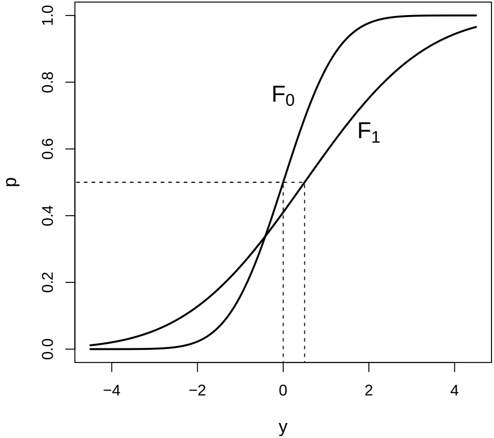


Figure 1: Heterogeneous horizontal shifts in distribution.

Stochastic dominance (SD)

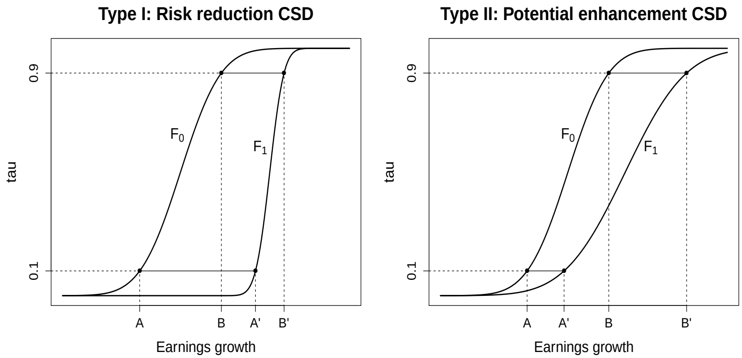
Conditional stochastic dominance (CSD)(Delgado and Escanciano, 2013) : Y_1 conditionally stochastically dominates Y_0:

$$F_{1|\boldsymbol{X}}(y|\boldsymbol{x}) \leq F_{0|\boldsymbol{X}}(y|\boldsymbol{x}) \quad \text{a.s. for all } y, \boldsymbol{x}, \tag{1}$$

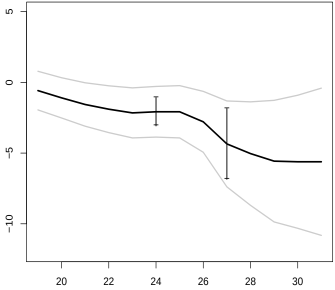
Take $y = F^{-1}_{0|\boldsymbol{X}}(\tau|\boldsymbol{x})$. Apply $F^{-1}_{1|\boldsymbol{X}}$ to (1):

$$F^{-1}_{0|\boldsymbol{X}}(\tau|\boldsymbol{x}) \leq F^{-1}_{1|\boldsymbol{X}}\{F_{0|\boldsymbol{X}}(y|\boldsymbol{x})\big|\boldsymbol{x}\} = F^{-1}_{1|\boldsymbol{X}}(\tau|\boldsymbol{x}) \quad \forall \boldsymbol{x}, \tau$$

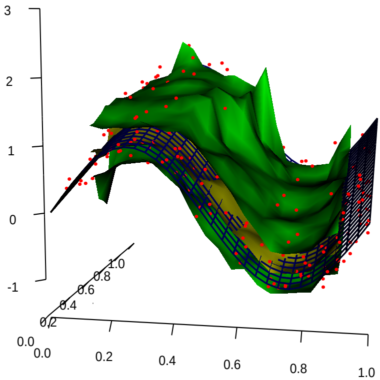
Which one helps more?



Confidence corridors (CC)



(a) $d = 1$

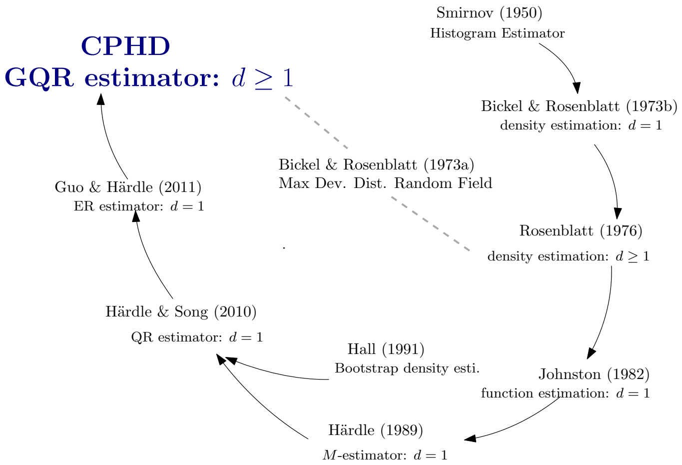


(b) $d = 2$

Distribution comparison & model diagnosis

- Compare (conditional) distributions or perform goodness-of-fit test
- Parametric inference: requires prior knowledge on the correct stochastic specification
- Nonparametric inference gives flexibility and avoids model bias

Confidence corridors: a history



The story line

- Claeskens and van Keilegom (2003): local polynomial mean estimator
- Gené and Nickl (2010): adaptive density estimation with wavelets and kernel
- Liu and Wu (2010): long memory, strictly stationary time series density estimation
- Fan and Liu (2013): one dimensional, generic (semi)parametric quantile estimator, avoid estimating conditional density
- The big question: How to statistically compare the nonparametric curve/surfaces?

Outline

1. Motivation ✓
2. Method and Theoretical Results
3. Bootstrap
4. Simulation
5. Application to National Supported Work (NSW) Demonstration data

Additive error model

- Let $(\boldsymbol{X}_1, Y_1), ..., (\boldsymbol{X}_n, Y_n)$ be a sequence of i.i.d. random vectors in $\mathbb{R}^{d+1}$ and consider the nonparametric regression model

$$Y_i = \theta(\boldsymbol{X}_i) + \varepsilon_i, \quad i = 1, ..., n, \tag{2}$$

  where θ is an aspect of Y conditional on $\boldsymbol{X}$ such as the τ-quantile, the τ-expectile regression curve, ε_i i.i.d. with τ-quantile/expectile 0.
- Heterogeneity: ε_i is allowed to be correlated with $\boldsymbol{X}$

Confidence intervals

$1-\alpha$-confidence interval

$$\mathrm{P}\left\{\hat{\theta}_n(\boldsymbol{x}) - B_n(\boldsymbol{x}) \le \theta_0(\boldsymbol{x}) \le \hat{\theta}_n(\boldsymbol{x}) + B_n(\boldsymbol{x})\right\} = 1-\alpha$$

- Confidence statement for one fixed $\boldsymbol{x}$.
- Only pointwise information!
- Cannot be used to check for global statements without a correction

Confidence corridors

Uniform $1-\alpha$-confidence corridor on a compact set $\mathcal{D}$

$$\mathrm{P}\left\{\hat{\theta}_n(\boldsymbol{x}) - \Phi_n(\boldsymbol{x}) \le \theta_0(\boldsymbol{x}) \le \hat{\theta}_n(\boldsymbol{x}) + \Phi_n(\boldsymbol{x}) \, \forall \, \boldsymbol{x} \in \mathcal{D}\right\} = 1-\alpha$$

- True values of $\theta_0(\boldsymbol{x})$ covered for all $\boldsymbol{x} \in \mathcal{D}$ simultaneously by the band with probability $1-\alpha$.
- Global information about θ_0 on $\mathcal{D}$.

Distribution of the maximal deviation

$$\mathrm{P}\left\{\hat{\theta}_n(\boldsymbol{x}) - \Phi_n(\boldsymbol{x}) \le \theta_0(\boldsymbol{x}) \le \hat{\theta}_n(\boldsymbol{x}) + \Phi_n(\boldsymbol{x})\,\forall\,\boldsymbol{x} \in \mathcal{D}\right\} = 1 - \alpha \tag{3}$$

Find Φ_n such that the equality (3) holds approximately. Suppose:

$$\sup_{\boldsymbol{x}\in\mathcal{D}} w_n(\boldsymbol{x})\,|\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})| \le \Phi_n$$

with probability $1 - \alpha$, this implies

$$|\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})| \le \frac{\varphi_n}{w_n(\boldsymbol{x})} \stackrel{\text{def}}{=} \Phi_n(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathcal{D}.$$

Approximation of the distribution of $\sup_{\boldsymbol{x}\in\mathcal{D}} w_n(\boldsymbol{x})\,|\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})|$.

Estimator and Bahadur representation

- Consider the local constant estimator

$$\hat{\theta}_n(\boldsymbol{x}) \stackrel{\text{def}}{=} \arg\min_{\theta} n^{-1} \sum_{i=1}^{n} K_h(\boldsymbol{x} - \boldsymbol{X}_i)\rho(Y_i - \theta)$$

with a kernel K and loss function ρ. ▸ Notations

- Uniform nonparametric Bahadur representation:

$$\sup_{\boldsymbol{x}\in\mathcal{D}} \left| \hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x}) - \frac{1}{nS_{n,0,0}(\boldsymbol{x})} \sum_{i=1}^{n} K_h(\boldsymbol{x} - \boldsymbol{X}_i)\psi_\tau\{Y_i - \theta_0(\boldsymbol{x})\} \right|$$
$$= \mathcal{O}\left\{(\log n/nh^d)^{3/4}\right\}, \quad a.s.[\mathrm{P}]$$

Bahadur representation

$$S_{n,0,0}(\boldsymbol{x})\{\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})\} \approx \frac{1}{n}\sum_{i=1}^{n} K_h(\boldsymbol{x} - \boldsymbol{X}_i)\psi\{Y_i - \theta_0(\boldsymbol{x})\}$$

$$\psi_\tau(u) = \begin{cases} \mathbf{1}(u \le 0) - \tau, & \text{Quantile;} \\ 2\{\mathbf{1}(u \le 0) - \tau\}|u|, & \text{Expectile.} \end{cases}$$

$$S_{n,0,0}(\boldsymbol{x}) = \begin{cases} f_{Y|\boldsymbol{X}}(\theta_0(\boldsymbol{x})|\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) + \mathcal{O}(h^s), & \text{Q;} \\ 2\left[\tau - F_{Y|\boldsymbol{X}}\left(\theta_0(\boldsymbol{x})\right|\boldsymbol{x})(2\tau - 1)\right] f_{\boldsymbol{X}}(\boldsymbol{x}) + \mathcal{O}(h^s), & \text{E.} \end{cases}$$

where s indicates the smoothness of $\theta(\boldsymbol{x})$, $s \ge d$.

Approximating empirical process

$$
\begin{aligned}
&V_n^{-1/2} S_{n,0,0}(\boldsymbol{x}) \left\{ \hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x}) \right\} \\
&\approx V_n^{-1/2} \frac{1}{n} \sum_{i=1}^{n} K_h(\boldsymbol{x} - \boldsymbol{X}_i) \psi\{ Y_i - \theta_0(\boldsymbol{x}) \} \\
&\approx \underbrace{\frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x}) \sigma^2(\boldsymbol{x})}} \int \int K\left(\frac{\boldsymbol{x} - \boldsymbol{u}}{h} \right) \psi\{ y - \theta_0(\boldsymbol{x}) \} dZ_n(y, \boldsymbol{u})}_{Y_n(\boldsymbol{x})}
\end{aligned}
$$

- with the centered empirical process

$$
Z_n(y, \boldsymbol{u}) \stackrel{\text{def}}{=} n^{1/2} \{ F_n(y, \boldsymbol{u}) - F(y, \boldsymbol{u}) \}.
$$

The empirical processes of QR

$Y_n(\boldsymbol{x})$ —— $\mathcal{O}_p\{(\log n)^{-1}\}$ —— $Y_{0,n}(\boldsymbol{x})$ —— $\mathcal{O}_p\{(\log n)^{-1}\}$ strong approx. —— $Y_{1,n}(\boldsymbol{x})$ Gaussian

$Y_{1,n}(\boldsymbol{x})$ —— $\mathcal{O}_p(h^{d/2})$ —— $Y_{2,n}(\boldsymbol{x})$

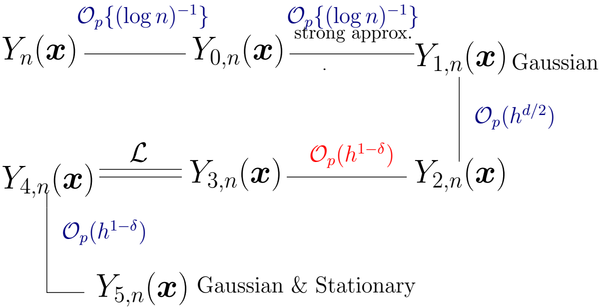
$Y_{4,n}(\boldsymbol{x}) \overset{\mathcal{L}}{=\!=} Y_{3,n}(\boldsymbol{x})$ —— $\mathcal{O}_p(h^{1/2-\delta})$ —— $Y_{2,n}(\boldsymbol{x})$

$Y_{4,n}(\boldsymbol{x})$ —— $\mathcal{O}_p(h^{1-\delta})$ —— $Y_{5,n}(\boldsymbol{x})$ Gaussian & Stationary

Rosenblatt (1976): $\sup_{\boldsymbol{x}} Y_{5,n}(\boldsymbol{x}) \xrightarrow{\mathcal{L}}$ Gumbel

▸ Assumptions

The empirical process of ER



Rosenblatt (1976): $\sup_{\boldsymbol{x}} Y_{5,n}(\boldsymbol{x}) \xrightarrow{\mathcal{L}}$ Gumbel

▸ Assumptions

Step 1: Support truncation

$$Y_0(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma^2(\boldsymbol{x})}} \int\int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\left\{y-\theta_0(\boldsymbol{x})\right\} dZ_n(y,\boldsymbol{u}),$$

- $\Gamma_n = \{y : |y| \le a_n\}$
- $\sigma_n^2(\boldsymbol{x}) = \mathrm{E}[\psi^2(Y-\theta_0(\boldsymbol{x}))\mathbf{1}(Y_i \le a_n)|\boldsymbol{X}=\boldsymbol{x}]$
- Claim: $\|Y_0 - Y_{n,0}\| = \mathcal{O}_P\left((\log n)^{-1/2}\right)$

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Step 1: Support truncation

- Show $\left(Y_{n,0} - Y_{n,0}\right)(x) \xrightarrow{P} 0$ for each x and tightness.
  ▸ Tightness Lemma
- Necessary to control the decay of the tail of distribution of Y
- Watch out for difference in quantile and expectile regression:
  - Quantile: very weak assumption (A2)
  - Expectile: exploding boundary deteriorates the strong approximation rate → requiring at least finite forth conditional moment (EA2)

Step 2: Strong approximation

$$Y_{0,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int \int_{\Gamma_{\boldsymbol{n}}} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \psi\left\{y-\theta_0(\boldsymbol{x})\right\} dZ_n(y,\boldsymbol{u}),$$

where

$$T(y,\boldsymbol{u}) = \left\{F_{X_1|Y}(u_1|y), F_{X_2|Y}(u_2|u_1,y), \ldots, F_{X_d|X_{d-1},\ldots,X_1,Y}(u_d|u_{d-1},\ldots,u_1,y), F_Y(y)\right\}$$

is the Rosenblatt transformation and

$$B_n\big(T(y,\boldsymbol{u})\big) = W_n\big(T(y,\boldsymbol{u})\big) - F(y,\boldsymbol{u})W(1,\ldots,1)$$

a multivariate Brownian bridge.

Claim: $\|Y_{0,n} - Y_{1,n}\| = \mathcal{O}_p\left\{(\log n)^{-1}\right\}$, a.s.

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Claim: $\|Y_{0,n} - Y_{1,n}\| = \mathcal{O}_p\big\{(\log n)^{-1}\big\}$, a.s.

Step 3: Brownian bridge → Wiener sheet

$$Y_{2,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int \int_{\Gamma_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \psi\left\{y - \theta_0(\boldsymbol{x})\right\} dB_n\big(T(y,\boldsymbol{u})\big).$$

Claim: $\|Y_{1,n} - Y_{2,n}\| = \mathcal{O}_p\big(h^{d/2}\big)$

- by integration by parts
- since $|W(1, ..., 1)| \leq \mathcal{O}_P(1)$.

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Step 4: Stationarise the process

$$Y_{2,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int \int_{\Gamma_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \psi\left\{y - \theta_0(\boldsymbol{x})\right\} dW_n\big(T(y,\boldsymbol{u})\big)$$

Claim: $\|Y_{2,n} - Y_{3,n}\| = \mathcal{O}_P(h^{1-\delta})$, for any $\delta > 0$

A supremum concentration inequality for Gaussian field is applied.
Meerschaert et al. (2013)

Step 4: Stationarise the process

$$Y_{3,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int \int_{\Gamma_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \psi\left\{y - \theta_0(\boldsymbol{u})\right\} dW_n\big(T(y,\boldsymbol{u})\big)$$

Claim: $\|Y_{2,n} - Y_{3,n}\| = \mathcal{O}_P(h^{1-\delta})$, for any $\delta > 0$

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Meerschaert et al. (2013)

Step 5: Equally distributed

$$Y_{3,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int\int_{\Gamma_n} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\left\{y-\theta_0(\boldsymbol{u})\right\} dW_n\big(T(y,\boldsymbol{u})\big).$$

Claim: $Y_{3,n} \stackrel{\mathrm{d}}{=} Y_{4,n}$

A computation of the covariance functions gives the result.

Step 5: Equally distributed

$$Y_{4,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \sqrt{\sigma_n^2(\boldsymbol{u}) f_{\boldsymbol{X}}(\boldsymbol{u})} dW(\boldsymbol{u}).$$

Claim: $Y_{3,n} \overset{\mathrm{d}}{=} Y_{4,n}$

A computation of the covariance functions gives the result.

Step 6: Final stationarisation

$$Y_{4,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) \sqrt{\sigma_n^2(\boldsymbol{u}) f_{\boldsymbol{X}}(\boldsymbol{u})}\, dW(\boldsymbol{u}).$$

$$Y_{5,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d}} \int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) dW(\boldsymbol{u}).$$

Claim: $\| Y_{4,n} - Y_{5,n} \| = \mathcal{O}_p(h^{1-\delta})$ for $\delta > 0$.

Supremum concentration inequality for Gaussian field is again applied.

▸ Meerschaert et al. (2013)

Maximal deviation for nonparametric QR

Theorem (1)

Under regularity conditions, vol$(\mathcal{D}) = 1$,

▸ Notations ▸ Assumptions

$$\mathrm{P}\left\{(2d\kappa \log n)^{1/2}\left(\sup_{\boldsymbol{x}\in\mathcal{D}}\left[r(\boldsymbol{x})|\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})|\right]/\|K\|_2 - d_n\right) < a\right\}$$
$$\to \exp\left\{-2\exp(-a)\right\},$$

as $n \to \infty$, where $\hat{\theta}_n(\boldsymbol{x})$ and $\theta_0(\boldsymbol{x})$ are the local constant quantile estimator and the true quantile function.

Corollary (RQ-CC)

Under the assumptions of Theorem 1, an approximate $(1-\alpha)\times 100\%$ *confidence corridor over* $\alpha \in (0,1)$ *is*

$$\hat{\theta}_n(\boldsymbol{t}) \pm (nh^d)^{-1/2}\left\{\tau(1-\tau)\|K\|_2/\hat{f}_{\boldsymbol{X}}(\boldsymbol{t})\right\}^{1/2}\hat{f}_{\varepsilon|\boldsymbol{X}}\left\{0|\boldsymbol{t}\right\}^{-1} \left\{d_n + c(\alpha)(2d\kappa\log n)^{-1/2}\right\},$$

where $c(\alpha) = \log 2 - \log\left|\log(1-\alpha)\right|$ *and* $\hat{f}_{\boldsymbol{X}}(\boldsymbol{t})$, $\hat{f}_{\varepsilon|\boldsymbol{X}}\left\{0|\boldsymbol{t}\right\}$ *are consistent estimates for* $f_{\boldsymbol{X}}(\boldsymbol{t})$, $f_{\varepsilon|\boldsymbol{X}}\left\{0|\boldsymbol{t}\right\}$.

Maximal deviation for nonparametric ER

Theorem (2)

Under regularity conditions, vol$(\mathcal{D}) = 1$, ▸ Notations ▸ Assumptions

$$\mathrm{P}\left\{(2d\kappa \log n)^{1/2}\left(\sup_{\boldsymbol{x}\in\mathcal{D}}\left[r(\boldsymbol{x})|\hat{\theta}_n(\boldsymbol{x}) - \theta_0(\boldsymbol{x})|\right]/\|K\|_2 - d_n\right) < a\right\} \to \exp\left\{-2\exp(-a)\right\},$$

as $n \to \infty$*, where* $\hat{\theta}_n(\boldsymbol{x})$ *and* $\theta_0(\boldsymbol{x})$ *are the local constant expectile estimator and the true expectile function.*

Corollary (RE-CC)

Under the assumptions of Theorem 3, an approximate $(1-\alpha)\times 100\%$ *confidence corridor over* $\alpha \in (0,1)$ *is*

$$\hat{\theta}_n(\boldsymbol{t}) \pm (nh^d)^{-1/2}\left\{\hat{\sigma}^2(\boldsymbol{x})\|K\|_2/\hat{f}_{\boldsymbol{X}}(\boldsymbol{t})\right\}^{1/2} 0.5\left[\tau - \hat{F}_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})(2\tau-1)\right]^{-1} \left\{d_n + c(\alpha)(2d\kappa\log n)^{-1/2}\right\},$$

where $c(\alpha) = \log 2 - \log\left|\log(1-\alpha)\right|$ *and* $\hat{f}_{\boldsymbol{X}}(\boldsymbol{t})$, $\hat{\sigma}^2(\boldsymbol{x})$ *and* $\hat{F}_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})$ *are consistent estimates for* $f_{\boldsymbol{X}}(\boldsymbol{t})$, $\sigma^2(\boldsymbol{x})$ *and* $F_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})$

Estimating scaling factors

we propose to estimate $F_{\varepsilon|\boldsymbol{X}}$, $f_{\varepsilon|\boldsymbol{X}}$ and $\sigma^2(\boldsymbol{x})$ based on residuals $\hat{\varepsilon}_i = Y_i - \hat{\theta}_n(\boldsymbol{x}_i)$:

$$\hat{F}_{\varepsilon|\boldsymbol{X}}(v|\boldsymbol{x}) = n^{-1}\sum_{i=1}^{n} G\left(\frac{v-\hat{\varepsilon}_i}{h_0}\right) L_{\bar{h}}(\boldsymbol{x}-\boldsymbol{X}_i)/\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}) \quad (4)$$

$$\hat{f}_{\varepsilon|\boldsymbol{X}}(v|\boldsymbol{x}) = n^{-1}\sum_{i=1}^{n} g_{h_0}\left(v-\hat{\varepsilon}_i\right) L_{\bar{h}}(\boldsymbol{x}-\boldsymbol{X}_i)/\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}) \quad (5)$$

$$\hat{\sigma}^2(\boldsymbol{x}) = n^{-1}\sum_{i=1}^{n} \psi^2(\hat{\varepsilon}_i) L_{\bar{h}}(\boldsymbol{x}-\boldsymbol{X}_i)/\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}) \quad (6)$$

where G is a CDF, g and L are a kernel functions, $h_0, \bar{h} \to 0$ and $nh_0\bar{h}^d \to \infty$

Lemma

Under regularity conditions, we have ▸ Assumptions

1. $\sup_{v \in I} \sup_{\mathbf{x} \in \mathcal{D}} \left| \hat{F}_{\varepsilon|\mathbf{X}}(v|\mathbf{x}) - F_{\varepsilon|\mathbf{X}}(v|\mathbf{x}) \right| = \mathcal{O}_p\left(n^{-\lambda}\right)$
2. $\sup_{v \in I} \sup_{\mathbf{x} \in \mathcal{D}} \left| \hat{f}_{\varepsilon|\mathbf{X}}(v|\mathbf{x}) - f_{\varepsilon|\mathbf{X}}(v|\mathbf{x}) \right| = \mathcal{O}_p\left(n^{-\lambda}\right)$
3. $\sup_{\mathbf{x} \in \mathcal{D}} \left| \hat{\sigma}^2(\mathbf{x}) - \sigma^2(\mathbf{x}) \right| = \mathcal{O}_p\left(n^{-\lambda_1}\right)$

where
$n^{-\lambda} = \mathcal{O}(h_0^2 + h^s + \bar{h}^2 + (nh_0\bar{h}^d)^{-1/2}\log n + (nh^d)^{-1/2}\log n)$, *and*
$n^{-\lambda_1} = \mathcal{O}(h^s + \bar{h}^2 + (n\bar{h}^d)^{-1/2}\log n + (nh^d)^{-1/2}\log n)$.

Bootstrap

- Smooth bootstrap:

$$\hat{f}_{\varepsilon,\boldsymbol{X}}(v,\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} g_{h_0}\left(v-\hat{\varepsilon}_i\right) L_{\bar{h}}(\boldsymbol{x}-\boldsymbol{X}_i),, \tag{7}$$

where g and L are kernels and $h_0, \bar{h} \to 0$, $nh_0\bar{h}^d \to \infty$

- Define

$$\begin{aligned} &\hat{\theta}^*(\boldsymbol{x}) - \hat{\theta}_n(\boldsymbol{x}) \\ &= \frac{1}{n\hat{S}_{n,0,0}(\boldsymbol{x})}\sum_{i=1}^{n} K_h(\boldsymbol{x}-\boldsymbol{X}_i^*)\psi(\varepsilon_i^*)\underbrace{-\mathrm{E}^*\left[K_h(\boldsymbol{x}-\boldsymbol{X}_i^*)\psi(\varepsilon_i^*)\right]}_{\text{Remove the bias}}, \end{aligned} \tag{8}$$

$$\hat{S}_{n,0,0}(\boldsymbol{x}) = \begin{cases} \hat{f}_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}), & \text{quantile case;} \\ 2\left[\tau - \hat{F}_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})(2\tau-1)\right]\hat{f}_{\boldsymbol{X}}(\boldsymbol{x}), & \text{expectile case.} \end{cases}$$

Theorem (Bootstrap)

Under regularity conditions, let ▸ Assumptions

$$r^*(\boldsymbol{x}) = \sqrt{\frac{nh^d}{\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})\sigma_*^2(\boldsymbol{x})}}\hat{S}_{n,0,0}(\boldsymbol{x}),$$

Then as $n \to \infty$,

$$\mathrm{P}^* \left\{ (2d\kappa \log n)^{1/2} \left(\sup_{\boldsymbol{x} \in \mathcal{D}} \left[r^*(\boldsymbol{x}) |\hat{\theta}^*(\boldsymbol{x}) - \hat{\theta}_n(\boldsymbol{x})| \right] / \|K\|_2 - d_n \right) < a \right\}$$
$$\to \exp\left[-2\exp(-a) \right], \ a.s.$$

Lemma

Under regularity conditions, $\|\sigma_*^2(\boldsymbol{x}) - \hat{\sigma}^2(\boldsymbol{x})\| = \mathcal{O}_p^*\big((\log n)^{-1/2}\big)$, *a.s.*

Corollary

Under the regularity conditions, the bootstrap confidence set is defined by

$$\left\{ \theta : \sup_{\boldsymbol{x}\in\mathcal{D}} \left| \frac{\hat{S}_{n,0,0}(\boldsymbol{x})}{\sqrt{\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})\hat{\sigma}^2(\boldsymbol{x})}} \left[\hat{\theta}_n(\boldsymbol{x}) - \theta(\boldsymbol{x})\right] \right| \le \xi_\alpha \right\}, \tag{10}$$

where ξ_α satisfies

$$\mathrm{P}^* \left(\sup_{\boldsymbol{x}\in\mathcal{D}} \left| \frac{\hat{S}_{n,0,0}(\boldsymbol{x})}{\sqrt{\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})\hat{\sigma}^2(\boldsymbol{x})}} \left[\hat{\theta}^*(\boldsymbol{x}) - \hat{\theta}_n(\boldsymbol{x})\right] \right| \le \xi_\alpha \right) = 1 - \alpha,$$

where α is the level of the test and $\hat{S}_{n,0,0}$ is defined as in (9).

Inplementation problem for QR: The CC (10) for QR tends to be too narrow

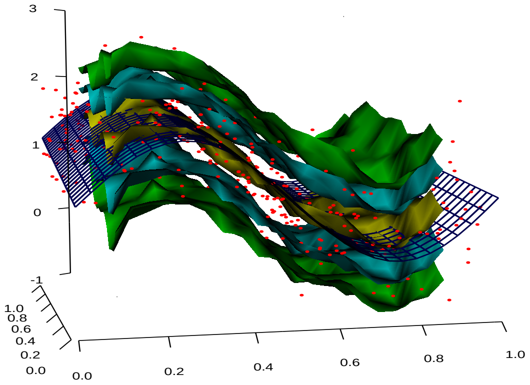


Figure 2: Confidence corridors: regression quantiles $\tau = 50\%$. Green: Asymptotic confidence band. Blue: Bootstrap confidence band.

Bootstrap CC for QR

Observation:

$$\hat{f}_{\varepsilon|\boldsymbol{X}}(0|x) = n^{-1}\sum_{i=1}^{n} g_{h_0}\left(\hat{\varepsilon}_i\right) L_{\bar{h}}(x - \boldsymbol{X}_i)/\hat{f}_{\boldsymbol{X}}(x) \tag{11}$$

$$\hat{f}_{Y|\boldsymbol{X}}(\hat{\theta}_n(x)|x) = n^{-1}\sum_{i=1}^{n} g_{h_1}\left(Y_i - \hat{\theta}_n(x)\right) L_{\tilde{h}}(x - \boldsymbol{X}_i)/\hat{f}_{\boldsymbol{X}}(x), \tag{12}$$

are NOT equivalent in finite sample, and $\hat{f}_{Y|\boldsymbol{X}}(\hat{\theta}_n(x)|x)$ accounts more for the bias

Bootstrap CC for QR

Hence, we propose to construct CC for QR by

$$\left\{\theta : \sup_{\boldsymbol{x}\in\mathcal{D}} \left|\sqrt{\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})}\hat{f}_{Y|\boldsymbol{X}}\{\hat{\theta}_n(\boldsymbol{x})|\boldsymbol{x}\}\left[\hat{\theta}_n(\boldsymbol{x}) - \theta(\boldsymbol{x})\right]\right| \leq \xi_\alpha^\dagger\right\},$$

where $\xi_\alpha^\dagger$ satisfies

$$\mathsf{P}^*\left(\sup_{\boldsymbol{x}\in\mathcal{D}}\left|\hat{f}_{\boldsymbol{X}}(\boldsymbol{x})^{-1/2}\frac{\hat{f}_{Y|\boldsymbol{X}}\{\hat{\theta}_n(\boldsymbol{x})|\boldsymbol{x}\}}{\hat{f}_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})}\left[A_n^*(\boldsymbol{x}) - \mathsf{E}^*A_n^*(\boldsymbol{x})\right]\right| \leq \xi_\alpha^\dagger\right) \approx 1-\alpha.$$

Simulated coverage probabilities

Generating process: $d = 2$

$$Y_i = f(X_{1,i}, X_{2,i}) + \sigma(X_{1,i}, X_{2,i})\varepsilon_i,$$

- $f(x_1, x_2) = \sin(2\pi x_1) + x_2$.
- (X_1, X_2) supported on $[0,1]^2$ with corr. $= 0.2876$ ▸ Sample Method
- $\varepsilon_i \sim N(0,1)$ i.i.d.
- Specification for $\sigma(X_1, X_2)$:
  - Homogeneity: $\sigma(X_1, X_2) = \sigma_0$, for $\sigma_0 = 0.2, 0.5, 0.7$
  - Heterogeneity:

$$\sigma(X_1, X_2) = \sigma_0 + 0.8X_1(1 - X_1)X_2(1 - X_2)$$

for $\sigma_0 = 0.2, 0.5, 0.7$

Simulated coverage probabilities

- Quantile regression bandwidth choice:
  - Rule-of-thumb for conditional density in R package np
  - Yu and Jones (1998) quantile regression adjustment (not applied to expectile)
  - Undersmoothed by $n^{-0.05}$
- Expectile bandwidth choice: Rule-of-thumb for conditional density and undersmoothed by $n^{-0.05}$
- $n = 100, 300, 500.$
  2000 simulation runs are carried out.

Table 1: Nonparametric quantile model asymptotic coverage probability. Nominal coverage is 95%. The "()" indicate the volume.

	Homogeneous			Heterogeneous		
n	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$
			$\sigma_0 = 0.2$			
100	.000(0.366)	.109(0.720)	.104(0.718)	.000(0.403)	.120(0.739)	.122(0.744)
300	.000(0.304)	.130(0.518)	.133(0.519)	.002(0.349)	.136(0.535)	.153(0.537)
500	.000(0.262)	.117(0.437)	.142(0.437)	.008(0.296)	.156(0.450)	.138(0.450)
			$\sigma_0 = 0.5$			
100	.070(0.890)	.269(1.155)	.281(1.155)	.078(0.932)	.300(1.193)	.302(1.192)
300	.276(0.735)	.369(0.837)	.361(0.835)	.325(0.782)	.380(0.876)	.394(0.877)
500	.364(0.636)	.392(0.711)	.412(0.712)	.381(0.669)	.418(0.743)	.417(0.742)
			$\sigma_0 = 0.7$			
100	.160(1.260)	.381(1.522)	.373(1.519)	.155(1.295)	.364(1.561)	.373(1.566)
300	.438(1.026)	.450(1.109)	.448(1.110)	.481(1.073)	.457(1.155)	.472(1.152)
500	.533(0.888)	.470(0.950)	.480(0.949)	.564(0.924)	.490(0.984)	.502(0.986)

Table 2: Nonparametric quantile model bootstrap coverage probability. Nominal coverage is 95%. The "()" the volume.

	Homogeneous			Heterogeneous		
n	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$
			$\sigma_0 = 0.2$			
100	.325(0.676)	.784(0.954)	.783(0.954)	.409(0.717)	.779(0.983)	.778(0.985)
300	.442(0.457)	.896(0.609)	.894(0.610)	.580(0.504)	.929(0.650)	.922(0.649)
500	.743(0.411)	.922(0.502)	.921(0.502)	.839(0.451)	.950(0.535)	.952(0.536)
			$\sigma_0 = 0.5$			
100	.929(1.341)	.804(1.591)	.818(1.589)	.938(1.387)	.799(1.645)	.773(1.640)
300	.950(0.920)	.918(1.093)	.923(1.091)	.958(0.973)	.919(1.155)	.923(1.153)
500	.988(0.861)	.968(0.943)	.962(0.942)	.990(0.902)	.962(0.986)	.969(0.987)
			$\sigma_0 = 0.7$			
100	.976(1.811)	.817(2.112)	.808(2.116)	.981(1.866)	.826(2.178)	.809(2.176)
300	.986(1.253)	.919(1.478)	.934(1.474)	.983(1.308)	.930(1.537)	.920(1.535)
500	.996(1.181)	.973(1.280)	.968(1.278)	.997(1.225)	.969(1.325)	.962(1.325)

Table 3: Nonparametric expectile model asymptotic coverage probability. Nominal coverage is 95%. The "()" indicate the volume.

	Homogeneous			Heterogeneous		
n	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$
			$\sigma_0 = 0.2$			
100	.000(0.428)	.000(0.333)	.000(0.333)	.000(0.463)	.000(0.362)	.000(0.361)
300	.049(0.341)	.000(0.273)	.000(0.273)	.079(0.389)	.001(0.316)	.002(0.316)
500	.168(0.297)	.000(0.243)	.000(0.243)	.238(0.336)	.003(0.278)	.002(0.278)
			$\sigma_0 = 0.5$			
100	.007(0.953)	.000(0.776)	.000(0.781)	.007(0.997)	.000(0.818)	.000(0.818)
300	.341(0.814)	.019(0.708)	.017(0.709)	.355(0.862)	.017(0.755)	.018(0.754)
500	.647(0.721)	.067(0.645)	.065(0.647)	.654(0.759)	.061(0.684)	.068(0.684)
			$\sigma_0 = 0.7$			
100	.012(1.324)	.000(1.107)	.000(1.107)	.010(1.367)	.000(1.145)	.000(1.145)
300	.445(1.134)	.021(1.013)	.013(1.016)	.445(1.182)	.017(1.062)	.016(1.060)
500	.730(1.006)	.062(0.928)	.078(0.929)	.728(1.045)	.068(0.966)	.066(0.968)

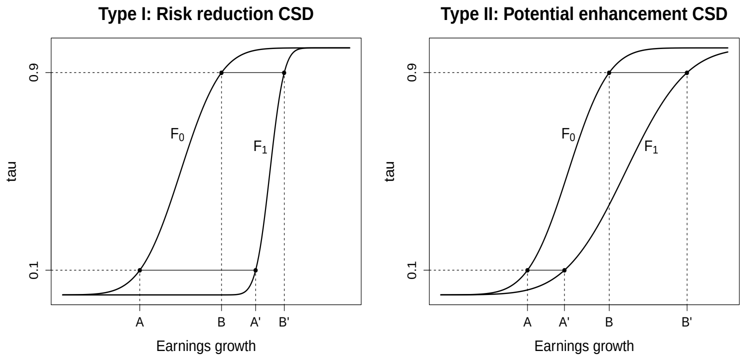
Table 4: Nonparametric expectile model bootstrap coverage probability. Nominal coverage is 95%. The "()" indicate the volume.

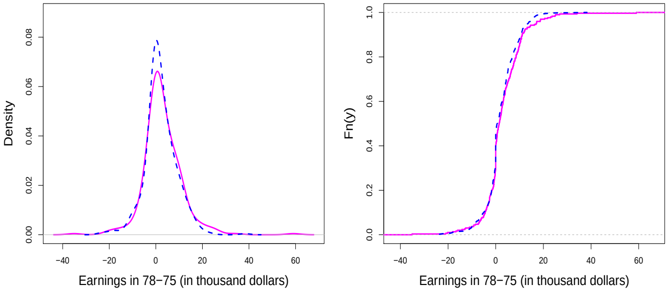
	Homogeneous			Heterogeneous		
n	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.5$	$\tau = 0.2$	$\tau = 0.8$
			$\sigma_0 = 0.2$			
100	.686(2.191)	.781(2.608)	.787(2.546)	.706(2.513)	.810(2.986)	.801(2.943)
300	.762(0.584)	.860(0.716)	.876(0.722)	.788(0.654)	.877(0.807)	.887(0.805)
500	.771(0.430)	.870(0.533)	.875(0.531)	.825(0.516)	.907(0.609)	.904(0.615)
			$\sigma_0 = 0.5$			
100	.886(5.666)	.906(6.425)	.915(6.722)	.899(5.882)	.927(6.667)	.913(6.571)
300	.956(1.508)	.958(1.847)	.967(1.913)	.965(1.512)	.962(1.866)	.969(1.877)
500	.968(1.063)	.972(1.322)	.972(1.332)	.972(1.115)	.971(1.397)	.974(1.391)
			$\sigma_0 = 0.7$			
100	.913(7.629)	.922(8.846)	.935(8.643)	.929(8.039)	.935(9.057)	.932(9.152)
300	.969(2.095)	.969(2.589)	.971(2.612)	.974(2.061)	.972(2.566)	.979(2.604)
500	.978(1.525)	.976(1.881)	.967(1.937)	.981(1.654)	.978(1.979)	.974(2.089)

Application to NSW demonstration data

- National Supported Work (NSW): a randomized, temporary employment program carried out in the US in 1970s to help the disadvantages workers
- 297 obs. treatment group; 425 obs. control group, all male
- Lalonde (1986), Dehejia and Wahba (1999)
- Delgado and Escanciano (2013): heterogeneity effect in age; nonnegative treatment effect
- X_1: Age; X_2: schooling in years; Y: Earning difference 78-75 (in thousand $)
- Bootstrap: 10,000 repetition

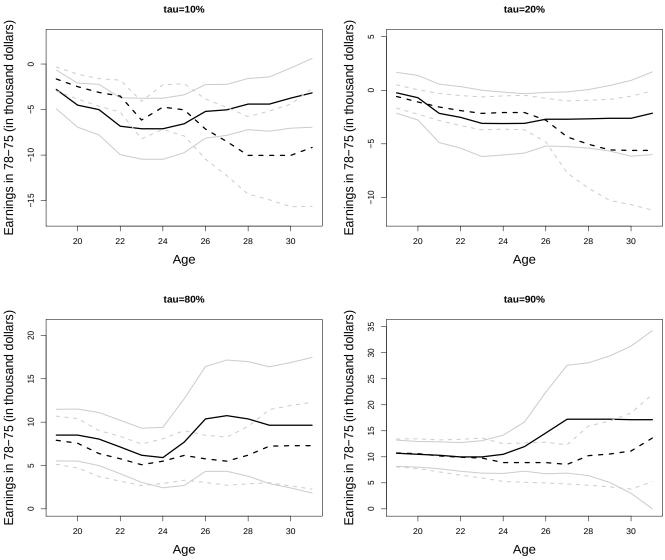
CSD revisits

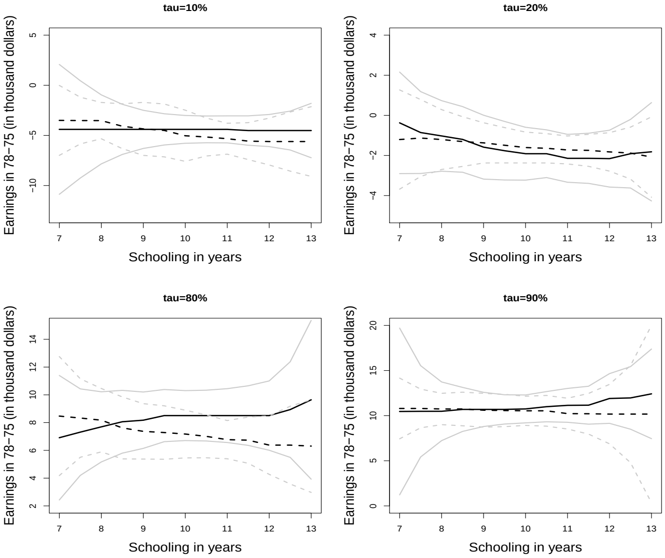


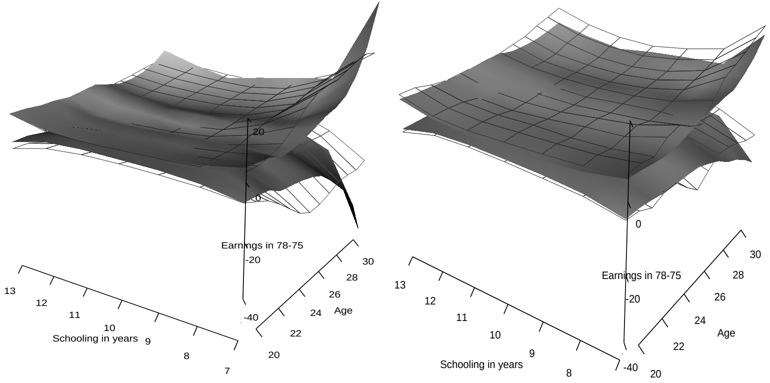


$\tau(\%)$	10	20	30	50	70	80	90
Treatment	-4.38	-1.55	0.00	1.40	5.48	8.50	11.15
Control	-4.91	-1.73	-0.17	0.74	4.44	7.16	10.56

Unconditional kernel densities. Magenta: treatment group. Blue: control group. $h_{tr} = 1.652$. $h_{co} = 1.231$.

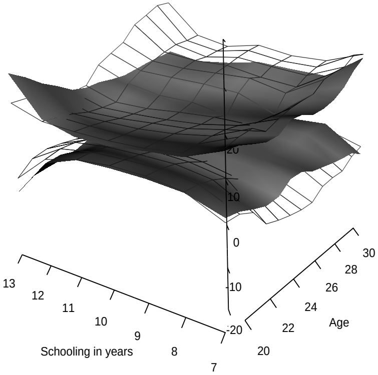

tau=10%
tau=20%
tau=80%
tau=90%
Earnings in 78−75 (in thousand dollars)
Age


tau=10%
tau=20%
tau=80%
tau=90%
Earnings in 78−75 (in thousand dollars)
Schooling in years

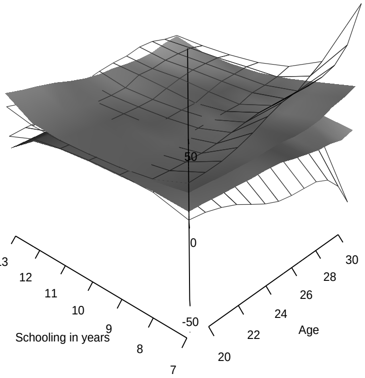


(a) $\tau = 10\%$ (b) $\tau = 20\%$

Figure 3: Confidence corridors for treatment (net surfaces) and control group (solid surfaces).



(a) $\tau = 80\%$



(b) $\tau = 90\%$

Figure 4: Confidence corridors for treatment (net surfaces) and control group (solid surfaces).

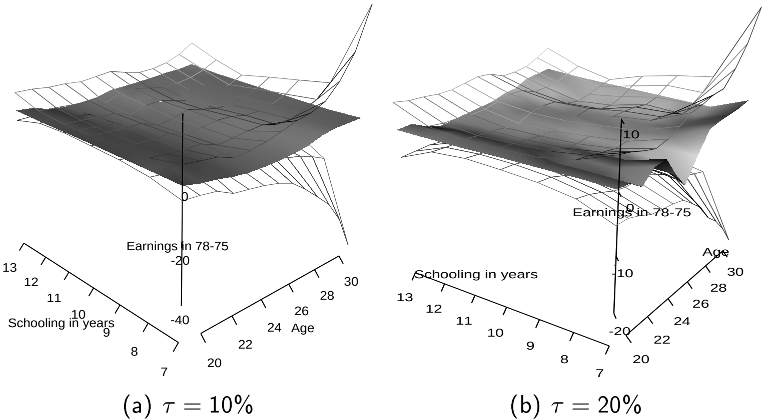
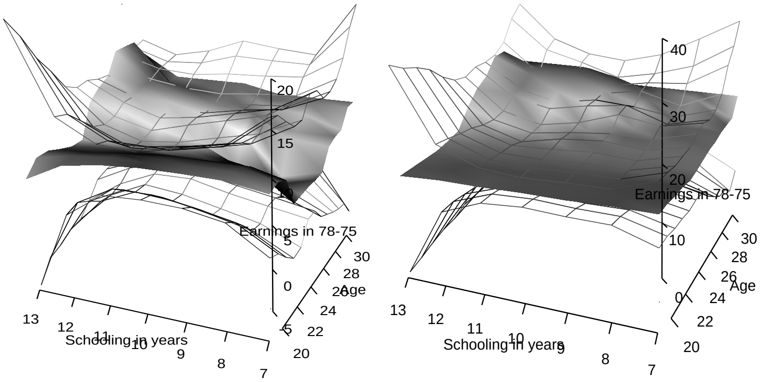


Figure 5: Quantile estimates for treatment (solid surfaces) and confidence corridors control group (net surfaces).



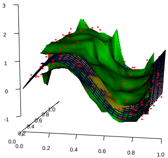
(a) $\tau = 80\%$ (b) $\tau = 90\%$

Figure 6: Quantile estimates for treatment (solid surfaces) and confidence corridors control group (net surfaces).

Summary

- The nonnegative CSD is not rejected, confirming the findings of Delgado and Escanciano (2013)
- Heterogeneous effect in age and schooling in years: individuals who are older and spend more time in the school benefit more from the treatment
- We show: treatment raises the potential for realizing higher earnings growth, but does little in reducing the risk of realizing low earnings growth

Confidence Corridors for Multivariate Generalized Quantile Regression



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Assumptions

(A1) K is of order $s-1$ (see (A3))has bounded support $[-A, A]^d$, continuously differentiable up to order d (and are bounded); i.e. $\partial^\alpha K \in L^1(\mathbb{R}^d)$ exists and is continuous for all multi-indices $\in \{0,1\}^d$

(A2) The increasing sequence $\{a_n\}_{n=1}^\infty$ satisfies

$$(\log n) h^{-3d} \int_{|y|>a_n} f_Y(y) dy = \mathcal{O}(1) \tag{13}$$

and

$$(\log n) h^{-d} \int_{|y|>a_n} f_{Y|\boldsymbol{X}}(y|\boldsymbol{x}) dy = \mathcal{O}(1), \text{ for all } \boldsymbol{x} \in \mathcal{D}$$

as $n \to \infty$ hold.

(A3) The true function $\theta_0(\boldsymbol{x})$ is continuously differentiable and is in Hölder class with order $s > d$.

Assumptions

(A4) $f_{\boldsymbol{X}}(x)$ is continuously differentiable and its gradient is uniformly bounded. In particular, $\inf_{\boldsymbol{x}\in\mathcal{D}} f_{\boldsymbol{X}}(x) > 0$.

(A5) The joint probability density function $f(y, \boldsymbol{u})$ is positive and continuously differentiable up to sth order (needed for Rosenblatt transform), and the conditional density $f_{Y|\boldsymbol{X}}(y|\boldsymbol{X} = x)$ is continuouly differentiable with respect to $\boldsymbol{x}$.

(A6) h satisfies $\sqrt{nh^d}h^s\sqrt{\log n} \to 0$ (undersmoothing), and $nh^{3d} \to \infty$ as $n \to \infty$

Thm RQ-Band
Emp. process QR

Assumptions

(EA2) $\sup_{\boldsymbol{x}\in\mathcal{D}} \left| \int v^{b_1} f_{\varepsilon|\boldsymbol{X}}(v|\boldsymbol{x})dv \right| < \infty$, where b_1 satisfies

$$n^{-1/6} h^{-d/2-3d/(b_1-2)} = \mathcal{O}(n^{-\nu}), \quad \nu > 0.$$

e.g. when $h = n^{-1/(2s+d)}$, then $b_1 > (4s + 14d)/(2s + d - 3)$.

Thm RE-Band
Emp. process ER

Assumptions

(B1) L is a Lipschitz, bounded, symmetric kernel. G is Lipschitz continuous cdf, and g is the derivative of G and is also a density, which is Lipschitz continuous, bounded, symmetric and five times continuously differentiable kernel.

(B2) $F_{\varepsilon|\boldsymbol{X}}(v|\boldsymbol{x})$ is in $s'+1$ order Hölder class with respect to v and continuous in $\boldsymbol{x}$, $s' > \max\{2, d\}$. $f_{\boldsymbol{X}}(\boldsymbol{x})$ is in second order Hölder class with respect to $\boldsymbol{x}$ and v. $\mathrm{E}[\psi^2(\varepsilon_i)|\boldsymbol{x}]$ is second order continuously differentiable with respect to $\boldsymbol{x} \in \mathcal{D}$.

(B3) $nh_0\bar{h}^d \to \infty$, $h_0, \bar{h} = \mathcal{O}(n^{-\nu})$, where $\nu > 0$.

▸ Scaling factors

Assumptions

(C1) There exist an increasing sequence c_n, $c_n \to \infty$ as $n \to \infty$ such that

$$(\log n)^3 (nh^{6d})^{-1} \int_{|v|>c_n/2} f_\varepsilon(v)dv = \mathcal{O}(1), \qquad (14)$$

as $n \to \infty$.

(EC1) $\sup_{\boldsymbol{x}\in\mathcal{D}} \left| \int v^b f_{\varepsilon|\boldsymbol{X}}(v|\boldsymbol{x})dv \right| < \infty$, where b satisfies

$$n^{-\frac{1}{6}+\frac{4}{b^2}-\frac{1}{b}} h^{-\frac{d}{2}-\frac{6d}{b}} = \mathcal{O}(n^{-\nu}), \quad \nu > 0, \text{ (Thm. 41)}$$

and

$$b > 2(2s' + d + 1)/(2s' + 3). \text{ (Lemma 7)}$$

Bootstrap

Quantile regression notations

$$h = n^{-\kappa}, \quad \rho_\tau(u) = \left|\tau - \mathbf{1}(u < 0)\right||u|, \quad \psi(u) = \mathbf{1}(u \le 0) - \tau$$

$$d_n = (2d\kappa \log n)^{1/2} + \left(2d\kappa(\log n)\right)^{-1/2} \left[\frac{1}{2}(d-1)\log\log n^{\kappa} + \log\left\{(2\pi)^{-1/2} H_2 (2d)^{(d-1)/2}\right\}\right],$$

$$H_2 = \left(2\pi\|K\|_2^2\right)^{-d/2} \det(\Sigma)^{1/2}, \ \Sigma_{ij} = \int \frac{\partial K}{\partial u_i}\frac{\partial K}{\partial u_j} d\boldsymbol{u},$$

$$r(x) = \sqrt{\frac{nh^d f_{\boldsymbol{X}}(x)}{\tau(1-\tau)}} f_{\varepsilon|\boldsymbol{X}}(0|x),$$

▸ Bahadur ▸ RQ-Band

Expectile regression notations

$$
h = n^{-\kappa}, \quad \rho_\tau(u) = |\tau - \mathbf{1}(u < 0)|u^2, \quad \varphi(u) = -2\{\tau - \mathbf{1}(u < 0)\}|u|
$$

$$
\begin{aligned}
d_n &= (2d\kappa \log n)^{1/2} + \big(2d\kappa(\log n)\big)^{-1/2} \Bigg[\frac{1}{2}(d-1)\log\log n^{\kappa} \\
&\qquad + \log\left\{(2\pi)^{-1/2} H_2 (2d)^{(d-1)/2}\right\}\Bigg],
\end{aligned}
$$

$$
H_2 = \big(2\pi \|K\|_2^2\big)^{-d/2} \det(\Sigma)^{1/2}, \; \Sigma_{ij} = \int \frac{\partial K}{\partial u_i}\frac{\partial K}{\partial u_j} d\boldsymbol{u},
$$

$$
r(\boldsymbol{x}) = \sqrt{\frac{nh^d f_{\boldsymbol{X}}(\boldsymbol{x})}{\sigma^2(\boldsymbol{x})}}\, 2\big[\tau - F_{\varepsilon|\boldsymbol{X}}(0|\boldsymbol{x})(2\tau - 1)\big],
$$

$$
\sigma^2(\boldsymbol{x}) = \mathsf{E}[\varphi^2(Y - \theta_0(\boldsymbol{x}))|\boldsymbol{X} = \boldsymbol{x}].
$$

Bahadur RE-Band

Approximations

$$Y_n(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma^2(\boldsymbol{x})}} \int\int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\{y-\theta_0(\boldsymbol{x})\}dZ_n(y,\boldsymbol{u})$$

$$Y_{0,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int\int_{\Gamma_{\boldsymbol{n}}} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\{y-\theta_0(\boldsymbol{x})\}dZ_n(y,\boldsymbol{u})$$

$$Y_{1,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int\int_{\Gamma_{\boldsymbol{n}}} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\{y-\theta_0(\boldsymbol{x})\}dB_n\left(T(y,\boldsymbol{u})\right)$$

$$Y_{2,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int\int_{\Gamma_{\boldsymbol{n}}} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\{y-\theta_0(\boldsymbol{x})\}dW_n\left(T(y,\boldsymbol{u})\right)$$

$$Y_{3,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int\int_{\Gamma_{\boldsymbol{n}}} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right)\psi\{y-\theta_0(\boldsymbol{u})\}dW_n\left(T(y,\boldsymbol{u})\right)$$

▸ Method

Approximations

$$Y_{4,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d f_{\boldsymbol{X}}(\boldsymbol{x})\sigma_n^2(\boldsymbol{x})}} \int \sqrt{\sigma_n^2(\boldsymbol{u}) f_{\boldsymbol{X}}(\boldsymbol{u})} K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) dW(\boldsymbol{u})$$

$$Y_{5,n}(\boldsymbol{x}) = \frac{1}{\sqrt{h^d}} \int K\left(\frac{\boldsymbol{x}-\boldsymbol{u}}{h}\right) dW(\boldsymbol{u})$$

Method

Lemma (Bickel and Wichura (1971))

If $\{X_n\}_{n=1}^{\infty}$ *is a sequence in* $D[0,1]^d$, $\mathrm{P}(X \in [0,1]^d) = 1$. *For* B, C *neighboring blocks in* $[0,1]^d$, *constants* $\lambda_1 + \lambda_2 > 1$, $\gamma_1 + \gamma_2 > 0$, $\{X_n\}_{n=1}^{\infty}$ *is tight if*

$$\mathrm{E}[|X_n(B)|^{\gamma_1}|X_n(C)|^{\gamma_2}] \leq \mu(B)^{\lambda_1}\mu(C)^{\lambda_2}, \tag{15}$$

where $\mu(\cdot)$ *is a finite nonnegative measure on* $[0,1]^d$ *(for example, Lebesgue measure), and the increment of* X_n *on the block* B, *denoted* $X_n(B)$, *is defined by*

$$X_n(B) = \sum_{\boldsymbol{\alpha}\in\{0,1\}^d} (-1)^{d-|\boldsymbol{\alpha}|} X_n\big(\boldsymbol{s} + \odot(\boldsymbol{t} - \boldsymbol{s})\big) \tag{16}$$

▸ Neighboring blocks ▸ Step 1

Neighboring Blocks

Definition

A block $B \subset \mathcal{D}$ is a subset of $\mathcal{D}$ of the form $B = \Pi_i(s_i, t_i]$ with s and t in $\mathcal{D}$; the pth-face of B is $\Pi_{i \neq p}(s_i, t_i]$. Disjoint blocks B and C are p-neigbbors if they abut and have the same pth face; they are neighbors if they are p-neighbors for some p (for example, when $d = 3$, the blocks $(s, t] \times (a, b] \times (c, d]$ and $(t, u] \times (a, b] \times (c, d]$ are 1-neighbors for $s \leq t \leq u$).

Examples

- $d = 1$: $B = (s, t]$, $X_n(B) = X_n(t) - X_n(s)$;
- $d = 2$: $B = (s_1, t_1] \times (s_2, t_2]$.
  $X_n(B) = X_n(t_1, t_2) - X_n(t_1, s_2) + X_n(s_1, s_2) - X_n(s_1, t_2)$;
- For general d, $B = \Pi_{i=1}^{d}(s_i, t_i]$, let $\boldsymbol{s} = (s_1, ..., s_d)^\top$, $\boldsymbol{t} = (t_1, ..., t_d)^\top$, then where $\odot$ denotes the vector of componentwise products.

▸ Bickel & Wichura (1971)

Lemma (Meerschaert et al. (2013))

Suppose that $Y = \{Y(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^d\}$ *is a centered Gaussian random field with values in* $\mathbb{R}$, *and denote* $d(\boldsymbol{s}, \boldsymbol{t}) \stackrel{\text{def}}{=} d_Y(\boldsymbol{s}, \boldsymbol{t}) = \left(\mathsf{E}|Y(\boldsymbol{t}) - Y(\boldsymbol{s})|^2\right)^{1/2}, \quad \boldsymbol{s}, \boldsymbol{t} \in \mathbb{R}^d$. *Let* $\mathcal{D}$ *be a compact set contained in a cube with length* r *in* $\mathbb{R}^d$ *and let* $\sigma^2 = \sup_{\boldsymbol{t} \in \mathcal{D}} \mathsf{E}[Y(\boldsymbol{t})^2]$. *For any* $m > 0$, $\epsilon > 0$, *define*

$$\gamma(\epsilon) = \sup_{\boldsymbol{s}, \boldsymbol{t} \in \mathcal{D}, \|\boldsymbol{s} - \boldsymbol{t}\| \le \epsilon} d(\boldsymbol{s}, \boldsymbol{t}), \quad Q(m) = (2 + \sqrt{2}) \int_1^\infty \gamma(m 2^{-y^2}) dy.$$

Then for all $a > 0$ *which staistfy* $a \ge (1 + 4d \log 2)^{1/2}(\sigma + a^{-1})$,

$$\mathsf{P}\left\{\sup_{\boldsymbol{t} \in S} |Y(\boldsymbol{t})| > a\right\} \le 2^{2d+2} \left(\frac{r}{Q^{-1}(1/a)} + 1\right)^d \frac{\sigma + a^{-1}}{a} \exp\left\{-\frac{a^2}{2(\sigma + a^{-1})^2}\right\}$$

where $Q^{-1}(a) = \sup\{m : Q(m) \le a\}$.

Step 4 Step 6

Generate Bivariate Uniform Samples

The bivariate samples (X_1, X_2) are generated as follows:

1. Generate n pairs of bivariate normal variables (Z_1, Z_2) with correlation ρ_N and variance 1
2. Transform the normal r.v.: $(X_1, X_2) = (\Phi(Z_1), \Phi(Z_2))$, where $\Phi(\cdot)$ is the standard normal distribution function
3. Let ρ_U be the correlation of (X_1, X_2), the following relation is true:

$$\rho_U = \frac{6}{\pi} \arcsin \frac{\rho_N}{2}.$$

Details: Falk (1999)

▸ Simulation

References

Bickel, P. J. & Rosenblatt, M.
On Some Global Measures of the Deviations of Density Function Estimates,
Annals of Statistics, 1973, 1, 1071-1095.

Bickel, P. J. & Rosenblatt, M.
Two-dimensional random fields,
Multivariate Analysis III, 1973, 3-15.

Dedecker, J.; Merlevéde, F. & Rio, E.
Strong Approx. of the empi. distri. function for abs. regular seq. in $\mathbb{R}^d$
Working Paper, 2013.

References

Delgado, M. A. & Escanciano, J. C.
Conditional Stochastic Dominance Testing
Journal of Business Economic Statistics, 2013, 31, 16-28.

Efron, B.
Regression Percentiles using Asymmetric squared error loss,
Statistica Sinica, 1991, 1, 93-125.

Falk, M.
A simple approach to the generation of uniformly distributed random variables with prescribed correlations.
Comm. Statist. - Simulation Comp. 1999, 28, 785-791.

Guo, M. & Härdle, W.
Simultaneous confidence bands for expectile functions,
AStA Advances in Statistical Analysis, 2012, 96, 517-541.

References

Härdle, W.
Asymptotic maximal deviation of M-smoothers
J. Multivariate Anal, 1989, 29, 163 - 179.

Härdle, W. & Song, S.
Confidence Bands in Quantile Regression,
Econometric Theory, 2010, 26, 1180-1200.

Johnston, G. J.
Prob. of max. dev. for nonp. regression function estimates
J. Multivariate Anal, 1982, 12, 402-414

Kong, E.; Linton, O. Xia, Y. Uniform Bahadur representation for
local polynomial estimates of M-reg. and its application to the additive model
Econometric Theory, 2010, 26, 1529-1564

Li, Q., Lin, J. & Racine, J. S.
Optimal Bandwidth Selection for Nonp. Cond. Dist. and Quantile Functions,
Journal of Business & Economic Statistics, 2013, 31, 57-65.

References

Mas-Colell, A., Whinston, M. D. Green, J. R.
Microeconomic Theory
Oxford University Press, USA, 1995.

Meerschaert, M. M., Wang, W. and Xiao, Y.
Fernique-Type Inequalities and Moduli of Continuity for Anisotropic Gaussian Random Fields
Transations of the American Mathematical Society, 2013, 365, 1081-1107

Owen, A. B.
Multidim. Variation for quasi-Monte Carlo
Contemporary Multivariate Analysis and Design of Experiments, World Sci. Publi., 2005, 2, 49-74

References

Rosenblatt, M.
On the Max. Dev. of k-Dimensional Density Estimates,
The Annals of Probability, 1976, 4, 1009-1015.

Smirnov, N. V.
On the construction of confidence regions for the dens. of distri. of random variables,
Doklady Akad. Nauk SSSR, 1950, 74, 189-191 (Russian)