Estimating Expected Shortfall using Expectiles

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Haindorf Seminar 2014

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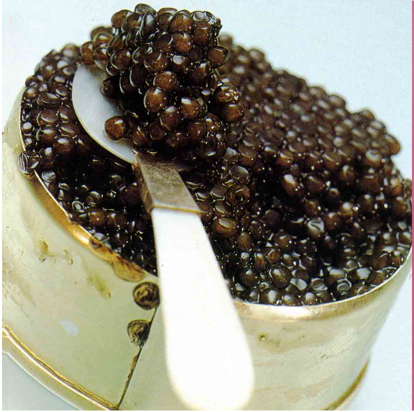



Time
to
Care

CARE history

- Roger Joseph Boscovich (1760) Idea for "median regression" (LS was introduced 1805 by Legendre)
- Koenker and Bassett (1978) Quantile regression (QR)
- Engle and Manganelli (2004) QR to estimate VaR
  - CAViaR (Conditional Autoregressive VaR)
- Aigner, Amemiya and Poirier (1976) Asymmetric LS regression
- Taylor (2008) Expectiles and expected shortfall
  - CARE (Conditional AutoRegressive Expectiles)

ES computation

- Empirical quantiles
- Empirical distributions
- Saddlepoint approximation of distributions
- EVT

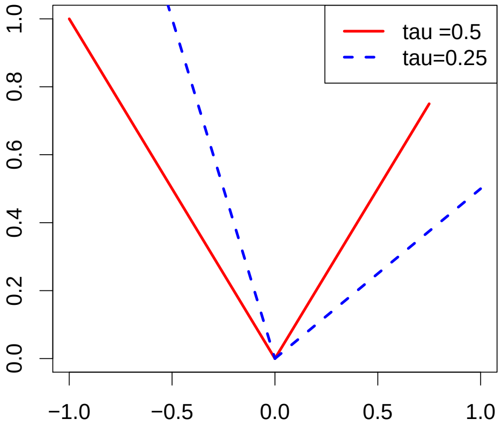
Outline

- Motivation ✓
- Asymmetric regression & expectiles
- Relation of expectiles and expected shortfall
- CARE
- Empirical findings

Notation

Symbol	
$\mu(\tau)$	$\tau-$Expectile
$Q(\theta)$	Theta quantile
$\mathcal{I}_t$	Information set available in t
ES	Expected Shortfall
ρ	Loss function

Quantile regression weights



- RQ model:

$$Q_n(\beta;\theta) = \sum_{i=1}^{n} \rho_\theta(Y_i - X_i^\top \beta) \tag{1}$$

ρ_θ is a convex function

- Loss function $r_\theta(\lambda) = |\theta - \mathbf{I}_{\lambda<0}||\lambda|$

- ALS (Expectile) loss function:

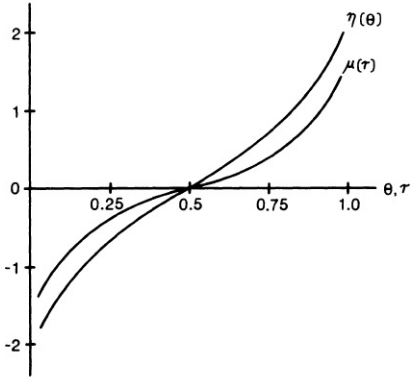
$$\rho_\tau(\lambda) = |\tau - \mathbf{I}_{\lambda<0}|\lambda^2 \tag{2}$$

(Newey & Powell 1987)

Expectiles are Quantiles

- M.C. Jones (1993)
- Expectiles are quantiles (but not of F(y))
- $G(y) = \frac{P(y)-yF(y)}{2(P(y)-yF(y))+(y-\mu)}$ Where P(y) is the partial moment.

Example: Standard Normal



(Newey and Powell 1987)

Relating Expectiles and Expected Shortfall

Newey and Powell 1987:

$$\arg\min_m \mathsf{E}\{|\tau - \mathbf{I}_{\{Y<m\}}|(Y-m)^2\}$$

$$\Rightarrow \frac{1-2\tau}{\tau}\mathsf{E}[\{Y-\mu(\tau)\}\,\mathbf{I}_{\{Y<\mu(\tau)\}}] = \mu(\tau) - \mathsf{E}(Y) \qquad (3)$$

For scalar $\mu(\tau)$ holds, Taylor (2008):

$$\Leftrightarrow \mathsf{E}\{Y|Y<\mu(\tau)\} = \mu(\tau) + \{\mu(\tau) - \mathsf{E}(Y)\}\frac{\tau}{(1-2\tau)F\{\mu(\tau)\}} \qquad (4)$$

Without loss of generality $\mathsf{E}(Y) = 0$

$$\Rightarrow \mathsf{ES} = \mu(\tau) + \mu(\tau)\frac{\tau}{(1-2\tau)\theta} \tag{5}$$

ES estimation needs:

- τ-expectile: $\mu(\tau)$
- θ-quantile that coincides with the τ expectile: $\theta = \mathsf{F}\{\mu(\tau)\}$

If μ depends on the information set $\mathcal{I}_t$:

$$\mathsf{ES}|\theta \propto \mu_t(\tau)$$

CAViaR structure

Engle and Manganelli model quantiles with the following structure (CAViaR):

$$f_t(\beta) = \beta_0 + \sum_{i=1}^{q} \beta_i f_{t-i}(\beta) + \sum_{j=1}^{r} \beta_j l(x_{t-j})$$

l is a function of lagged observables.

The model of Taylor (CARE) assumes a similar dynamic for the expectile process.

Model structures recommended by Engle and Manganelli (CAViaR):

- Symmetric Absolute:

$$Q_t(\theta) = \beta_0 + \beta_1 Q_{t-1}(\theta) + \beta_2 |y_{t-1}| \tag{6}$$

- Asymmetric Slope:

$$Q_t(\theta) = \beta_0 + \beta_1 Q_{t-1}(\theta) + \beta_2 (y_{t-1})^+ + \beta_3 (y_{t-1})^- \tag{7}$$

- Indirect GARCH(1,1) CAViaR, $\beta_i > 0$:

$$Q_t(\theta) = (1 - 2\,\mathbf{I}_{\{\theta<0.5\}})\{\beta_0 + \beta_1 Q_{t-1}(\theta)^2 + \beta_2 y_{t-1}^2\}^{\frac{1}{2}} \tag{8}$$

Where y_t is defined as the residual $r_t - \mathsf{E}(r_t|\mathcal{I}_{t-1})$

Indirect AR(1) GARCH(1,1) CAViaR specification, Kuester, Mittnik and Paolella (2006):

$$
\begin{aligned}
Q_t(\theta) = \alpha_1 r_{t-1} + \{1 - 2\mathbf{I}(\theta < 0.5)\}[\beta_0 + \beta_1\{Q_{t-1}(\theta) \\
-\alpha_1 r_{t-2}\}^2 + \beta_2(r_{t-1} - \alpha_1 r_{t-2}^2)]^{\frac{1}{2}}
\end{aligned}
\tag{9}
$$

The last missing piece

$$
\begin{aligned}
\mathsf{ES} &= \mu(\tau) + \mu(\tau)\frac{\tau}{(1-2\tau)\theta} \\
\Leftrightarrow \mu(\tau) &= \mathsf{ES}\left\{1 + \frac{\tau}{(1-2\tau)\theta}\right\}^{-1}
\end{aligned}
\tag{10}
$$

Plug-into the chosen expectile dynamic

Example: symmetric absolute value CARE

$$\mu_t(\tau) = \beta_0 + \beta_1 \mu_{t-1}(\tau) + \beta_2 |y_{t-1}|$$

$$\Rightarrow \mathrm{ES}_t(\theta) = \gamma_0 + \gamma_1 \mu_{t-1}(\tau) + \gamma_2 |y_{t-1}|$$

with

$$\gamma_i = \left\{ 1 + \frac{\tau}{(1-2\tau)\theta} \right\} \beta_i \text{ for } i \in \{0, 2\} \text{ and } \gamma_1 = \beta_1$$

Advantages of CARE

- Simplicity
- Specific usage of the realized data to estimate ES and not the whole distribution
- Easy to extend, other model's predictions can be implemented as additional regressor

VaR estimation

CARE uses

$$\text{ES}|\theta \propto \mu_t(\tau)$$

$\Rightarrow$ Necessary to estimate θ in a first step.

Find the right τ so that τ percent of the sample lie above $\mu(\tau)$

Numerical solution, still faster than CAViaR

Empirical comparison

- Empirical quantiles ✓
- Empirical distributions ✓
  - GARCH(1,1) & Student-t
  - GARCH(1,1) & Generalized asymmetric Student-t
- Saddlepoint approximation of distributions
- EVT ✓
  - POT (with the parametric residuals, 10% threshold)

Tests

- Dynamic quantile, test Cristofferson 1998
  - The percentages of exceedances should be close to θ (for VaR).
  - Test whether exceedances (minus the threshold θ) are i.i.d. binomial(θ)
- ES testing, McNeil & Frey
  - When threshold is exceeded measure how far it is exceeded
  - Bootstrap test (Efron & Tibshirani, 1993) because CARE does not estimate conditional vola

Data

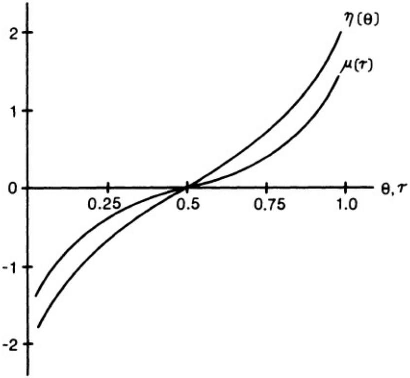
- September 1997 to May 2005
- CAC40
- DAX30
- FTSE100
- Nikkei225
- US SP500

Max rejections = 5

	VaR Hit % Test					VaR DQ Test					ES Bootstrap Test				
	$\theta\,(\times 100)$					$\theta\,(\times 100)$					$\theta\,(\times 100)$				
	1	5	95	99	Total	1	5	95	99	Total	1	5	95	99	Total
CAViaR models															
Sym Abs Value CAViaR	0	0	0	1	1	0	1	0	1	2	-	-	-	-	-
Indirect GARCH CAViaR	0	0	0	0	0	0	0	0	1	1	-	-	-	-	-
Indirect ARGARCH CAViaR	0	1	0	1	2	1	0	0	1	2	-	-	-	-	-
CARE models															
Sym Abs Value CARE	0	1	0	2	3	1	2	0	2	5	0	0	1	0	1
Indirect GARCH CARE	0	1	0	0	1	0	2	1	1	4	0	0	1	1	2
Indirect ARGARCH CARE	1	0	0	2	3	1	1	0	1	3	0	1	1	0	2

(Small is good)

VaR is only a byproduct



	VaR Hit % Test					VaR DQ Test					ES Bootstrap Test				
	$\theta\,(\times 100)$					$\theta\,(\times 100)$					$\theta\,(\times 100)$				
	1	5	95	99	Total	1	5	95	99	Total	1	5	95	99	Total
Benchmark methods															
Hist Sim 1000	1	0	2	4	7	4	5	4	4	17	0	0	3	0	3
Hist Sim 500	0	1	1	1	3	3	5	4	3	15	0	0	1	1	2
Hist Sim 250	0	0	1	0	1	4	4	1	2	11	0	0	0	1	1
GARCH Student-*t*	1	2	5	3	11	1	0	4	1	6	0	0	2	1	3
GARCH Student-*t* EVT	0	0	0	1	1	0	0	0	0	0	0	0	1	1	2
GARCH Skew-*t*	1	0	0	1	2	0	0	2	2	4	0	0	2	1	3
GARCH Skew-*t* EVT	1	0	0	1	2	0	0	2	2	4	0	0	0	1	1
CARE models															
Sym Abs Value CARE	0	1	0	2	3	1	2	0	2	5	0	0	1	0	1
Indirect GARCH CARE	0	1	0	0	1	0	2	1	1	4	0	0	1	1	2
Indirect ARGARCH CARE	1	0	0	2	3	1	1	0	1	3	0	1	1	0	2

Research outlook

- Local estimation of θ
- What are the implications of the tail imlied by the expectiles?
- How does CARE perform in different asset classes?

Thank you for your time and feel free to join me in a game of Kuhhandel later on!

For Further Reading

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