

M-Quantiles and Tail-Behavior

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International Research training group 1792 -
High Dimensional Non Stationary Time Series

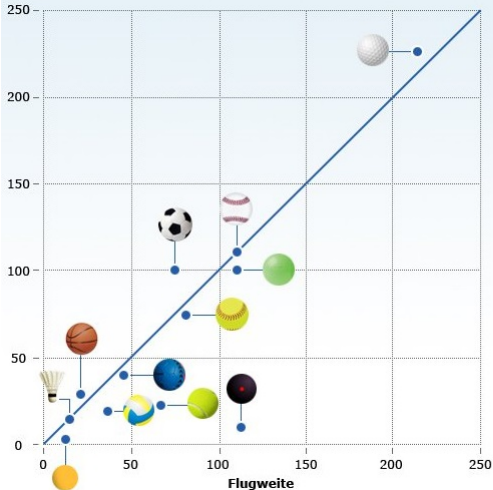
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Spielfeldlänge

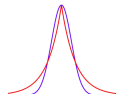


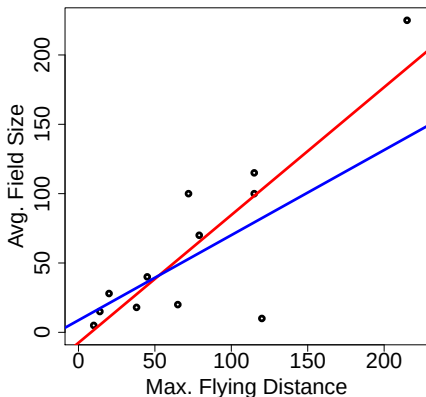
- Badminton
- Baseball
- Basketball
- Fußball
- Golf
- Handball
- Lacrosse
- Tennis
- Tischtennis
- Softball
- Squash
- Volleyball

SPIEGEL ONLINE

Quelle: Texier, et al./IOP Publishing Ltd

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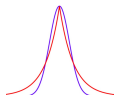
OLS regression of average field size on maximum flying distance

► Model definition and results

Slope full sample = 0.9215

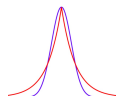
Slope excluding outlier = 0.6137

Figure 1: The effect of tail events



Motivation

- Economics and Finance
 - ▶ Risk Management
 - ▶ Observe 100 observations, find the 99.9% Quantile
- Statistics
 - ▶ Expectiles and Quantiles
 - ▶ Robust estimation
 - ▶ Robustness in ε -Neighborhood



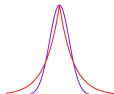
Research Goals

□ M-Quantiles

- ▶ Idea
- ▶ Optimality

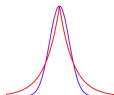
□ Expectiles

- ▶ Overview
- ▶ Relation to Expected Shortfall and Quantiles



Outline

1. Motivation ✓
2. M-Quantiles
3. Expectiles
4. Conclusion



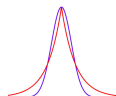
Location Model

$$Y_i = \theta + \varepsilon_i, \quad i = 1, \dots, n$$

$$\varepsilon_i \sim F_\varepsilon$$

(Observe Data $\{y_i\}_{i=1}^n$)

- ▣ M-Estimator, Huber (1964)
- ▣ M-Quantile, Brackling and Chambers (1988)



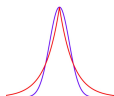
Estimation

$$\theta_n = \arg \min_{\theta} \sum_{i=1}^n \rho(Y_i - \theta)$$

- Different Possibilities for the loss function $\rho(u)$

M-Estimator		M-Quantile	
$\rho(u) = u^2$	Mean	$\rho(u) = \tau - \mathbf{I}\{u < 0\} u^2$	Expectile
$\rho(u) = u $	Median	$\rho(u) = \tau - \mathbf{I}\{u < 0\} u $	Quantile

Table 1: M-Quantiles use asymmetric weights



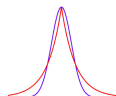
Setup

- Mixture distribution with parameter ε , Huber (1964) and Huber (2006)

$$F_\varepsilon = (1 - \varepsilon)\Phi + \varepsilon H \quad (1)$$

H is an unknown distribution

- ε -Neighborhood



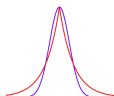
Minimax principle

$$\min_{\theta_n} \left[\sup_{n \rightarrow \infty} \{\text{Var}(\theta_n)\} \right]$$
$$\rho(u)_{opt} = \begin{cases} \frac{1}{2}u^2, & \text{if } |u| < k \\ -k|u| - \frac{1}{2}k^2, & \text{else} \end{cases}$$

Where $k \in [0, \infty)$ and decreasing in ε

$$\frac{2\phi(k)}{k} - 2\Phi(-k) = \frac{\varepsilon}{1 - \varepsilon}$$

ϕ is the standard normal probability density function



Special cases:

- ▶ For $\varepsilon = 1$ (total contamination) the sample median is obtained

$$\left. \frac{\partial \rho(u)_{opt}}{\partial u} \right|_{\varepsilon=1} = -k \cdot \text{Sign}(u)$$

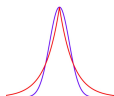
- ▶ For $\varepsilon = 0$ the sample mean is obtained:

$$\left. \frac{\partial \rho(u)_{opt}}{\partial u} \right|_{\varepsilon=0} = u$$

Least favorable density

$$f_0(t) = (1 - \varepsilon)(2\pi)^{-\frac{1}{2}} \exp\{-\rho(u)_{opt}\} \quad (2)$$

- ▶ The worst possible distribution has all contaminating mass outside the interval $[u - k, u + k]$



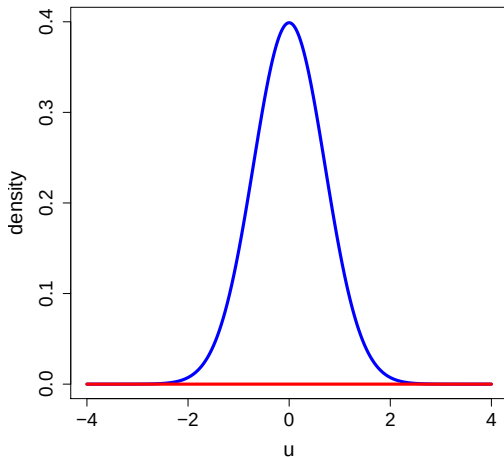
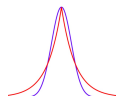


Figure 2: $f_0(u_{opt})$ for $\epsilon = 0$ and $\epsilon = 1$

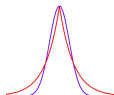


Expectiles

$$\min_{e_\tau} \left\{ (1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau)^2 f(u) du + \tau \int_{e_\tau}^{\infty} (u - e_\tau)^2 f(u) du \right\}$$

$$(1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du + \tau \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du = 0 \quad (3)$$

□ e_τ is the τ -expectile

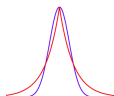


Reformulations

Newey-Powell (1987)

$$e_\tau - E[u] = \frac{(2\tau - 1)}{1 - \tau} \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du \quad (4)$$

► Proof



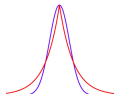
Expectiles are quantiles, Jones (1993)

$$G(e_\tau) = \tau = \frac{\text{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)}{2 \{ \text{LPM}_{e_\tau}(u) - e_\tau F(e_\tau) \} + e_\tau - \mu_u} \quad (5)$$

$$\text{With } \text{LPM}_{e_\tau}(u) = \int_{-\infty}^{e_\tau} uf(u)du$$

$$\text{And } \mu_u = \int_{-\infty}^{\infty} uf(u)du$$

► Proof



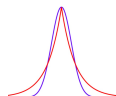
Expectiles and Expected Shortfall

(4) can be rewritten

$$\frac{1-2\tau}{\tau} E[(Y - e_\tau) I\{Y < e_\tau\}] = e_\tau - E[Y]$$

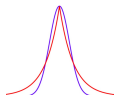
Scalar $e(\tau)$, Taylor (2008)

$$E[Y|Y < e_\tau] = e_\tau + \frac{\{e_\tau - E[Y]\}\tau}{(1-2\tau)F(e_\tau)} \quad (6)$$



Idea

- $\exists$ one to one mapping from e_τ to q_τ , Jones (1993)
- Can relations of $F(u)$ and $G(u)$ be used?
 - ▶ What can be said about the relation of e_τ and q_τ



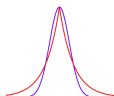
▣ Pareto-like distribution

$$F(x) = 1 - L(x)x^{-\beta}$$

$L(x)$ is slowly varying at infinity:

$$\lim_{x \rightarrow \infty} \frac{L(tx)}{L(x)} = 1 \quad \forall t > 0$$

▣ $\beta = 2$ then $e_\tau = q_\tau$ Koenker (1993)



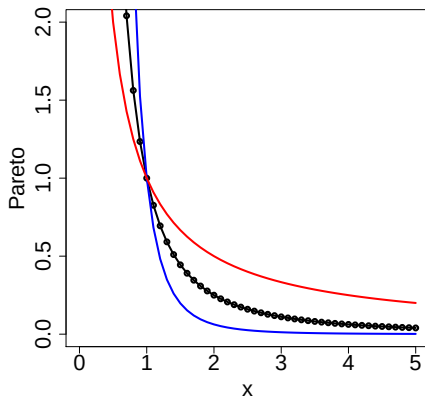
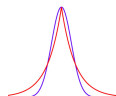


Figure 3: A Pareto-like distribution family, $\beta = 4$, $\beta = 2$ and $\beta = 1$



- ▣ $\beta = 2$ then $e_\tau = q_\tau$ Koenker (1993)
- ▣ For $\tau > 0.5$, Bellini et al. (2014)

Heavy tail	Light tail
$1 < \beta < 2$	$2 < \beta$
$e_\tau > q_\tau$	$e_\tau < q_\tau$

Table 2: Expectile quantile comparison

- ▣ Simulation: $F_\varepsilon = (1 - \varepsilon)\Phi + \varepsilon Par(1, 1)$
 - $Par(\alpha, \beta)$ is the asymmetric Pareto distribution with tail parameter α and skewness parameter β

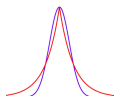
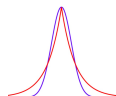


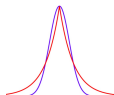
Figure 4: $q(\tau)$, $e(\tau)$ of a Normal-Pareto mixture

M-Quantiles and Tail-Behavior



Conclusion

- Heavy tailed distributions:
 - Expectile is a more conservative risk measure
- But: Other case not unlikely
- Distributional relations arise from (2)
 - ▶ Quantiles - Laplace distribution
 - ▶ Expectiles - Normal Distribution



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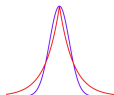
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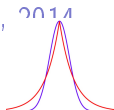
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Insurance: Mathematics and Economics 54, pp. 41-48, 2014



Influence of an extreme observation

Assume the structure: $Y_i = c + X_i\beta_1 + \varepsilon$, $\varepsilon \sim N$

Where $Y = \{(\text{Mean}) \text{ Length of the playing field}\}$

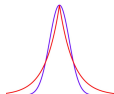
Where $X = \{\text{Maximal flight distance}\}$

Two regressions: Golf included in the sample and excluded (below)

$\hat{\beta}_1$	Standard deviation	95 %Confidence interval
0.9215	0.1874	[0.56, 1.29]
0.6137	0.2620	[0.1, 1.12]

Table 3: Omitting the extreme observation emphasizes it's influence on the estimated slope parameter

Please note that all data was entered via optical approximation from the original graph. [▶ Back](#)



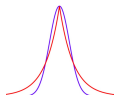
Appendix: Proof of Newey and Powell's (2.7)

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$$\begin{aligned}(1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du + (1 - \tau) \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du = \\ (-\tau) \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du + (1 - \tau) \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du\end{aligned}$$

$$(1 - \tau) \{E(Y) - e_\tau\} = (1 - 2\tau) \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du$$

$$e_\tau - E(Y) = \frac{(2\tau - 1)}{1 - \tau} \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du$$

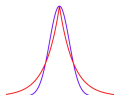


Implicit definition

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$$(1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du = \\ \tau \int_{e_\tau}^{\infty} (e_\tau - u) f(u) du + \tau \int_{-\infty}^{e_\tau} e_\tau f(u) du - \tau \int_{-\infty}^{e_\tau} e_\tau f(u) du$$

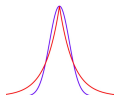
$$(1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du = \\ \tau \int_{-\infty}^{\infty} e_\tau f(u) du - \tau \int_{e_\tau}^{\infty} u f(u) du - \tau \int_{-\infty}^{e_\tau} e_\tau f(u) du$$



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$$(1 - \tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du + \tau \int_{e_\tau}^{\infty} u f(u) du = \\ \tau \int_{-\infty}^{\infty} e_\tau f(u) du - \tau \int_{-\infty}^{e_\tau} e_\tau f(u) du$$

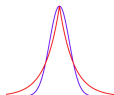
$$(1 - \tau) \int_{-\infty}^{e_\tau} u f(u) du + \tau \int_{e_\tau}^{\infty} u f(u) du = \\ (1 - \tau) \int_{-\infty}^{e_\tau} e_\tau f(u) du + \tau \int_{-\infty}^{\infty} e_\tau f(u) du - \tau \int_{-\infty}^{e_\tau} e_\tau f(u) du$$



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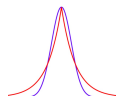
$$(1 - \tau) \int_{-\infty}^{e_\tau} uf(u)du + \tau \int_{e_\tau}^{\infty} uf(u)du = \\ e_\tau \left\{ (1 - \tau) \int_{-\infty}^{e_\tau} f(u)du + \tau \int_{\infty}^{\infty} f(u)du - \tau \int_{-\infty}^{e_\tau} f(u)du \right\}$$

$$\frac{(1 - \tau) \int_{-\infty}^{e_\tau} uf(u)du + \tau \int_{e_\tau}^{\infty} uf(u)du}{(1 - \tau) \int_{-\infty}^{e_\tau} f(u)du + \tau \int_{\infty}^{\infty} f(u)du - \tau \int_{-\infty}^{e_\tau} f(u)du} = e_\tau$$



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$$e_\tau = \frac{(1 - \tau) \int_{-\infty}^{e_\tau} u f(u) du + \mu + \tau \int_{-\infty}^{e_\tau} u f(u) du}{(1 - \tau) F(e_\tau) + \tau \{1 - F(e_\tau)\}}$$



► Back

From (3):

$$\begin{aligned}
 & \tau \left(e_\tau - 2 \int_{-\infty}^{e_\tau} e_\tau f(u) du \right) + \int_{-\infty}^{e_\tau} e_\tau f(u) du = \\
 & \tau \left(\int_{-\infty}^{\infty} u f(u) du - 2 \int_{-\infty}^{e_\tau} u f(u) du \right) + \int_{-\infty}^{e_\tau} u f(u) du \\
 & \tau \left\{ 2 \left(\int_{-\infty}^{e_\tau} u f(u) du - e_\tau \int_{-\infty}^{e_\tau} f(u) du \right) + e_\tau - \mu_u \right\} = \\
 & \int_{-\infty}^{e_\tau} u f(u) du - \int_{-\infty}^{e_\tau} e_\tau f(u) du \\
 & \tau = \frac{\text{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)}{2 \{ \text{LPM}_{e_\tau}(u) - e_\tau F(e_\tau) \} + e_\tau - \mu_u}
 \end{aligned}$$

