M-Quantiles and Tail-Behavior

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International Research training group 1792 -
High Dimensional Non Stationary Time Series

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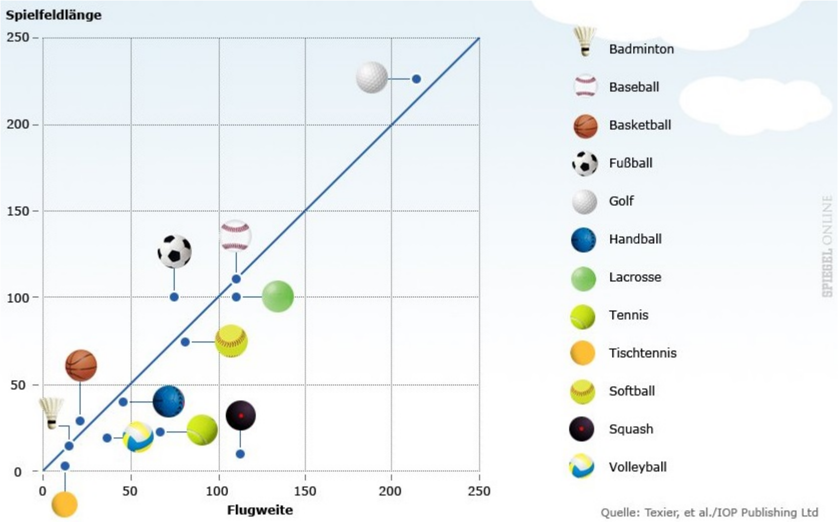

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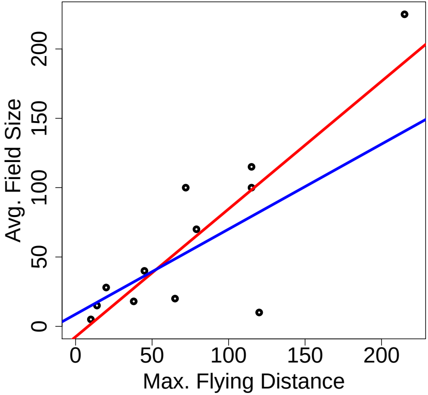

Figure 1: The effect of tail events

OLS regression of average field size on maximum flying distance

Model definition and results

Motivation

- Economics and Finance
  - Risk Management
  - Observe 100 observations, find the 99.9% Quantile
- Statistics
  - Expectiles and Quantiles
  - Robust estimation
  - Robustnes in ε-Neighborhood

Research Goals

- M-Quantiles
  - Idea
  - Optimality
- Expectiles
  - Overview
  - Relation to Expected Shortfall and Quantiles

Outline

1. Motivation ✓
2. M-Quantiles
3. Expectiles
4. Conclusion

Location Model

$$
\begin{aligned}
&Y_i = \theta + \varepsilon_i, \quad i = 1, \ldots, n \\
&\varepsilon_i \sim F_\varepsilon \\
&(\text{Observe Data } \{y_i\}_{i=1}^n)
\end{aligned}
$$

- M-Estimator, Huber (1964)
- M-Quantile, Brackling and Chambers (1988)

Estimation

$$\theta_n = \arg\min_{\theta} \sum_{i=1}^{n} \rho(Y_i - \theta)$$

- Different Possibilities for the loss function $\rho(u)$

M-Estimator		M-Quantile	
$\rho(u) = u^2$	Mean	$\rho(u) = \lvert\tau - \mathbf{I}\{u < 0\}\rvert u^2$	Expectile
$\rho(u) = \lvert u\rvert$	Median	$\rho(u) = \lvert\tau - \mathbf{I}\{u < 0\}\rvert \lvert u\rvert$	Quantile

Table 1: M-Quantiles use asymmetric weights

Setup

- Mixture distribution with parameter ε, Huber (1964) and Huber (2006)

$$F_\varepsilon = (1-\varepsilon)\Phi + \varepsilon H \tag{1}$$

  H is an unkonwn distribution

- ε-Neighborhood

Minimax principle

$$\min_{\theta_n}\left[\sup_{n\to\infty}\{\mathrm{Var}(\theta_n)\}\right]$$

$$\rho(u)_{opt}=\begin{cases}\frac{1}{2}u^2, & \text{if } |u|<k\\ -k|u|-\frac{1}{2}k^2, & \text{else}\end{cases}$$

Where $k\in[0,\infty)$ and decreasing in ε

$$\frac{2\phi(k)}{k}-2\Phi(-k)=\frac{\varepsilon}{1-\varepsilon}$$

ϕ is the standard normal probability density function

- Special cases:
  - For $\varepsilon = 1$ (total contamination) the sample median is obtained

$$\left.\frac{\partial \rho(u)_{opt}}{\partial u}\right|_{\varepsilon=1} = -k \cdot \mathrm{Sign}(u)$$

  - For $\varepsilon = 0$ the sample mean is obtained:

$$\left.\frac{\partial \rho(u)_{opt}}{\partial u}\right|_{\varepsilon=1} = u$$

- Least favorable density

$$f_0(t) = (1-\epsilon)(2\pi)^{-\frac{1}{2}} \exp\{-\rho(u)_{opt}\} \tag{2}$$

  - The worst possible distribution has all contaminating mass outside the interval $[u-k, u+k]$

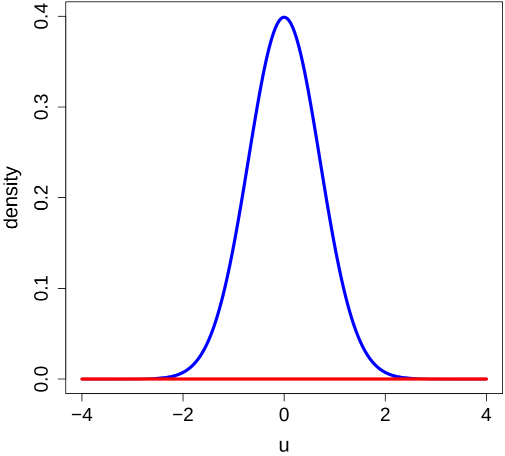

Figure 2: $f_0(u_{opt})$ for $\epsilon = 0$ and $\epsilon = 1$

Expectiles

$$\min_{e_\tau} \left\{ (1-\tau) \int_{-\infty}^{e_\tau} (u - e_\tau)^2 f(u) du + \tau \int_{e_\tau}^{\infty} (u - e_\tau)^2 f(u) du \right\}$$

$$(1-\tau) \int_{-\infty}^{e_\tau} (u - e_\tau) f(u) du + \tau \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du = 0 \tag{3}$$

- e_τ is the τ-expectile

Reformulations

Newey-Powell (1987)

$$e_\tau - \mathsf{E}[u] = \frac{(2\tau - 1)}{1 - \tau} \int_{e_\tau}^{\infty} (u - e_\tau) f(u) du \tag{4}$$

▸ Proof

Expectiles are quantiles, Jones (1993)

$$G(e_\tau) = \tau = \frac{\mathrm{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)}{2\left\{\mathrm{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)\right\} + e_\tau - \mu_u} \tag{5}$$

$$\text{With } \mathrm{LPM}_{e_\tau}(u) = \int_{-\infty}^{e_\tau} u f(u) du$$

$$\text{And } \mu_u = \int_{-\infty}^{\infty} u f(u) du$$

▸ Proof

Expectiles and Expected Shortfall

(4) can be rewritten

$$\frac{1-2\tau}{\tau}\,\mathsf{E}\left[(Y-e_\tau)\,\mathbf{I}\{Y<e_\tau\}\right]=e_\tau-\mathsf{E}[Y]$$

Scalar $e(\tau)$, Taylor (2008)

$$\mathsf{E}[Y|Y<e_\tau]=e_\tau+\frac{\{e_\tau-\mathsf{E}[Y]\}\tau}{(1-2\tau)F(e_\tau)} \tag{6}$$

Idea

- $\exists$ one to one mapping from e_τ to q_τ , Jones (1993)
- Can relations of $F(u)$ and $G(u)$ be used?
  - What can be said about the relation of e_τ and q_τ

- Pareto-like distribution

$$F(x) = 1 - L(x)x^{-\beta}$$

$$L(x) \text{ is slowly varying at infinity:}$$

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 \; \forall t > 0$$

- $\beta = 2$ then $e_\tau = q_\tau$ Koenker (1993)

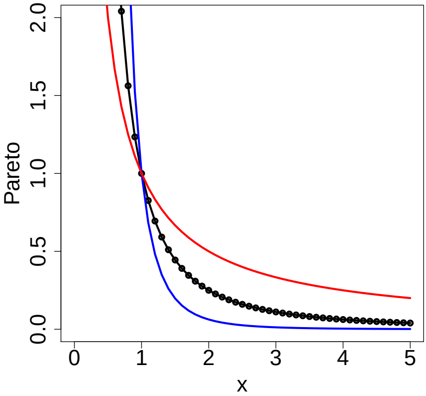

Figure 3: A Pareto-like distribution family, $\beta = 4$, $\beta = 2$ and $\beta = 1$

- $\beta = 2$ then $e_\tau = q_\tau$ Koenker (1993)
- For $\tau > 0.5$, Bellini et al. (2014)

Heavy tail	Light tail
$1 < \beta < 2$	$2 < \beta$
$e_\tau > q_\tau$	$e_\tau < q_\tau$

Table 2: Expectile quantile comparison

- Simulation: $F_\varepsilon = (1-\varepsilon)\Phi + \varepsilon Par(1,1)$
  - $Par(\alpha, \beta)$ is the asymmetric Pareto distribution with tail parameter α and skewness parameter β

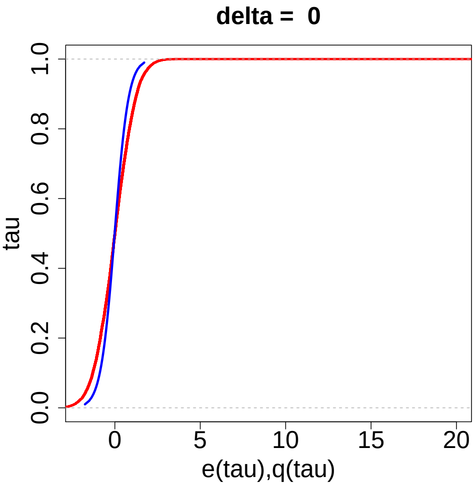

Figure 4: $q(\tau)$, $e(\tau)$ of a Normal-Pareto mixture

Conclusion

- Heavy tailed distributions:

  Expectile is a more conservative risk measure
- But: Other case not unlikely
- Distributional relations arise from (2)
  - Quantiles - Laplace distribution
  - Expectiles - Normal Distribution

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Influence of an extreme observation

Assume the structure: $Y_i = c + X_i\beta_1 + \varepsilon, \quad \varepsilon \sim N$
Where $Y = \{$(Mean) Length of the playing field$\}$
Where $X = \{$Maximal flight distance$\}$

Two regressions: Golf included in the sample and excluded (below)

$\hat{\beta}_1$	Standard deviation	95 %Confidence interval
0.9215	0.1874	$[0.56, 1.29]$
0.6137	0.2620	$[0.1, 1.12]$

Table 3: Omitting the extreme observation emphasizes it's influence on the estimated slope parameter

Please note that all data was entered via optical approximation from the original graph. Back

Appendix: Proof of Newey and Powell's (2.7)

Back

$$
\begin{aligned}
(1-\tau)\int_{-\infty}^{e_\tau}(u-e_\tau)f(u)du + (1-\tau)\int_{e_\tau}^{\infty}(u-e_\tau)f(u)du = \\
(-\tau)\int_{e_\tau}^{\infty}(u-e_\tau)f(u)du + (1-\tau)\int_{e_\tau}^{\infty}(u-e_\tau)f(u)du
\end{aligned}
$$

$$
\begin{aligned}
(1-\tau)\{\mathsf{E}(Y)-e_\tau\} &= (1-2\tau)\int_{e_\tau}^{\infty}(u-e_\tau)f(u)du \\
e_\tau - \mathsf{E}(Y) &= \frac{(2\tau-1)}{1-\tau}\int_{e_\tau}^{\infty}(u-e_\tau)f(u)du
\end{aligned}
$$

Implicit definition

Back

$$(1-\tau)\int_{-\infty}^{e_\tau}(u-e_\tau)f(u)du =$$

$$\tau\int_{e_\tau}^{\infty}(e_\tau-u)f(u)du + \tau\int_{-\infty}^{e_\tau}e_\tau f(u)du - \tau\int_{-\infty}^{e_\tau}e_\tau f(u)du$$

$$(1-\tau)\int_{-\infty}^{e_\tau}(u-e_\tau)f(u)du =$$

$$\tau\int_{\infty}^{\infty}e_\tau f(u)du - \tau\int_{e_\tau}^{\infty}uf(u)du - \tau\int_{-\infty}^{e_\tau}e_\tau f(u)du$$

Back

$$(1-\tau)\int_{-\infty}^{e_\tau}(u-e_\tau)f(u)du+\tau\int_{e_\tau}^{\infty}uf(u)du=$$
$$\tau\int_{\infty}^{\infty}e_\tau f(u)du-\tau\int_{-\infty}^{e_\tau}e_\tau f(u)du$$

$$(1-\tau)\int_{-\infty}^{e_\tau}uf(u)du+\tau\int_{e_\tau}^{\infty}uf(u)du=$$
$$(1-\tau)\int_{-\infty}^{e_\tau}e_\tau f(u)du+\tau\int_{\infty}^{\infty}e_\tau f(u)du-\tau\int_{-\infty}^{e_\tau}e_\tau f(u)du$$

Back

$$(1-\tau)\int_{-\infty}^{e_\tau} uf(u)du + \tau \int_{e_\tau}^{\infty} uf(u)du =$$
$$e_\tau \left\{ (1-\tau)\int_{-\infty}^{e_\tau} f(u)du + \tau \int_{\infty}^{\infty} f(u)du - \tau \int_{-\infty}^{e_\tau} f(u)du \right\}$$

$$\frac{(1-\tau)\int_{-\infty}^{e_\tau} uf(u)du + \tau \int_{e_\tau}^{\infty} uf(u)du}{(1-\tau)\int_{-\infty}^{e_\tau} f(u)du + \tau \int_{\infty}^{\infty} f(u)du - \tau \int_{-\infty}^{e_\tau} f(u)du} = e_\tau$$

Back

$$e_\tau = \frac{(1-\tau)\int_{-\infty}^{e_\tau} uf(u)du + \mu + \tau\int_{-\infty}^{e_\tau} uf(u)du}{(1-\tau)F(e_\tau) + \tau\{1-F(e_\tau)\}}$$

Back

From (3):

$$
\begin{aligned}
\tau\left(e_\tau - 2\int_{-\infty}^{e_\tau} e_\tau f(u)du\right) + \int_{-\infty}^{e_\tau} e_\tau f(u)du =&\\
\tau\left(\int_{-\infty}^{\infty} uf(u)du - 2\int_{-\infty}^{e_\tau} uf(u)du\right) + \int_{-\infty}^{e_\tau} uf(u)du&\\
\tau\left\{2\left(\int_{-\infty}^{e_\tau} uf(u)du - e_\tau\int_{-\infty}^{e_\tau} f(u)du\right) + e_\tau - \mu_u\right\} =&\\
\int_{-\infty}^{e_\tau} uf(u)du - \int_{-\infty}^{e_\tau} e_\tau f(u)du&\\
\tau = \frac{\mathrm{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)}{2\left\{\mathrm{LPM}_{e_\tau}(u) - e_\tau F(e_\tau)\right\} + e_\tau - \mu_u}&
\end{aligned}
$$