TERES - Tail Event Risk Expected Shortfall

Philipp Gschöpf
Wolfgang Karl Härdle
Andrija Mihoci

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics
and Economics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de/
http://case.hu-berlin.de
http://irtg1792.hu-berlin.de

Motivation

Risk Management

- Regulation, Basel II and III
- Quantiles (q_α), VaR_α - not coherent, level α
- Small sample size

▸ Coherence

Quantiles and Tail Risk

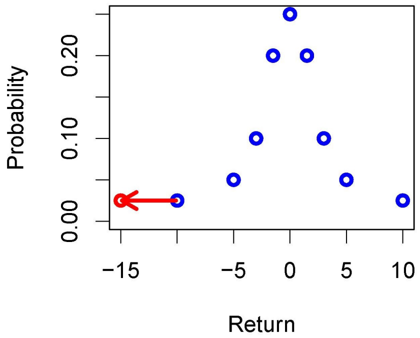

Figure 1: Discrete distribution of returns, $q_{0.05}$ remains unchanged if tail structure changes

Coherent Tail Measuring

- Expectiles
  - Not co-monotonic additive
  - Challenges in risk aggregation
- Expected Shortfall (ES)
  - Small sample size
  - Expectiles connect ES and VaR

Objectives

(i) Expected Shortfall (ES)

- Expectiles, Quantiles
- TERES

(ii) Estimating Expected Shortfall

- Distributional robustness, Huber (1964)
- Lengthening the distribution tails

Example

Expected Shortfall

An investor has a long position in Cisco Inc. (CSCO)

Calculate $ES_{0.01}$ assuming time stationarity

Distribution of de-GARCHed returns

(a) Normal

(b) Laplace

Example

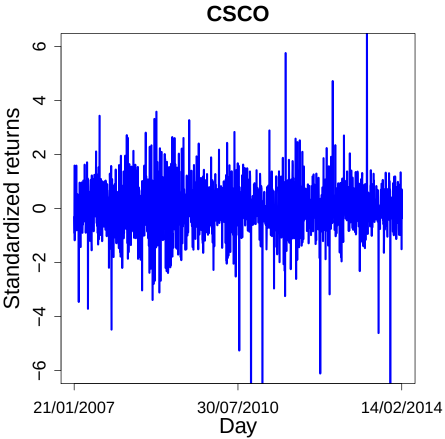

Figure 2: Standardized returns of Cisco Inc. (CSCO)

Example

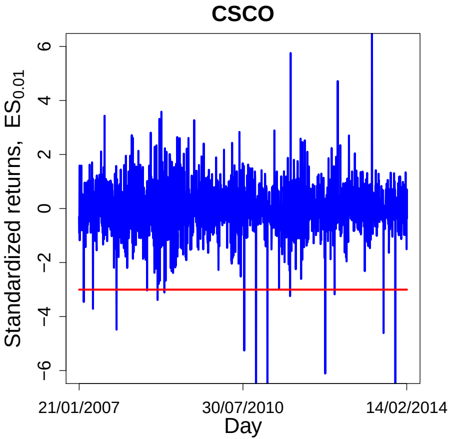

Figure 3: Standardized returns of Cisco Inc. (CSCO),
(a) Normal $ES_{0.01}$(solid)

Example

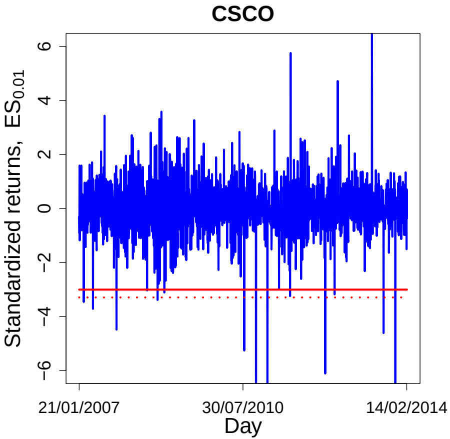

Figure 4: Standardized returns of Cisco Inc. (CSCO),
(b) Laplace $ES_{0.01}$(dashed)

Outline

1. Motivation ✓
2. Expected Shortfall (ES)
3. Distributional Robustness
4. Empirical Results
5. Conclusions

Value at Risk

- Standardized returns Y_i, $i = 1, \ldots, n$

  ▸ Definitions

- Quantile

$$
\begin{aligned}
q_\alpha &= F^{-1}(\alpha), \quad \alpha \in [0,1] \\
&= \arg\min_\theta \mathsf{E}\, \rho_\alpha(Y_i - \theta)
\end{aligned}
$$

$$\rho_\alpha^q(u) = |\alpha - \mathbf{I}\{u < 0\}||u|$$

Value at Risk

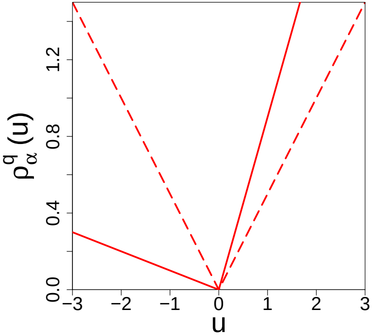

Figure 5: Quantile loss function ρ^q_α.
Solid (dashed) lines depict $\alpha = 0.75$ ($\alpha = 0.50$)

LQRcheck

Expected Shortfall

- Expected shortfall

$$ES_\alpha = \mathsf{E}[Y|Y < q_\alpha]$$

- Expectile

$$e_\alpha = \arg\min_\theta \mathsf{E}\,\rho_\alpha(Y_i - \theta)$$

$$\rho^e_\alpha(u) = \rho_\alpha(u) = |\alpha - \mathbf{I}\{u < 0\}||u|^2$$

M-Quantiles

Expectiles

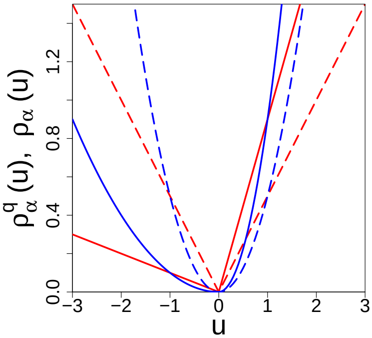

Figure 6: Expectile and quantile loss functions ρ_α and ρ^q_α.
Solid (dashed) lines depict $\alpha = 0.75$ ($\alpha = 0.50$)

Expected Shortfall and Expectiles

- Expectiles
  - Level α, e_α
  - Level τ, $e_\tau = q_\alpha$, $F(q_\alpha) = F(e_\tau) = \alpha$

- Taylor (2008)

$$ES_\alpha = e_\tau + \frac{e_\tau - \mathsf{E}[Y]}{1-2\tau}\frac{\tau}{\alpha} = e_\tau + \frac{(e_\tau - \mathsf{E}[Y])\,\tau}{(1-2\tau)\,F(e_\tau)}$$

Proof

Expectiles and Quantiles

- Jones (1993), Guo and Härdle (2011)

$$\tau(\alpha) = \frac{LPM_Y(q_\alpha) - q_\alpha \alpha}{2\{LPM_Y(q_\alpha) - q_\alpha \alpha\} + q_\alpha - \mathrm{E}[Y]}$$

$$LPM_Y(u) = \int_{-\infty}^{u} y f(y) dy$$

▸ Proofs

Example: $LPM_Y(q_\alpha) = -\varphi(q_\alpha)$ for $\mathrm{N}(0,1)$

Distributional Robustness

- Huber (1964), mixture distribution

$$F_\delta = (1-\delta)\mathrm{N}(0,1) + \delta H$$

  - H is an unknown symmetric distribution
    Example: standard Laplace distribution
  - δ-Neighborhood

Tail Event Risk

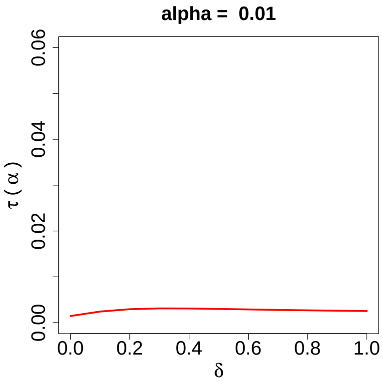

Figure 7: $\tau(\alpha)$ for F_δ

Re-scaled results

Expected Shortfall

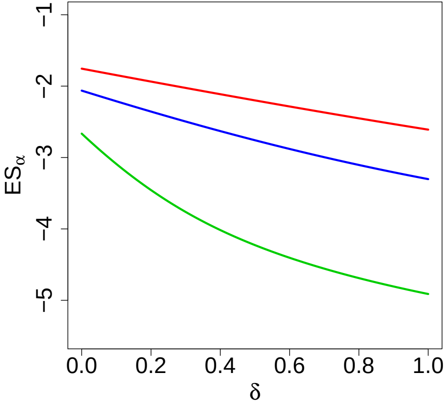

Figure 8: $ES_{0.1}$, $ES_{0.05}$ and $ES_{0.01}$in a δ-neighborhood

Data

- Datastream: Cisco Inc. (CSCO)
- Span: 20070121-20140218 (1748 trading days)
- Standardized daily returns

Data

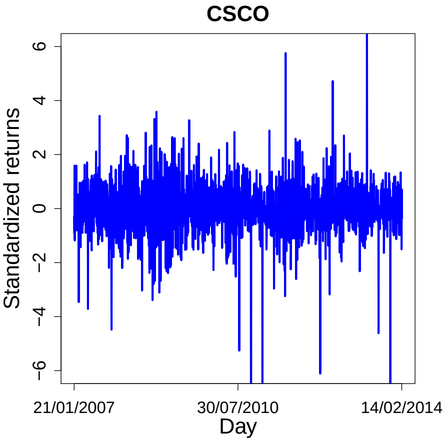

Figure 9: Standardized returns of Cisco Inc. (CSCO)

Expected Shortfall

- Risk level α: 0.01, 0.05 and 0.10
- Sample quantiles $\widehat{q}_\alpha$: -2.62, -1.43 and -1.03
- Contamination level

$$\delta \in \{0, 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.10, 0.15, 0.25, 0.5, 1\}$$

Scaled results

Expected Shortfall

δ	$ES_{0.10}$
0.0	-1.41
0.001	-1.41
0.002	-1.41
0.005	-1.41
0.01	-1.41
0.02	-1.42

δ	$ES_{0.10}$
0.05	-1.43
0.10	-1.45
0.15	-1.48
0.25	-1.52
0.50	-1.60
1.00	-1.67

Table 1: *ES* for Cisco Inc. (CSCO) at $\alpha = 0.10$

Expected Shortfall

δ	$ES_{0.05}$
0.0	-1.79
0.001	-1.79
0.002	-1.79
0.005	-1.79
0.01	-1.80
0.02	-1.81

δ	$ES_{0.05}$
0.05	-1.83
0.10	-1.87
0.15	-1.90
0.25	-1.96
0.50	-2.04
1.00	-2.05

Table 2: *ES* for Cisco Inc. (CSCO) at $\alpha = 0.05$

Expected Shortfall

δ	$ES_{0.01}$
0.0	-3.00
0.001	-3.01
0.002	-3.01
0.005	-3.02
0.01	-3.03
0.02	-3.07

δ	$ES_{0.01}$
0.05	-3.14
0.10	-3.26
0.15	-3.34
0.25	-3.43
0.50	-3.41
1.00	-3.29

Table 3: *ES* for Cisco Inc. (CSCO) at $\alpha = 0.01$

Conclusions

(i) Expected Shortfall (ES)

- M-Quantiles applied successfully to estimate *ES*
- Interaction between α and τ illustrated

(ii) Estimating Expected Shortfall

- Distributional robustness: δ-neighborhood
- TERES: Cisco Inc. (CSCO) - $ES_{0.01}$, $ES_{0.05}$ and $ES_{0.10}$

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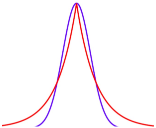

Ladislaus von Bortkiewicz Chair of Statistics
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Definitions

- Log-return r_i of a portfolio, $i = 1, \dots, n$
- Standardized returns with cdf F and pdf f

$$Y_i = \frac{r_i - \mathsf{E}[r_i]}{\sigma_i}$$

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Coherent Risk Measures

- Let $\rho(Y)$ be a risk measure
- Subadditivity, $\rho(< Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2)$
- Translation invariance, $\rho(Y + c) = \rho(Y)$ for constant c
- Monotonicity, $\rho(Y_1) > \rho(Y_2) \quad \forall Y_1 < Y_2$
- Positive homogeneity $\rho(kY) = k\rho(Y) \quad \forall k > 0$

Back

Subadditivity

- $\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2)$
- Diversification never increases risk
- Quantiles are not subadditive
- Expected shortfall is subadditive, Delbaen (1998)

Back

The expectile is defined as

$$\widehat{\theta} = \arg\min_{\theta} \sum_{i=1}^{n} \rho(Y_i - \theta)$$

$$\rho(u) = |\tau - \mathrm{I}\{u < 0\}| u^2$$

Or more generally using

$$\widehat{\theta} = \arg\min_{\theta} \int \rho(Y - \theta)$$

- Quadratic convex problem, F.O.C.

$$(1-\tau)\int_{-\infty}^{s} (y-s)f(y)dy + \tau \int_{s}^{\infty} (y-s)f(y)dy = 0$$

Back

$$
\begin{aligned}
&(1-\tau)\int_{-\infty}^{e_\tau}(y-e_\tau)f(y)dy+(1-\tau)\int_{e_\tau}^{\infty}(y-e_\tau)f(y)dy\\
=&(-\tau)\int_{e_\tau}^{\infty}(y-e_\tau)f(y)dy+(1-\tau)\int_{e_\tau}^{\infty}(y-e_\tau)f(y)dy
\end{aligned}
$$

$$
\begin{aligned}
(1-\tau)\{\mathrm{E}(Y)-e_\tau\} &= (1-2\tau)\int_{e_\tau}^{\infty}(y-e_\tau)f(y)dy\\
e_\tau-\mathrm{E}(Y) &= \frac{(2\tau-1)}{1-\tau}\int_{e_\tau}^{\infty}(y-e_\tau)f(y)dy
\end{aligned}
$$

This result is equal to (2.7) in Newey and Powell (1987)

Back

This is equal to, Taylor (2008)

$$
\begin{aligned}
e_\tau - \mathsf{E}[Y] &= \frac{1-2\tau}{\tau}\,\mathsf{E}\left[(Y-e_\tau)\,\mathbf{I}\{Y>e_\tau\}\right] \\
\mathsf{E}[Y|Y>e_\tau] &= e_\tau + \frac{\tau(e_\tau - \mathsf{E}[Y])}{(1-2\tau)F(e_\tau)} \\
\mathsf{E}[Y|Y>q_\alpha] &= e_\tau + \frac{(e_\tau - \mathsf{E}[Y])\tau}{(1-2\tau)\alpha} \\
&= \mathrm{ES}(e_\tau, \tau|\alpha)
\end{aligned}
$$

▸ Back

M-Quantiles

- Breckling and Chambers (1988), M-Quantiles

$$\theta^{(M)} = \arg\min_{\theta} \mathsf{E}\, \rho_\alpha(Y_i - \theta)$$

$$\rho_\alpha(u) = |\alpha - \mathbf{I}\{u < 0\}||u|^\gamma$$

- Quantile q_α, $\gamma = 1$; Expectile e_α, $\gamma = 2$

▸ Back

M-Quantiles

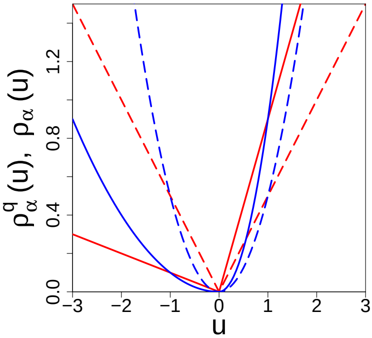

Figure 10: Expectile ($\gamma = 2$) and quantile ($\gamma = 1$) loss functions $\rho_\alpha(u)$. Solid (dashed) lines depict $\alpha = 0.75$ ($\alpha = 0.50$)

LQRcheck

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Generalized Error Distribution

- Let $\kappa > 0$ and $g(x)$ be a symmetric distribution
- An asymmetric distribution $f(x)$ can be obtained as:

$$f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} g(x\kappa) & , 0 \leq x \\ g(\frac{x}{\kappa}) & , \text{ else} \end{cases} \tag{1}$$

- The Generalized Error Distribution (GED, Exponential Power distr.) is defined as

$$g(x|\gamma,\sigma,\theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})} \exp\left\{ - \left| \frac{x-\theta}{\sigma} \right|^{\gamma} \right\} \tag{2}$$

Back

Combining (1) and (2) yields a skew GED:

$$f(x|\gamma,\kappa,\sigma,\theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})}\frac{\kappa}{1+\kappa^2}\exp\left\{-\frac{\kappa^\gamma}{\sigma^\gamma}|x-\theta|_+^\gamma - \frac{1}{\kappa^\gamma\sigma^\gamma}|x-\theta|_-^\gamma\right\}$$

- Parameter
  - γ Shape, $\gamma = 1$ Laplace, $\gamma = 2$ Normal
  - κ Skewness, $\kappa = 1$ is symmetric
  - σ Variance
  - θ Mean

Back

- Part of $-\ln\{f(\cdot)\}$ that depends on x

$$\frac{\kappa^\gamma}{2\sigma^\gamma}\,|x-\theta|^\gamma\,\mathbf{I}\{x-\theta\le 0\}+\frac{1}{2\kappa^\gamma\sigma^\gamma}\,|x-\theta|^\gamma\,\mathbf{I}\{x-\theta<0\}$$

- M-quantile loss function

$$\begin{aligned}\rho(x-\theta) &= |\tau-\mathbf{I}\{x-\theta<0\}||x-\theta|^\gamma\\ &= \tau|x-\theta|^\gamma\mathbf{I}\{x-\theta\le 0\}+(1-\tau)|x-\theta|^\gamma\mathbf{I}\{x-\theta<0\}\end{aligned}$$

- M-Quantile-GED relation: $\tau=\frac{\kappa^\gamma}{2\sigma^\gamma}$

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Back

F.O.C. of M-Quantiles:

$$0 = (1-\tau)\int_{-\infty}^{s}(y-s)f(y)dy + \tau\int_{s}^{\infty}(y-s)f(y)dy$$

Reformulation yields

$$\tau\left(e_\tau - 2\int_{-\infty}^{e_\tau} e_\tau f(y)dy\right) + \int_{-\infty}^{e_\tau} e_\tau f(y)dy$$

$$= \tau\left(\int_{-\infty}^{\infty} yf(y)dy - 2\int_{-\infty}^{e_\tau} yf(y)dy\right) + \int_{-\infty}^{e_\tau} yf(y)dy$$

Back

$$
\begin{aligned}
&\tau \left\{ 2 \left(\int_{-\infty}^{e_\tau} y f(y) dy - e_\tau \int_{-\infty}^{e_\tau} f(y) dy \right) + e_\tau - \mathrm{E}[Y] \right\} \\
&= \int_{-\infty}^{e_\tau} y f(y) dy - \int_{-\infty}^{e_\tau} e_\tau f(y) dy
\end{aligned}
$$

And finally

$$
\tau = \frac{\mathrm{LPM}_{e_\tau}(y) - e_\tau F(e_\tau)}{2 \left\{ \mathrm{LPM}_{e_\tau}(y) - e_\tau F(e_\tau) \right\} + e_\tau - \mathrm{E}[Y]}
$$

Tail Event Risk

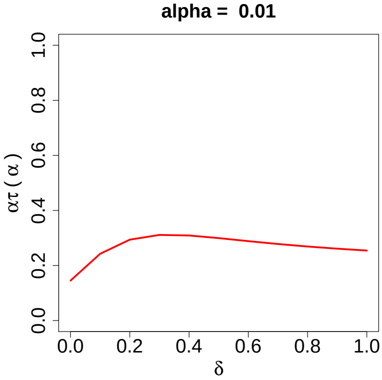

Figure 11: $\alpha\tau(\alpha)$ for F_δ

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Standardization

- $\widehat{\sigma}_i$ from GARCH(1,1)

$$\begin{aligned} y_i =& \beta_0 + \beta_1 y_{i-1} + \varepsilon_i \\ \sigma_i^2 =& \alpha_0 + \alpha_1 \varepsilon_{i-1}^2 + \alpha_2 \sigma_{i-1}^2 \end{aligned}$$

- $\widehat{q}_{0.5}$ is assumed time constant
- $\widehat{Y}_i = \frac{r_i - \widehat{q}_{0.5}}{\widehat{\sigma}_i}$

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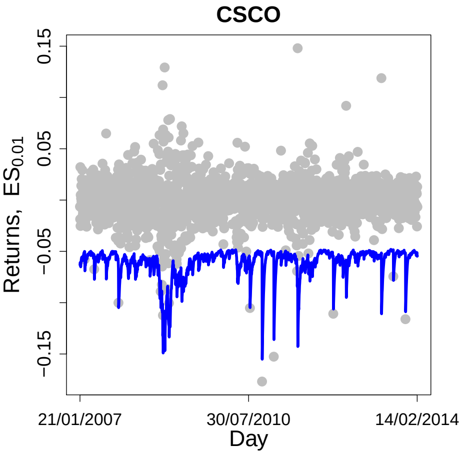
Rescaled Expected Shortfall



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