Robust Inference for Quantile Predictive Regressions with Persistent Predictors

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Predictability of stock returns

- Test efficient market hypothesis
- Check policy effectiveness
- Look for potential investment opportunities
- Foresee financial crisis

Mean Predictive Regression

$$y_t = \beta_0 + \beta_1 x_{t-1} + u_{0t},\ x_t = R_T x_{t-1} + u_{xt},$$
$$R_T = 1 + c/T^{\alpha},\ t = 1, \ldots, T \tag{1}$$

- y_t - stock returns
- x_{t-1} - predictors, e.g. monthly dividend-price ratio (DP), earnings-price ratio (EP), or macroeconomic variables etc.
- highly persistent and even non-stationary predictors, R_T is close to 1
- test the predictability of stock return, $H_0 : \beta_1 = 0$

Two Issues due to Persistent Predictors

- Embedded endogeneity
  - x_{t-1} and u_{0t} may be correlated
  - OLS estimator and its standard error are biased in finite sample
  - t-statistics tends to over-reject H_0

- Unknown degree of persistency
  - non-pivotal statistics, depending on nuisance parameter
  - robust inference theory across different persistence categories is required

persistence categories

Corrections on the Issues

- Embedded endogeneity
  - first-order bias-correction estimator (Stambaugh, 1999)
  - linear projection of u_{0t} onto u_{xt} (Amihud et al., 2009; Cai and Wang, 2014)
- Unknown degree of persistency
  - approximations to finite-sample distributions, construct confidence intervals by Bonferroni's method (Cavanagh et al., 1995; Campbell and Yogo, 2006)
  - IVX filtering (Magdalinos and Phillips, 2009): $\tilde{z}_t = R_{Tz}\tilde{z}_{t-1} + \Delta x_t$, $R_{Tz} = 1 + \frac{c_z}{T^\delta}$
  - include one more redundant lag term x_{t-2} (Ren et al., 2015)

Quantile Predictive Regresson

- Advantages
  - examine the predicability under the entire conditional distribution
  - skewed and heavy tailed errors in financial asset returns
  - VaR in risk management

- Existing works
  - unit root predictor (Xiao, 2009) and nearly integrated case (Maynard et al., 2011)
  - IVX-QR (Lee, 2016)

Our Contributions

- New two-step estimation procedure to deal with the two issues in turn
  - remove the endogeneity in the first step
  - construct a pivotal statistic in the second step
- Consistency and asymptotic normality of the estimator
- Robust inference theory across different persistence types to perform the tests

Outline

1. Motivation ✓
2. Model and Estimation
3. Inference Theory
4. Monte Carlo Simulation
5. Empirical Applications

Model Setting

- Univariate predictive regression model

$$
\begin{aligned}
y_t &= \beta_0 + \beta_1 u_{xt} + \beta_2 x_{t-1} + v_t, t = 1, \ldots, T \\
\mathrm{E}\left(v_t | \mathcal{F}_{t-1}\right) &= 0, \mathrm{E}\left(v_t^2 | \mathcal{F}_{t-1}\right) = \sigma_v^2
\end{aligned} \tag{2}
$$

where $\mathcal{F}_t = \{(v_j, u_{xj})^\top, j \leq t\}$ is a natural filtration.

- Persistent predictor

$$
\begin{aligned}
x_t &= R_T x_{t-1} + u_{xt} \\
R_T &= 1 + c/T^\alpha
\end{aligned} \tag{3}
$$

with $u_{xt} = \sum_{j=0}^{\infty} F_{xj}\varepsilon_{t-j}, \varepsilon_t \sim \text{m.d.s.}(0, \sigma_\varepsilon^2), \mathrm{E}\,|\varepsilon_1|^{2+\nu} < \infty, \forall \nu > 0,$
$F_{x0} = 1, \sum_{j=0}^{\infty} j|F_{xj}| < \infty, F_x(z) = \sum_{j=0}^{\infty} F_{xj} z^j, F_x(1) = \sum_{j=0}^{\infty} F_{xj} > 0$

Two Step Quantile Regression - 2SQR(1)

- Quantile Predictive Regression

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \beta_{0\tau} + \beta_{1\tau}u_{xt} + \beta_{2\tau}x_{t-1}, \tau \in (0,1)$$

  - step1:

$$(\hat{\beta}_{0\tau}, \hat{\beta}_{1\tau}, \hat{\hat{\beta}}_{2\tau})^\top = \arg \min_{\beta_0,\beta_1,\beta_2} \sum_{t=2}^{T} \rho_\tau \left(y_t - \beta_0 - \beta_1 x_t - \beta_2 x_{t-1} \right),$$

    where $\rho_\tau(e) = e\left[\tau - \mathbf{I}\{e < 0\}\right]$.

  - step2:

$$(\hat{\beta}_{2\tau}, \hat{\beta}_{3\tau})^\top = \arg \min_{\beta_2,\beta_3} \sum_{t=3}^{T} \rho_\tau \left(y_{t\tau} - \beta_2 x_{t-1} - \beta_3 x_{t-2} \right),$$

    with $y_{t\tau} = y_t - \hat{\beta}_{0\tau} - \hat{\beta}_{1\tau}\hat{u}_{xt}$, and $\hat{u}_{xt}$ are the errors from OLS on (3).

Two Step Quantile Regression - 2SQR(2)

- Quantile Predictive Regression

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \beta_{0\tau} + \beta_{1\tau} u_{xt} + \beta_{2\tau} x_{t-1}, \tau \in (0,1)$$

- step1:

$$(\hat{\beta}_{0\tau}, \hat{\beta}_{1\tau}, \hat{\hat{\beta}}_{2\tau})^\top = \arg \min_{\beta_0, \beta_1, \beta_2} \sum_{t=2}^{T} \rho_\tau \left(y_t - \beta_0 - \beta_1 x_t - \beta_2 x_{t-1}\right),$$

where $\rho_\tau(e) = e\left[\tau - \mathbf{I}\{e < 0\}\right]$.

- step2:

$$\hat{\beta}_{2\tau} = \arg \min_{\beta_2} \sum_{t=2}^{T} \rho_\tau \left(y_{t\tau} - \beta_2 x_{t-1}\right),$$

with $y_{t\tau} = y_t - \hat{\beta}_{0\tau} - \hat{\beta}_{1\tau}\hat{u}_{xt}$, and $\hat{u}_{xt}$ are the errors from OLS on (3).

Asymptotic Distributions

Theorem 1: Asymptotic distribution of $\hat{\beta}_{2\tau}$ by 2SQR(1)

$$\sqrt{T}\left(\hat{\beta}_{2\tau} - \beta_{2\tau}\right) \xrightarrow{\mathcal{L}} \frac{\sqrt{\tau(1-\tau)}}{f_{v\tau}(0)} \mathrm{N}\left(0, \sigma_{xx}^{-1}\right),$$

where $\sigma_{xx} = \Sigma_{h=-\infty}^{\infty} \mathrm{E}\left(u_{xt} u_{xt-h}\right)$, $f_{v\tau}$ *is the pdf of the residuals in the 2nd step regression.*

Test Statistics

Theorem 2: t-statistics of $\hat{\beta}_{2\tau}$ by 2SQR(1)

Under $H_0 : \beta_{2\tau} = 0$,

$$t = \frac{\sqrt{T}\hat{f}_{v\tau}(0)\hat{\beta}_{2\tau}\hat{\sigma}_{xx}}{\sqrt{\tau(1-\tau)}} \xrightarrow{\mathcal{L}} \mathrm{N}(0,1),$$

where $\hat{f}_{v\tau}(0)$ *and* $\hat{\sigma}_{xx}$ *are consistent estimators of* $f_{v\tau}(0)$ *and* σ_{xx}.

Test Statistics

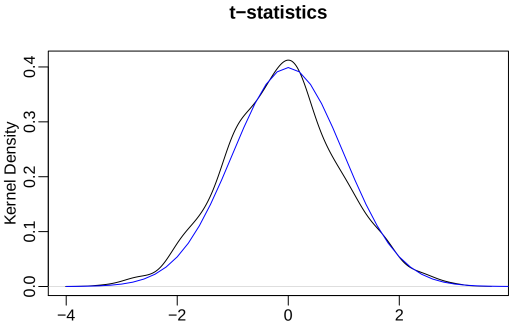

Figure 1: Kernel density of the t-statistics and standard normal pdf, with $R_T = 0.99$, $\tau = 0.1$, $\text{cov}(u_{xt}, u_{0t}) = 0.95$, normal errors.
BQPR_statdensity

Asymptotic Distributions

Theorem 3: Asymptotic distribution of $\hat{\beta}_{2\tau}$ by 2SQR(2)

$$D_T\left(\hat{\beta}_{2\tau}-\beta_{2\tau}\right)\overset{\mathcal{L}}{\to}\begin{cases}\frac{\sqrt{\tau(1-\tau)}}{f_{v\tau}(0)}\,\mathrm{N}\left(0,\sigma_{xx}^{-1}\right), & \textit{for I0;}\\ \frac{\sqrt{\tau(1-\tau)}}{f_{v\tau}(0)}\,\mathrm{N}\left(0,\omega_{xx}^{-1}\right), & \textit{for MI;}\\ \frac{\sqrt{\tau(1-\tau)}}{f_{v\tau}(0)}\,\mathrm{MN}\left(0,\left\{\int_0^\infty \bar{J}_x^c(r)^2dr\right\}^{-1}\right), & \textit{for NI or I1;}\\ \frac{\sqrt{\tau(1-\tau)}}{f_{v\tau}(0)}\,\mathrm{MN}\left(0,\tilde{\omega}_{xx}^{-1}\right), & \textit{for ME;}\end{cases}$$

where $\omega_{xx}=\int_0^\infty \exp(2rc)\sigma_{xx}dr$, $\tilde{\omega}_{xx}=\int_0^\infty \exp(-2rc)Y_c^2dr$, $Y_c\equiv \mathrm{N}\left(0,\omega_{xx}\right)$, $J_x^c(r)=\int_0^r \exp\{(r-s)c\}dB_x(s)$, $\frac{1}{\sqrt{T}}\sum_{j=1}^{[Ts]}u_{xj}\overset{\mathcal{L}}{\to}B_x(s)$,

$$\bar{J}_x^c(r)=J_x^c(r)-\int_0^1 J_x^c(r)dr,\ \textit{and}\ D_T=\begin{cases}\sqrt{T}, & \textit{for I0;}\\ T^{(1+\alpha)/2}, & \textit{for MI;}\\ T, & \textit{for NI or I1;}\\ T^\alpha R_T^T, & \textit{for ME.}\end{cases}$$

Test Statistics

Theorem 4: self normalized statistics of $\hat{\beta}_{2\tau}$ by 2SQR(2)

Under $H_0 : \beta_{2\tau} = 0$,

$$\frac{\sqrt{T}\hat{f}_{v\tau}(0)\hat{\beta}_{2\tau}\sqrt{\sum_{t=1}^{T-1} x_t^2 - \frac{1}{T-1}\left(\sum_{t=1}^{T-1} x_t\right)^2}}{\sqrt{\tau(1-\tau)}} \xrightarrow{\mathcal{L}} \mathrm{N}(0,1),$$

where $\hat{f}_{v\tau}(0)$ *is a consistent estimators of* $f_{v\tau}(0)$.

Note: Theorem 2 and Theorem 4 can be extended to Wald statistics following χ^2 distribution with multiple predictors.

Test Statistics

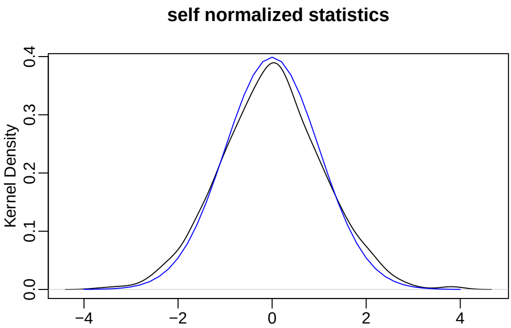

Figure 2: Kernel density of the self normalized statistics and standard normal pdf, with $R_T = 0.95$, $\tau = 0.1$, $\text{cov}(u_{xt}, u_{0t}) = 0.3$, normal errors.
BQPR_statdensity

Power Performance

One NI predictor:

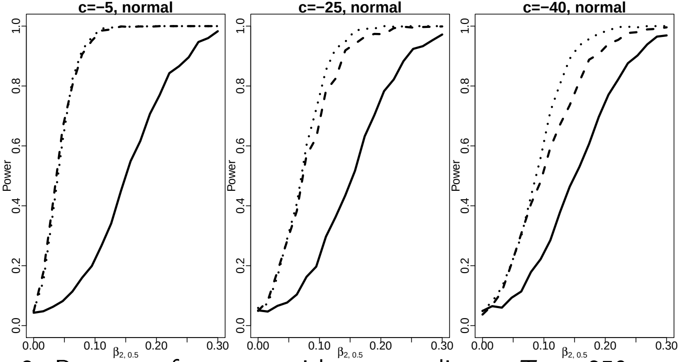

Figure 3: Power performance with one predictor, $T = 250$, $\tau = 0.5$, $\beta_1 = 0.3$, normal errors. Solid - 2SQR(1), dotted - 2SQR(2), dashed - IVX-QR.

BQPR_power

Power Performance

One NI predictor:

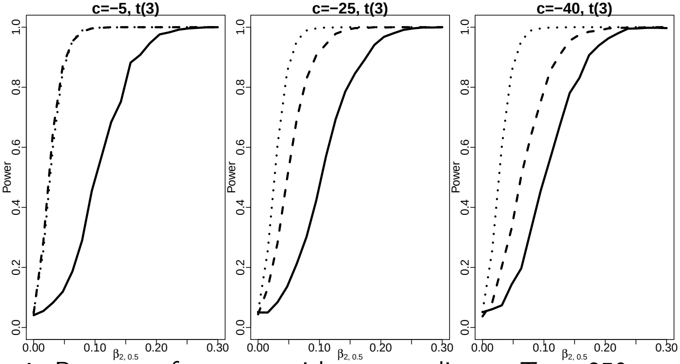

Figure 4: Power performance with one predictor, $T = 250$, $\tau = 0.5$, $\beta_1 = 0.3$, $t(3)$ errors. Solid - 2SQR(1), dotted - 2SQR(2), dashed - IVX-QR.

BQPR_power

Size Performance

With one predictor in more cases:

τ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
2SQR(1)											
I0: $c = -0.1$, $r = 0$	6.8	4.7	4.9	4.0	4.2	3.3	3.3	5.2	4.5	6.4	6.6
MI: $c = -0.1$, $r = 0.9$	6.2	5.0	3.9	4.3	3.6	5.4	4.6	3.8	5.0	5.3	7.8
NI: $c = -0.6$, $r = 1$	7.1	5.5	4.6	4.9	4.2	4.2	4.2	4.1	5.1	4.5	6.8
I1: $c = 0$, $r = 1$	6.9	6.3	4.8	5.0	4.6	3.9	3.9	4.3	4.9	5.2	6.4
ME: $c = 0.3$, $r = 0.6$	5.6	5.3	5.7	6.0	4.5	3.5	3.5	3.4	5.5	6.1	5.1
2SQR(2)											
I0: $c = -0.1$, $r = 0$	8.1	7.5	5.2	4.5	4.8	7.0	4.7	6.2	7.1	7.4	7.7
MI: $c = -0.1$, $r = 0.9$	8.4	6.3	7.3	7.1	4.9	5.8	4.9	6.0	6.2	7.2	10.5
NI: $c = -0.6$, $r = 1$	8.5	8.7	6.1	5.0	5.4	4.4	3.8	4.6	5.3	6.5	7.7
I1: $c = 0$, $r = 1$	7.5	6.5	5.5	6.0	3.9	6.0	4.8	5.1	5.2	7.9	8.0
ME: $c = 0.3$, $r = 0.6$	8.0	7.0	6.8	6.5	5.8	6.0	5.1	4.3	7.2	6.6	8.1
IVX-QR											
I0: $c = -0.1$, $r = 0$	7.8	7.0	3.6	4.6	4.3	4.2	3.6	4.9	4.6	5.4	7.5
MI: $c = -0.1$, $r = 0.9$	16.9	12.8	9.5	8.2	9.3	8.1	7.4	8.0	10.6	13.0	15.8
NI: $c = -0.6$, $r = 1$,	15.1	11.5	8.4	7.0	8.9	7.8	7.7	9.4	8.1	12.4	13.7
I1: $c = 0$, $r = 1$,	16.1	13.3	10.2	9.8	9.1	5.4	7.4	10.0	9.8	14.5	15.0
ME: $c = 0.3$, $r = 0.6$	18.4	15.3	12.9	12.6	12.7	13.0	13.6	12.7	13.8	15.1	17.8

Table 1: Size performance (%) with normal errors and $\beta_1 = 0.3$. BQPR_size

Size Performance

With one predictor in more cases:

τ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
2SQR(1)											
I0: $c = -0.1$, $r = 0$	8.4	5.0	5.6	3.6	3.6	4.6	3.6	6.3	6.1	6.8	7.8
MI: $c = -0.1$, $r = 0.9$	8.0	7.3	5.6	4.6	5.3	2.7	4.2	5.2	6.2	7.5	7.0
NI: $c = -0.6$, $r = 1$	7.0	7.0	4.8	5.5	4.3	3.8	4.2	4.8	6.2	6.7	7.7
I1: $c = 0$, $r = 1$	8.4	5.0	5.6	3.6	3.6	4.6	3.6	6.3	6.1	6.8	7.8
ME: $c = 0.3$, $r = 0.6$	7.8	8.0	6.2	3.9	4.4	3.7	4.2	5.6	6.6	8.1	7.9
2SQR(2)											
I0: $c = -0.1$, $r = 0$	10.8	6.7	6.9	7.2	6.1	5.8	6.4	6.1	6.8	9.6	10.3
MI: $c = -0.1$, $r = 0.9$	10.3	7.9	6.6	5.8	4.8	4.8	3.6	4.5	5.8	8.6	10.7
NI: $c = -0.6$, $r = 1$	9.4	7.4	6.4	6.4	5.3	5.9	3.9	4.1	5.6	8.1	9.8
I1: $c = 0$, $r = 1$	11.6	9.3	6.3	5.4	5.8	5.8	5.0	6.1	8.2	8.0	11.9
ME: $c = 0.3$, $r = 0.6$	9.5	9.8	6.6	6.3	5.2	5.1	6.1	5.9	7.3	8.1	12.8
IVX-QR											
I0: $c = -0.1$, $r = 0$	9.9	8.4	6.1	4.6	5.3	3.0	3.8	4.0	5.9	7.2	9.2
MI: $c = -0.1$, $r = 0.9$	16.8	14.3	12.6	10.2	10.9	7.7	9.9	11.6	11.0	14.2	16.3
NI: $c = -0.6$, $r = 1$	15.4	13.4	11.5	9.6	7.7	8.1	9.7	10.1	10.2	13.4	14.8
I1: $c = 0$, $r = 1$	17.9	15.3	12.1	10.1	8.6	9.7	10.6	11.4	12.3	15.4	16.6
ME: $c = 0.3$, $r = 0.6$	21.8	17.0	15.1	12.3	14.8	10.1	8.9	13.7	17.1	19.1	19.6

Table 2: Size performance (%) with $t(3)$ errors and $\beta_1 = 0.3$.

BQPR_size

Predictability of US Stock Returns

- Same dataset as Lee (2016): monthly data from 1927:01 to 2005:12
- Eight persistent predictors: d/p, e/p, b/m, ntis, d/e, tbl, dfy, tms
- Predict S&P 500 index including dividends minus one month Treasury bill rate

Predictability of US Stock Returns by 2SQR(1)

τ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
d/p	0.72	0.96	0.34	0.99	0.60	0.21	0.06	**0.00**	**0.00**	**0.00**	**0.00**
e/p	0.39	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
b/m	0.34	0.92	0.95	0.99	0.99	0.89	0.61	0.06	0.09	**0.01**	0.20
ntis	0.62	0.27	0.71	0.45	0.35	0.95	0.95	0.79	0.19	0.23	0.16
d/e	0.90	0.80	0.73	0.60	0.92	0.98	0.93	0.89	0.85	0.59	0.99
tbl	0.55	0.58	0.21	0.24	0.29	0.69	0.57	0.17	0.09	0.11	0.08
dfy	0.15	0.11	0.10	**0.04**	0.60	0.78	**0.00**	**0.04**	**0.00**	**0.02**	**0.00**
tms	0.99	0.85	0.10	0.17	0.12	0.24	0.65	0.68	0.46	0.15	0.37

Table 3: *p*-values in hypothesis tests on univariate quantile predictor. Values are in bold if they are less or equal to 0.05. BQPR_predictability

Predictability of US Stock Returns by 2SQR(2)

τ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
d/p	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**	0.10	0.36	0.85	0.75	0.14	**0.00**
e/p	**0.00**	**0.01**	0.23	0.13	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	0.63	0.21
b/m	0.41	0.40	0.29	0.94	0.22	**0.00**	**0.00**	**0.00**	**0.00**	**0.03**	0.59
ntis	**0.04**	**0.02**	**0.01**	0.07	**0.02**	**0.01**	0.13	0.17	0.30	0.73	0.93
d/e	**0.00**	**0.00**	**0.00**	**0.03**	0.50	0.72	0.95	0.09	**0.02**	**0.00**	**0.00**
tbl	0.41	0.76	0.44	0.15	**0.01**	**0.03**	**0.01**	**0.00**	**0.00**	**0.01**	**0.01**
dfy	**0.00**	**0.00**	**0.00**	**0.00**	**0.02**	0.46	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**
tms	0.97	0.51	0.10	0.38	0.17	0.36	0.47	0.83	0.26	**0.00**	**0.00**

Table 4: *p*-values in hypothesis tests on univariate quantile predictor. BQPR_predictability

Predictability of US Stock Returns by 2SQR(2)

τ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
e/p, dfy	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.03**	**0.04**	**0.00**
ntis, dfy	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0.06	0.23	**0.00**	**0.00**	**0.00**	**0.00**
d/e, dfy	**0.00**	**0.00**	**0.00**	**0.00**	0.08	0.76	**0.04**	**0.01**	**0.00**	**0.00**	**0.00**

Table 5: p-values in hypothesis tests on bivariate quantile predictors. BQPR_predictability

VaR Backtesting

	Violation frequency
d/p	0.0787
e/p	0.0694
ntis	0.0586
d/e	0.0540
dfy	0.0602
e/p, dfy	0.0571
ntis, dfy	0.0694
d/e, dfy	0.0648

Table 6: Violation frequency in backtesting on the estimated $VaR_{0.05}$ (one step ahead) by 2SQR(2), in rolling window samples with window width = 300.

BQPR_backtesting

VaR Backtesting

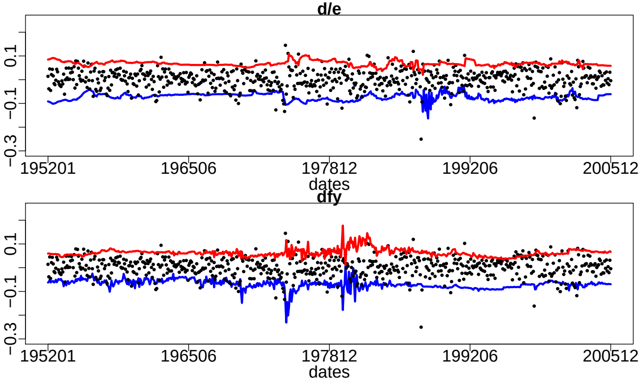

Figure 5: One step ahead estimated $\text{VaR}_{0.05}$, $\text{VaR}_{0.95}$ by 2SQR(2) with univariate predictor, and real observations. BQPR_backtesting

VaR Backtesting

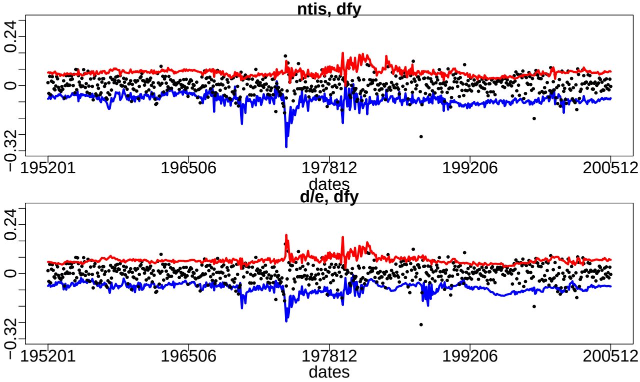

Figure 6: One step ahead estimated $VaR_{0.05}$, $VaR_{0.95}$ by 2SQR(2) with bivariate predictors, and real observations. BQPR_backtesting

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Persistence Categories

- (I0) Stationary: $\alpha = 0, |1 + c| < 1$
- (MI) Mildly integrated: $0 < \alpha < 1, c < 0$
- (NI) Nearly integrated: $\alpha = 1, c < 0$
- (I1) Unit root: $c = 0$
- (ME) Mildly explosive: $0 < \alpha < 1, c > 0$

Return