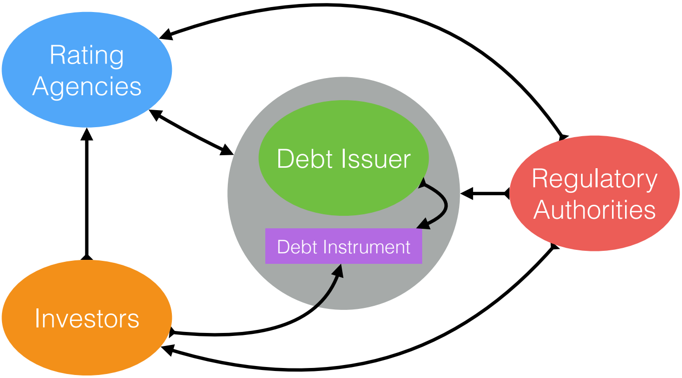
DFINC: Dynamic Forward INtensity Curves

Wolfgang Karl Härdle
Felix Jung
Dedy Dwi Prastyo

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics
and Economics
Humboldt-Universität zu Berlin
lvb.wiwi.hu-berlin.de
case.hu-berlin.de

Default risk

▸ Literature

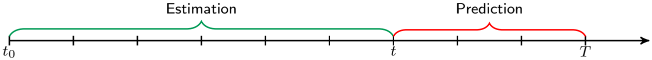


Duration analysis and defaults

- Model duration until default
- Stochastic counting process N_t
- Time of default τ: first jump time of N_t
- Mean arrival rate of jumps (intensity): λ_t
- Survival and default probabilities

$$S_t(T) = \mathsf{P}_t(\tau > T) = \mathsf{P}_t(N_T < 1) \tag{1}$$

$$PD_t(T) = \mathsf{P}_t(t < \tau \leq T) = \mathsf{P}_t(N_T - N_t \geq 1) \tag{2}$$



Counting via a Poisson Process

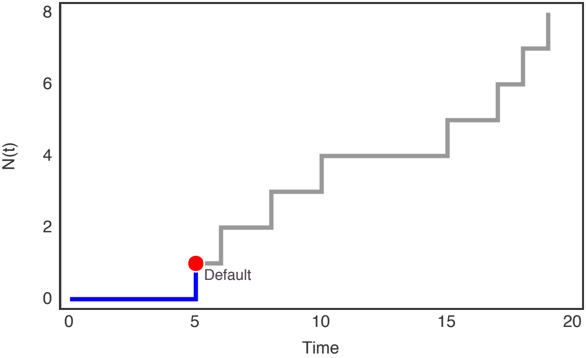


Figure 1: Homegeneous Poisson process. Default occurs at $\tau = 5$.

Standard reduced form model

▸ Duffie et al. (2007)

- N_t: doubly stochastic Poisson process
- Time-t-conditional survival and default probabilities:

$$S_t(T) = P_t(\tau > T) = E_t\left\{\exp\left(-\int_t^T \lambda_s\, ds\right)\right\} \tag{3}$$

$$PD_t(T) = P_t(t < \tau \leq T) = E_t\left\{\int_t^T \exp\left(-\int_t^s \lambda_u\, du\right)\lambda_s\, ds\right\} \tag{4}$$

- Two possibilities of computing (3) and (4)
  - λ_t affine/quadratic specification: closed form solution
  - Otherwise: simulation of λ_t

Forward intensity approach

▸ Derivation ▸ Duan et al. (2012)

- Same starting point as standard model
- Intensities $\lambda_s, s > t$ accessible through *forward* counterparts $\lambda_t(s)$
- Time-t-conditional survival and default probabilities:

$$S_t(T) = \exp\left[-\int_t^T \lambda_t(s)\,ds\right] \tag{5}$$

$$PD_t(T) = \int_t^T \exp\left[-\int_t^s \lambda_t(u)\,du\right] \lambda_t(s)\,ds \tag{6}$$

- Probabilities directly computable

Forward intensities

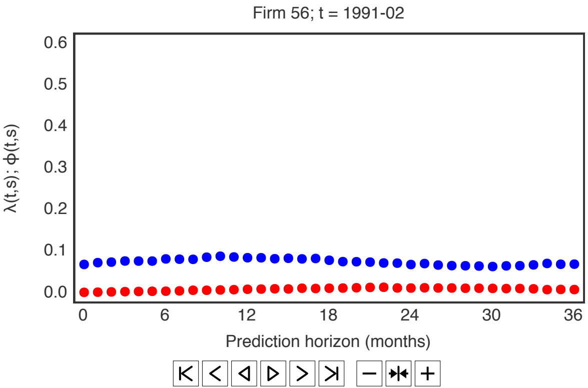


Figure 2: Estimated default and other-exit forward intensities for $s \in [0, \ldots, 36]$.

Default forward intensity curves

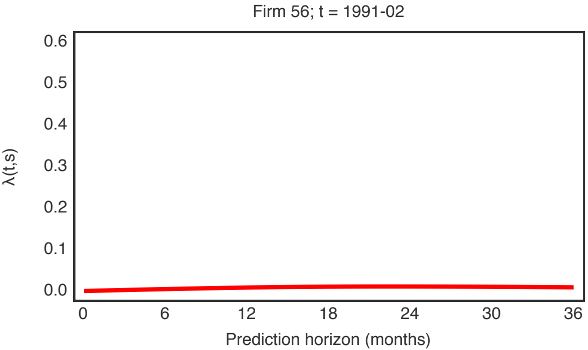


Figure 3: Default forward intensity curves.

Research goals

- Parsimonious forward intensity model via dynamic curves
- Curve functions depending on firm characteristics

Challenges

- Intensities are latent
- Curve model parameters determined by survival data alone
- Intensities must be positive

Outline

1. Motivation ✓
2. Modelling Approach
3. Estimation
4. Outlook

Basic forward intensity model

- Market exit scenarios: firm default, other reasons (i.e. takeover)
- Necessary model adjustments
  - Time of default τ_D: process M_t with intensity λ_t
  - Time of other exit τ_O: process L_t with intensity ϕ_t
  - Combined exit time τ_C: $\min(\tau_D, \tau_O)$

$$\mathrm{S}_t(T) = \exp\left[-\int_t^T \{\lambda_t(s) + \phi_t(s)\}\, ds\right] \tag{7}$$

$$\mathrm{PD}_t(T) = \int_t^T \exp\left[-\int_t^s \{\lambda_t(u) + \phi_t(u)\}\, du\right] \lambda_t(s)\, ds \tag{8}$$

Forward intensity curve

Details

- Various dynamic curve models applicable
- Choice of dynamic Nelson-Siegel (DNS) model

$$\begin{aligned} \lambda_t(s) = \; & [\alpha_{t1} + \alpha_{t2} \exp\{-(s-t)\,\gamma_t\} \\ & + \alpha_{t3}\,\gamma_t\,(s-t)\,\exp\{-(s-t)\,\gamma_t\}]^2 \end{aligned} \tag{9}$$

$$\begin{aligned} \phi_t(s) = \; & [\beta_{t1} + \beta_{t2} \exp\{-(s-t)\,\delta_t\} \\ & + \beta_{t3}\,\delta_t\,(s-t)\,\exp\{-(s-t)\,\delta_t\}]^2 \end{aligned} \tag{10}$$

Forward default intensity curves

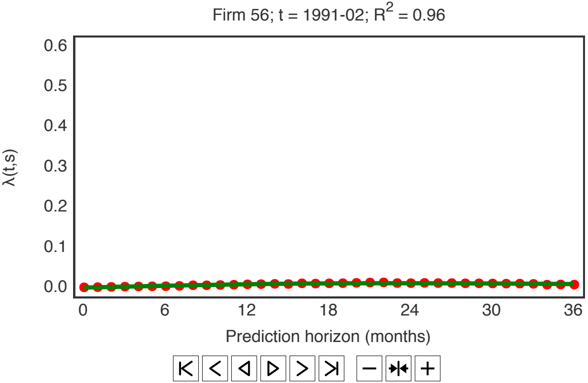


Figure 4: Forward default intensity curves and intensities. DNS fitted to (positive) square root intensities.

Forward other-exit intensity curves

Details

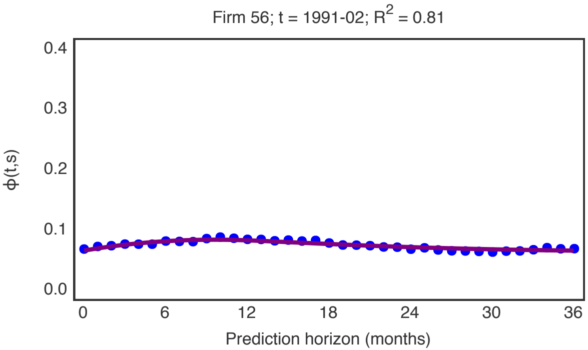


Figure 5: Forward other-exit intensity curves and intensities. DNS fitted to (positive) square root intensities.

Overall model fit

- Fit (9) and (10) to forward intensities of Duan et al. (2012)

Table 1: Descriptive statistics for R^2 across sample (time *and* firms).

	Mean	SD	Min	$q_{0.25}$	Median	$q_{0.75}$	Max
Default	0.86	0.16	0.02	0.83	0.92	0.97	1.00
Other-exit	0.69	0.21	0.01	0.56	0.75	0.85	0.99

Inclusion of observable covariates

- x_t: k macro and m firm-specific covariates
- Assumption: α_t, β_t affine, γ_t, δ_t exp-affine in x_t

$$\alpha_t = A \begin{pmatrix} 1 \\ x_t \end{pmatrix}, \qquad \beta_t = B \begin{pmatrix} 1 \\ x_t \end{pmatrix}$$

$$\gamma_t = \exp\left\{ c^\top \begin{pmatrix} 1 \\ x_t \end{pmatrix} \right\}, \qquad \delta_t = \exp\left\{ d^\top \begin{pmatrix} 1 \\ x_t \end{pmatrix} \right\}$$

- A, B: $3 \times (k+m+1)$ matrices
- c and d: $(k+m+1) \times 1$ vectors
- A, B, c, d: constant across firms and over time

Model properties

- Entire term structure of intensities given at any time t
  - Distribution of default times computable
- Dependence (only) implicit through macro covariates
- Potentially large number of parameters
- No dynamic dependence structure between intensities and covariates

Sample firm data

Sub-sample of Duan et al. (2012) data

- Size: 2000 U.S. firms
- Period: Feb 1991 to Dec 2011 (251 months)
- Frequency: monthly
- Defaults: 168
- Other exits: 1334

Active firms

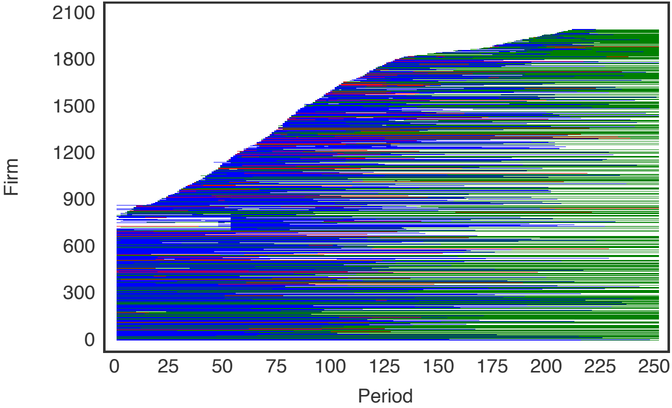


Figure 6: Active firms in sample over the sample period.

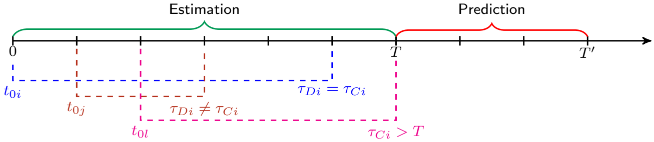
Macro and firm-specific covariates

Covariate	Explanation
*S&P*500	Trailing 1-year return
InterestRate	3-month US Treasury bill rate
DTD	Distance-to-default
CASH/TA	Ratio of the sum of cash and short-term investments to total assets
NI/TA	Ratio of net income to total assets
Size	Logarithm of the ratio of a firm's market equity value to the S& P500 average
M/B	Market-to-book asset ratio
Sigma	1-year idiosyncratic volatility

Table 2: Macroeconomic and firm-specific covariates. For DTD, CASH/TA, NI/TA, and Size both the level and trend are considered.

Likelihood notation

- Sample of N firms
- Overall sample period in months: $[0, T]$
- First observation for firm i: t_{0i}
- Month firm i exits sample: τ_{Ci}/τ_{Di}
- Prediction period: $(T, T']$



Pseudo-log-likelihood

- Discrete observations

$$\mathcal{L}\left\{A, B, c, d; \tau_C, \tau_D, X\right\} = \sum_{t=1}^{T-1} \sum_{i=1}^{N} \ell_{it}\left(A, B, c, d; \tau_{Ci}, \tau_{Di}, x_{it}\right) \quad (11)$$

- In the following: $\mathcal{L}\left\{A, B, c, d\right\}$ and $\ell_{it}\left(A, B, c, d\right)$
- Decomposed pseudo-log-likelihood

$$\mathcal{L}\left\{A, B, c, d\right\} = \sum_{t=1}^{T-1} \sum_{i=1}^{N} \left\{\ell_{it}^{\lambda}\left(A, c\right) + \ell_{it}^{\phi}\left(B, d\right)\right\} \quad (12)$$

Firm-specific log-likelihoods for $t_{0i} \leq t$

$$
\begin{aligned}
\ell_{it}^{\lambda}(A,c) = & -\mathbf{1}(\tau_{Ci} > T) \int_0^{T-t} \lambda_t(s)ds \\
& + \mathbf{1}(\tau_{Di} = \tau_{Ci} = t+1) \log\left[1 - \exp\left\{-\int_0^1 \lambda_t(s)ds\right\}\right] \\
& - \mathbf{1}(\tau_{Di} = \tau_{Ci}, t+1 < \tau_{Ci} \leq T) \int_0^{\tau_{Di}-t-1} \lambda_t(s)ds \\
& + \mathbf{1}(\tau_{Di} = \tau_{Ci}, t+1 < \tau_{Ci} \leq T) \log\left[1 - \exp\left\{-\int_{\tau_{Di}-t-1}^{\tau_{Di}-t} \lambda_t(s)ds\right\}\right] \\
& - \mathbf{1}(\tau_{Di} \neq \tau_{Ci} \geq T) \int_0^{\tau_{Ci}-t} \lambda_t(s)ds
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\ell_{it}^{\phi}(B,d) = & -\mathbf{1}(\tau_{Ci} > T)\int_0^{T-t}\phi_t(s)ds \\
& + \mathbf{1}(\tau_{Di} \neq \tau_{Ci} = t+1)\log\left[1-\exp\left\{-\int_0^1 \phi_t(s)ds\right\}\right] \\
& - \mathbf{1}(\tau_{Di} \neq \tau_{Ci}, t+1 < \tau_{Ci} \leq T)\int_0^{\tau_{Ci}-t-1}\phi_t(s)ds \\
& + \mathbf{1}(\tau_{Di} \neq \tau_{Ci}, t+1 < \tau_{Ci} \leq T)\log\left[1-\exp\left\{-\int_{\tau_{Ci}-t-1}^{\tau_{Ci}-t}\phi_t(s)ds\right\}\right] \\
& - \mathbf{1}(\tau_{Di} = \tau_{Ci}, t+1 < \tau_{Ci} \leq T)\int_0^{\tau_{Di}-t-1}\phi_t(s)ds
\end{aligned} \tag{14}
$$

Estimators and algorithm

- Decomposability: separate estimation

$$(\hat{A}, \hat{c}) = \arg\max_{A,c} \sum_{t=0}^{T-1} \sum_{i=1}^{N} \ell_{it}^{\lambda}(A, c) \tag{15}$$

$$(\hat{B}, \hat{d}) = \arg\max_{B,d} \sum_{t=0}^{T-1} \sum_{i=1}^{N} \ell_{it}^{\phi}(B, d) \tag{16}$$

- Pseudo-log-likelihood, gradient, and Hessian: closed form
- Algorithm options
  1. Treat c and d as hyper parameters
  2. Sequential estimation of A_{1j} and B_{1j}, $j = 1, 2, 3$
  3. Sequential estimation with increasing number of covariates
  4. Combination of above options

Roadmap

1. Implement log-likelihood and its gradient, Hessian in Julia
2. Choose best estimation algorithm
3. Evaluate default prediction performance
4. Finish project
5. Apply approach in CDS pricing

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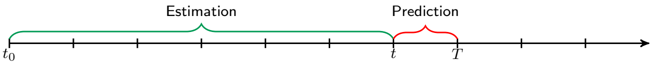
Prediction approaches

Back

- **Early empirical work**
  Discriminant analysis, static logit models, single-period prediction

  Altman (1968), Beaver (1966, 1968), and Ohlson (1980)
- **Duration analysis: single-period prediction**
  Proportional hazards models, dynamic logit models

  Campbell et al. (2008), Lee et al. (1996), and Shumway (2001)

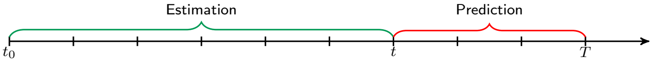


Prediction approaches

Back

- **Duration analysis: multi-period prediction**
  Duration analysis, default *and* other exit possibilities

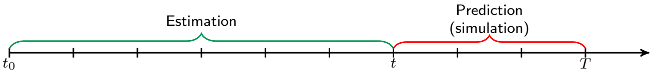
  Duan et al. (2013, 2012), Duffie et al. (2007), Orth (2013), and Prastyo et al. (2014)



Duffie et al. (2007)

Back

- Specify $\lambda_t = \exp(c + \alpha^\top u_t + \beta^\top y_t)$
  - Macro factors: $u_t = (u_{1t}, \dots, u_{kt})^\top$
  - Firm-specific factors: $y_t = (y_{1t}, \dots, y_{mt})^\top$
  - Intercept c, α, β constant over time and across firms
- Simulation of λ_t necessary for multi-period prediction
- Time series models for u_t and y_t required
- Models for u_t and y_t estimated for each company ($n \approx 3000$)



Duan et al. (2012)

Back

- Model forward intensity directly,

$$\lambda_t(s) = \exp\{c + \alpha(s)^\top u_t + \beta(s)^\top y_t\}$$

- Covariates u_t, and y_t equivalent to Duffie et al. (2007)
- No need for time series model of $(u_t, y_t)^\top$
- Loadings $\alpha(s)$ and $\beta(s)$ depend only on prediction horizon
- Direct estimation of $\alpha(s)$ and $\beta(s)$ via qML
- Parameters: $(1 + m + k) \cdot 2 \cdot 37$

Forward intensities

Back

Definition

$F_t(s)$: time-t conditional cdf of τ at $s > t$. Assume $F_t(s)$ as differentiable, then the forward intensity $\lambda_t(s)$ is

$$\lambda_t(s) = \frac{F_t'(s)}{1 - F_t(s)}. \tag{17}$$

- $S_t(T)$ and $PD_t(T)$ directly computable from $\lambda_t(s)$:

$$S_t(T) = \exp\left\{-\int_t^T \lambda_t(s)\,ds\right\} \tag{18}$$

$$PD_t(T) = \int_t^T \exp\left\{-\int_t^s \lambda_t(u)\,du\right\} \lambda_t(s)\,ds \tag{19}$$

Derivation

Back

- Define

$$\overline{\lambda}_t(s) = -\frac{\log\{1 - F_t(s)\}}{s - t} \tag{20}$$

$$= -\frac{\log \mathsf{E}_t \left\{\exp\left(-\int_t^s \lambda_u \, du\right)\right\}}{s - t} \tag{21}$$

- Survival probability over $[t, s]$:

$$P_t(\tau > s) = \exp\left\{-\overline{\lambda}_t(s)\,(s - t)\right\} \tag{22}$$

▸ Back

- Assume $\overline{\lambda}_t(s)$ differentiable, then

$$\lambda_t(s) \equiv \frac{F_t'(s)}{1 - F_t(s)}$$
$$= \overline{\lambda}_t(s) + \overline{\lambda}_t'(s)\,(s - t) \tag{23}$$

- Hence,

$$\exp\{-\lambda_t(s)\,(s - t)\} = \exp\left\{-\int_t^s \lambda_t(u)\,du\right\} \tag{24}$$

Censored forward intensities

Back

- Define

$$\overline{\kappa}_t(s) = -\frac{\log \mathrm{E}_t \left\{ \exp\left(-\int_t^s \lambda_u + \phi_u \, du\right) \right\}}{s-t} \tag{25}$$

- Forward default intensity censored by other forms of exit:

$$\lambda_t(s) = e^{\overline{\kappa}_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{P_t(s < \tau_D = \tau_C \leq s + \Delta t)}{\Delta t} \tag{26}$$

$$= e^{\overline{\kappa}_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{\mathrm{E}_t \left\{ \int_s^{s+\Delta t} \exp\left(-\int_t^u \lambda_z + \phi_z \, dz\right) \lambda_u \, du \right\}}{\Delta t} \tag{27}$$

- Forward intensity of other forms of exit censored by default:

$$\phi_t(s) = e^{\overline{\kappa}_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{P_t(s < \tau_{Oi} = \tau_C \leq s + \Delta t)}{\Delta t} \tag{28}$$

$$= e^{\overline{\kappa}_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{\mathrm{E}_t \left\{ \int_s^{s+\Delta t} \exp\left(-\int_t^u \lambda_z + \phi_z \, dz\right) \phi_u \, du \right\}}{\Delta t} \tag{29}$$

- Default probability over $[t, T]$

$$P_t(\tau_D \leq T) = \int_t^T \exp\left\{-\overline{\kappa}_t(s)\,(s-t)\right\} \lambda_t(s) ds \tag{30}$$

- Other-exit probability over $[t, T]$

$$P_t(\tau_O \leq T) = \int_t^T \exp\left\{-\overline{\kappa}_t(s)\,(s-t)\right\} \phi_t(s) ds \tag{31}$$

Dynamic Nelson-Siegel model

Back

- Dynamic version of Nelson et al. (1987) yield curve model by Diebold et al. (2006)
- Curve dynamics driven by three latent factors:
  - *level*: L_t
  - *slope*: S_t
  - *curvature*: C_t
- Latent factors identified as the first three principal components of yields

Back

- Spot yield curve given by

$$y_t(T) = L_t + S_t \left[\frac{1 - \exp\{-(T-t)\,\delta\}}{(T-t)\,\delta} \right] + C_t \left[\frac{1 - \exp\{-(T-t)\,\delta\}}{(T-t)\,\delta} - \exp\{-(T-t)\,\delta\} \right], \tag{32}$$

where δ is called *decay* factor and T is the maturity

- The forward curve is given by

$$F_t(T) = y_t(T) + y_t'(T)\,(T-t) \tag{33}$$

$$= L_t + S_t \exp\{-(T-t)\,\delta\} + C_t\,\delta\,(T-t)\exp\{-(T-t)\,\delta\}. \tag{34}$$

Forward other-exit intensity curves

Back

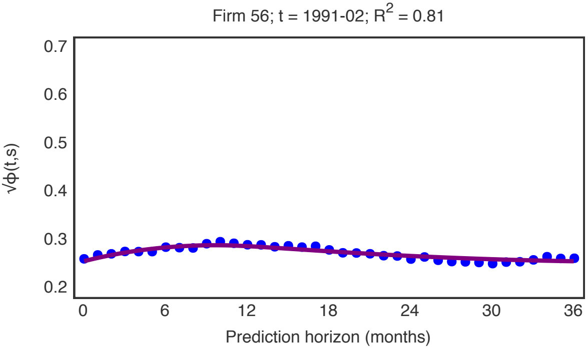


Figure 7: Dynamic Nelson-Siegel curves fitted to (positive) square root forward other-exit intensities.

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