t-Copula Based Factor Model for Credit Risk Analysis

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Systematic Risk

Figure 1: Credit Risk depends the state of economy.

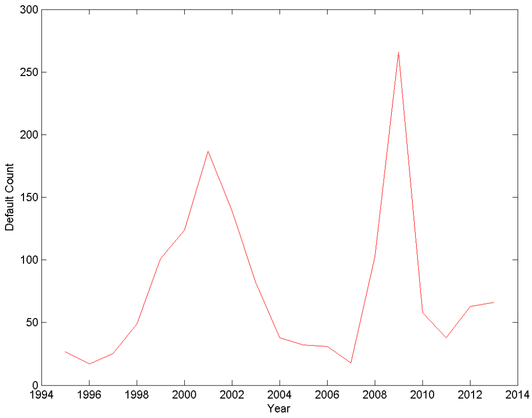


Figure 2: Annual Default Counts from 1995-2013.

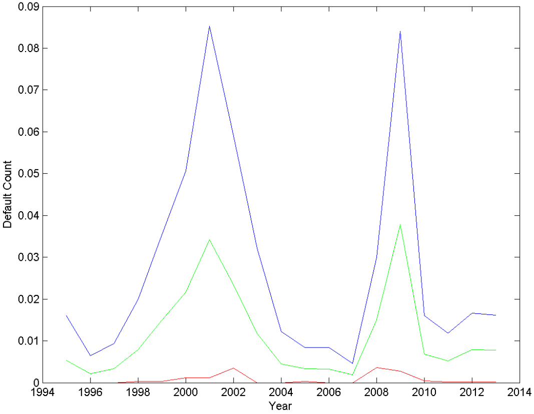
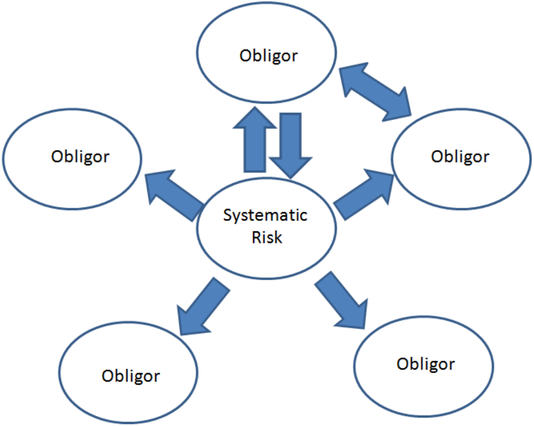


Figure 3: Annual average Loss Given Default rate: IG , SG and All , from 1995-2013.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Objectives

(i) Credit Risk Modeling

- Factor loading conditional on hectic and quiet state.
- State-dependent recovery rate.

(ii) Model Comparison

- Four models

Standard Technology

- Default event modeling
  - Latent variable is a linear combination of systematic and idiosyncratic shocks.
  - Copula enables flexible and realistic default dependence structure.
    - Gaussian Copula
    - t Copula

Outline

1. Motivation ✓
2. Factor Copulae & Stochastic Recoveries
3. Methodology
4. Empirical Results
5. Conclusions

Factor Copulae & Stochastic Recoveries

- Factor copula model is a flexible measurement of portfolio credit risk: Krupskii and Joe (2013)
- t copulas generate a greater likelihood of a clustering of defaults for companies: Hull and White (2004)
- Correlation breakdown structure: Ang and Bekaert (2002), Anderson et al. (2004)
- Recovery rate varies with the market conditions: Amraoui et al. (2012)

Candidate Models-Gaussian Copula

- FC model - One-factor Gaussian copula model with constant correlation structure and constant recoveries.
- RFL model - Conditional factor loading and constant recoveries.
- RR model - One-factor Gaussian copula and stochastic recoveries.
- RRFL model - Conditional factor loading and stochastic recoveries.

Candidate Models-t copula

- TFC model - One-factor t copula model with constant correlation structure and constant recoveries.
- TRFL model - One-factor t copula model with conditional factor loading and constant recoveries.
- TRR model - One-factor t copula and stochastic recoveries.
- TRRFL model - One-factor t copula model with conditional factor loading and stochastic recoveries.

Copulae

- For n dimensions distribution F with marginal distribution $F_{X_1}, \cdots, F_{X_n}$, Copula function:

$$F(x_1, \cdots, x_n) = C\{F_{X_1}(x_1), \cdots, F_{X_n}(x_n)\}$$

Hoeffding on BBI: 

One factor copula model

- Assume that $U = (U_1, \ldots, U_d)$ is a random vector. A factor copula model can be expressed as following.

$$C(u_1, \ldots, u_d) = \int_{[0,1]} \prod_{j=1}^{d} F_{j|V}(u_j \mid v) dv \qquad (1)$$

- C is a d-dimensional copula.
- $C(u_1, \ldots, u_d)$ is the joint cdf of the vector U.
- $F_{1|V}, \ldots, F_{d|V}$ denote joint distribution conditional on V.

Gaussian-copula based one factor model(I)

- Let $C_{j,v}$ be the bivariate Gaussian copula with correlation α_j. Then $C_{j,v}(u_j, v) = \Phi_2\{\Phi^{-1}(u_j), \Phi^{-1}(v); \alpha_j\}$, and $F_{j|V}(u_j \mid v) = C_{j|V}(u_j \mid v) = \frac{\partial C_{j,v}(u_j,v)}{\partial v}$

$$F_{j|V}(u_j \mid v) = \Phi\left[\frac{\{\Phi^{-1}(u_j) - \alpha_j \Phi^{-1}(v)\}}{\sqrt{1-\alpha_j^2}}\right] \tag{2}$$

- Φ denotes the Gaussian cdf and Φ_2 is the bivariate normal cdf.

Gaussian-copula based one factor model(II)

- Let $u_j = \Phi(z_j)$ and $v = \Phi(w)$

$$\begin{aligned} C(u_1, \ldots, u_d) &= \int_0^1 \prod_{j=1}^{d} \left\{ \Phi \left[\frac{\Phi^{-1}(u_j) - \alpha_j \Phi^{-1}(v)}{\sqrt{1-\alpha_j^2}} \right] \right\} dv \\ &= \int_{-\infty}^{\infty} \prod_{j=1}^{d} \left\{ \Phi \left[\frac{z_j - \alpha_j w}{\sqrt{1-\alpha_j^2}} \right] \right\} \psi(w) dw \end{aligned} \tag{3}$$

- ψ denotes the Gaussian pdf.

- Eq.3) comes from

$$Z_j = \alpha_j W + \sqrt{1-\alpha_j^2}\varepsilon_j \quad j = 1, \dots, d.$$

- W: systematic factor, ε_j: idiosyncratic factors.
- W and ε_j are independent, and ε_j are uncorrelated among each other
- Z_j: the proxies for firm asset and liquidation value.
- Correlation coefficient between Z_1 and Z_2 is

$$\rho_{12} = \frac{\alpha_1 \alpha_2 \sigma^2}{\sqrt{\alpha_1^2(\sigma^2-1)+1}\sqrt{\alpha_2^2(\sigma^2-1)+1}}.$$

t-copula based one factor model(I)

- Let $C_{j,v}(u_j, v) = \Phi_2(T_\nu^{-1}(u_j), \Phi^{-1}(v); \alpha_j)$ (McNeil and Frey, 2015), and

$$F_{j|V}(u_j \mid v) = \Phi\left[\frac{\{v_2^{-1} T_\nu^{-1}(u_j) - \alpha_j \Phi^{-1}(v)\}}{\sqrt{1-\alpha_j^2}}\right] \tag{4}$$

- where $V_2 \sim Ig(\frac{\nu}{2}, \frac{\nu}{2})$ (Ig is inverse gamma distribution), and ν represents degrees of freedom.

t-copula based one factor model(II)

- Let $u_j = T_\nu(z_j)$ and $v = \Phi(w)$

$$C(u_1, \ldots, u_d) = \int_0^1 \prod_{j=1}^{d} \left\{ \Phi \left[\frac{v_2^{-1} T_\nu^{-1}(u_j) - \alpha_j \Phi^{-1}(v)}{\sqrt{1-\alpha_j^2}} \right] \right\} dv$$

$$= \int_{-\infty}^{\infty} \prod_{j=1}^{d} \left\{ \Phi \left[\frac{v_2^{-1} z_j - \alpha_j w}{\sqrt{1-\alpha_j^2}} \right] \right\} \psi(w) dw \tag{5}$$

- Eq.(5) comes from

$$Z_j = V_2(\alpha_j W + \sqrt{1-\alpha_j^2}\varepsilon_j) \quad j = 1, \ldots, d.$$

- W is iid non-standard Gaussian distribution and ε_j are iid standard Gaussian.
- Correlation coefficient between Z_1 and Z_2 is

$$\rho_{12} = \frac{\alpha_1\alpha_2\sigma^2}{V_2\sqrt{\alpha_1^2(\sigma^2-1)+1}\sqrt{\alpha_2^2(\sigma^2-1)+1}}.$$

- The default indicator

$$\mathbf{I}\{\tau_j \leq t\} = \mathbf{I}\left[Z_j \leq F^{-1}\{P_j(t)\}\right].$$

- τ_j indicates the default time of each obligor.
- $F^{-1}(\cdot)$ denotes the inverse cdf of any distribution.
- $P_j(t)$: hazard rate and marginal probability that obligor j defaults before t.
  - From Moody's report.
  - Extract from Credit spreads.
  - Extract from Credit default swap spreads.

- Portfolio loss for each obligor

$$L = \sum_{j=1}^{N} G_j \mathbf{I}\left\{\tau_j \leq t\right\} = \sum_{j=1}^{N} G_j \mathbf{I}\left[Z_j \leq F^{-1}\{P_j(t)\}\right].$$

- G_j is the loss given default (LGD) (j-th obligor's exposure = 1).

Conditional Default Model-General Form

- Conditional factor copula model

$$Z_j|_{\mathrm{S=H}} = \alpha_j^H W + \sqrt{1-(\alpha_j^H)^2}\varepsilon_j$$

$$Z_j|_{\mathrm{S=Q}} = \alpha_j^Q W + \sqrt{1-(\alpha_j^Q)^2}\varepsilon_j$$

- α^H, α^Q are conditional factor loading.
- Conditional default probability

$$P(\tau_j < t|\mathrm{S}) = F\left[\frac{F^{-1}\{P_j(t)\} - \alpha_j^S W}{\sqrt{1-(\alpha_j^S)^2}}\right] = P_j(W|\mathrm{S}) \qquad \mathrm{S} \in \{\mathrm{H,Q}\}$$

- with P(S=H)=ω, and P(S=Q)=$1-\omega$

State-Dependent Recovery Rate

- The LGD on name j, $G_j(W)$ is related to common factor W and the marginal default probability P_j
- Given fixed expected loss, $(1 - R_j)P_j = (1 - \bar{R}_j)\bar{P}_j$

$$G_j(W|\mathrm{S{=}H}) = (1 - \bar{R}_j)\frac{F\left[\left\{F^{-1}\left(\overline{P}_j\right) - \alpha_j^H W\right\}/\sqrt{1-(\alpha_j^H)^2}\right]}{F\left[\left\{F^{-1}\left(P_j\right) - \alpha_j^H W\right\}/\sqrt{1-(\alpha_i^H)^2}\right]}.$$

$$G_j(W|\mathrm{S{=}Q}) = (1 - \bar{R}_j)\frac{F\left[\left\{F^{-1}\left(\overline{P}_j\right) - \alpha_j^Q Z\right\}/\sqrt{1-(\alpha_j^Q)^2}\right]}{F\left[\left\{F^{-1}\left(P_j\right) - \alpha_j^Q Z\right\}/\sqrt{1-(\alpha_j^Q)^2}\right]}.$$

- We set $\bar{R}_j = 0$ in the simplest case.

Conditional Expected Loss

- Conditional default probability $P_j(W|\text{S=H,Q})$ and conditional LGD, $G_j(W|\text{S=H,Q})$, conditional expected loss,

$$\text{E}(L_j|Z) = \omega G_j(W|\text{S=H})P_j(W|\text{S=H}) + (1-\omega)G_j(W|\text{S=Q})P_j(W|\text{S=Q}).$$

Monte Carlo Simulation and MSE

- One-factor non-standardized Gaussian Copula
  - $W \sim \mathrm{N}(-0.03, 3.05), Z, \varepsilon_i \sim N(0,1)$.
  - W and ε_i are generated 10000 observations.
- One-factor t Copula
  - $W \sim \mathrm{N}(-0.03, 3.05), \varepsilon_i \sim N(0,1)$.
  - Z follows t pdf with ν degrees of freedom.
- Conditional probability that date t was belonging to the hectic is $\pi(W = w)$.

$$\mathrm{P}(S = H|W = w) = \pi(W = w)$$
$$= \frac{\omega c(z_j, w|\theta^H)}{(1-\omega)c(z_j, w|\theta^Q) + \omega c(z_j, w|\theta^H)}$$

- where c is copula density.

Project to Default Time

- Using the definition of survival rate (Hull, 2006)

$$\tau_i|\mathsf{S} = -\frac{\log\{1 - F(Z_j|\mathsf{S})\}}{P_j}.$$

- P_j is the hazard rate and marginal probability that obligor j will default.
- $\tau_j|\mathsf{S}$ is corresponding to

$$\mathsf{E}[\mathsf{I}(\tau_j|\mathsf{S} < 1)] = \mathsf{P}(\tau_j|\mathsf{S} < 1) = P_j(Z|\mathsf{S}).$$

State-Dependent Recovery Rate Simulation

- $(1 - R_j)P_j = (1 - \bar{R}_j)\bar{P}_j$.
- $\overline{P}_j$ is a adjusted default probability calibrated by plugging hazard rate P_j.
- $\bar{R}_j$ is a lower bound for state-dependent recovery rates [0,1].
- We set $\bar{R}_j = 0$ in the simplest case.
- Given α_j^S and simulated Z, we generate $G_j(Z|S)$.

Expected Loss Function

- With these two specifications, we study the expected loss function under the given scenarios

$$
\begin{aligned}
\mathrm{E}(L_j|W) &= \pi(W=w)G_j(W|\mathrm{S{=}H})P_j(W|\mathrm{S{=}H}) \\
&+ \{1-\pi(W=w)\}G_j(W|\mathrm{S{=}Q})P_j(W|\mathrm{S{=}Q})
\end{aligned}
$$

- $\pi(W=w)$ is better than unconditional probability ω.

Estimation of the AE

- Absolute Error (AE)

$$\text{AE} = (\text{actual portfolio loss - expected portfolio loss}).$$

- Actual portfolio loss is from Moody's report.
- Exposure of each obligor is 100 million.
- Compare minimum AE, MAE to evaluate candidate models.

Data

- Forecast Period: 31 in 2008
- Daily USD S&P 500 and stock return of the defaults
- Estimated period: 3 years before the default year
- Source: Datastream

Data

- Recovery rate: Realized recovery rate R_j (weighted by volume) before default year by Moody's

- Hazard rate: Average historical default probability from Moody's report

Financial return data

- Considering the S&P 500 and 45 stock returns are following the AR(1)-GARCH(1,1) model

$$r_{jt} = \mu_j + \rho_j r_{j,t-1} + \sigma_{jt}\epsilon_{jt}$$
$$\sigma_{jt} = \omega_j + \alpha_j r^2_{j,t-1} + \beta_j \sigma^2_{j,t-1}$$

- where r_{jt} is stock return and $j = 1, \cdots, d, t = 1, \cdots, T$, ϵ_{jt} are i.i.d vectors with distribution,

$$F(z_1, \cdots, z_d) = C(F_v(z_1), \cdots, F_v(z_d))$$

- F_v denotes the cdf of t distribution with v degrees of freedom, used to model innovations in GARCH model

Conditional Factor Loading-Gaussian Copula

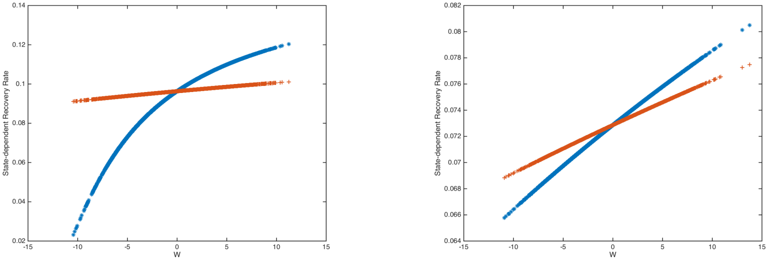
Company	Uncond.	Quiet	Hectic
Abitibi-Consolidated Com. of Can.	0.21	-0.26	0.31
Franklin Bank	0.39	0.66	-0.18
Glitnir Banki	0.24	-0.99	0.13
GMAC	0.24	0.16	0.98
Lehman Bros	-0.09	-0.33	0.56

Table 1: Correlation coefficients between S&P500 index returns and the return of default companies in 2008 are computed by Gaussian copula.

Conditional Factor Loading-t copula

Company	Uncond.	Quiet	Hectic
Abitibi-Consolidated Com. of Can.	-0.26	-0.50	0.22
Franklin Bank	0.39	0.66	-0.16
Glitnir Banki	0.13	-0.73	0.24
Kaupth. Bank	0.16	-0.25	0.31
Lehman Bros	-0.06	-0.17	0.85

Table 2: Correlation coefficients between S&P500 index returns and the return of default companies in 2008 are computed by t copula.



(a)Glitnir Banki (b) Washington Mutual Bank

Figure 4: The relationship between state-dependent recovery rates and S&P 500, Z by using t copula
'*' in blue illustrates the pattern of state-dependent recovery rate, and '+' in red plots the recoveries proposed by Amraoui et al.(2012)

Estimation of MAE-Gaussian Copula

	FC	RFL	RR	RRFL
2008				
APL	2035.02	2035.02	2035.02	2035.02
EPL	1158.61	1186.51	1550.05	1598.29
AE	878.89	575.36	484.97	436.73
MAE	27.47	17.98	15.16	13.65
EPL/APL	53.71%	58.30%	78.93%	77.60%

Table 3: The mean of actual portfolio loss (APL), expected portfolio loss (EPL) and AE, MAE (in million)

Estimation of MAE-t Copula

	TFC	TRFL	TRR	TRRFL
2008				
APL	2035.02	2035.02	2035.02	2035.02
EPL	1138.53	1483.23	1695.85	1921.80
AE	896.49	551.79	339.17	113.22
MAE	28.02	17.24	10.60	3.54
EPL/APL	55.95%	72.89%	83.33%	94.44%

Table 4: The actual portfolio loss (APL), expected portfolio loss (EPL), AE, and MAE (in million) for robustness

Conclusions

(i) Model the dependence in a more flexible and realistic way.
- Build the quiet and hectic regimes.
- Connect the recovery rate to the common factor.
- State-dependent describes the asymmetric thick tail.

(ii) The conditional factor copulae together with state-dependent recoveries model could predict the default event during the crisis period.

t-Copula Based Factor Model for Credit Risk Analysis

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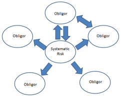

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