

# Shape Invariant Modeling and Risk Patterns

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## Financial Market

Riskless bond with constant interest rate  $r$ , stock price  $(S_t)_{t \in [0, T]}$

### □ risk neutral valuation principle

$$e^{-r\tau} \int_0^\infty \psi_t(s_T) \frac{q(s_T)}{p(s_T)} p(s_T) ds_T$$

where  $\psi_t(s_T)$  is the payoff function of a derivative with maturity  $\tau = T - t$ ,  $q$  is a risk neutral density and  $p$  is the probability density function of  $S_T$



## Pricing Kernels & Preferences

- the pricing kernel

$$\mathcal{K}(S_T) = \frac{q(S_T)}{p(S_T)}$$

- relationship between representative investor's preferences and pricing kernel (e.g. Leland 1980)

$$ARA(S_T) = \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)} = \frac{-\mathcal{K}'(S_T)}{\mathcal{K}(S_T)}$$



## Empirical Pricing Kernel (EPK)

- EPK ( $\hat{\mathcal{K}}$ ) any estimator of the pricing kernel  $\mathcal{K}$
- Black-Scholes model: EPK is decreasing
- general models for stock prices: Ait-Sahalia and Lo (2000), Engle and Rosenberg (2002), Brown and Jackwerth (2004), Härdle, Krättschmer and Moro (2009)



### EPK paradox



## EPK under the Black-Scholes Model I

Geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

The historical density  $p$  is log-normal:

$$p_t(S_T) = \frac{1}{S_T \sqrt{2\pi\sigma^2\tau}} \exp \left\{ -\frac{1}{2} \left( \frac{\log(S_T/S_t) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{t}} \right)^2 \right\}$$



## EPK under the Black-Scholes Model II

The risk neutral density  $q$  is log-normal as well with  $r$  instead of  $\mu$  s.t. the pricing kernel is a decreasing function of  $S_T$

$$\begin{aligned}\mathcal{K}(S_T) &= \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} \exp\left\{\frac{(\mu-r)(\mu+r-\sigma^2)\tau}{2\sigma^2}\right\} \\ &= a\left(\frac{S_T}{S_t}\right)^{-\gamma}\end{aligned}$$

where  $a = \exp\left\{\frac{(\mu-r)(\mu+r-\sigma^2)\tau}{2\sigma^2}\right\}$  and  $\gamma = \frac{\mu-r}{\sigma^2} \geq 0$  constant  
coefficient of relative risk aversion



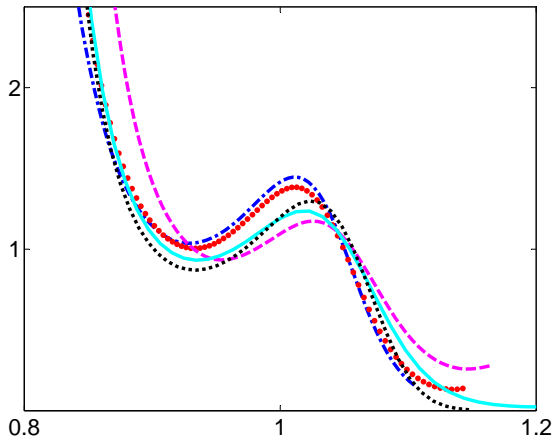


Figure 1: EPK vs. moneyness ( $K/S_t$ ) for maturities:  $\tau = 0.097y$  (5w) (blue),  $0.083y$  (4w) (red),  $0.069$  (3.5w) (magenta),  $0.061$  (3w) (cyan),  $0.047$  (2.5w) (black) Expiration date (for all): 20060602



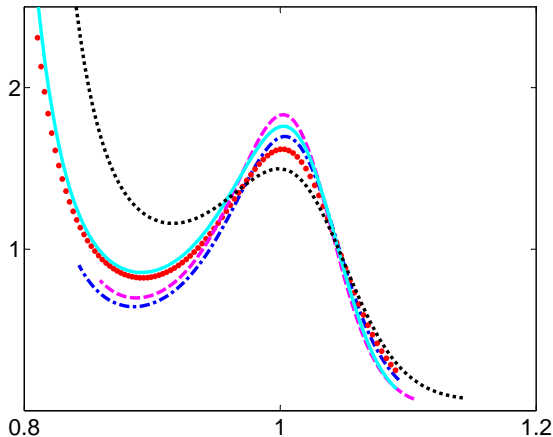


Figure 2: EPK vs moneyness for maturity  $\tau = 0.083$  (4w), observed on 20060118 (blue), 20060215 (red), 20060322 (magenta), 20060419 (cyan), 20060417 (black)





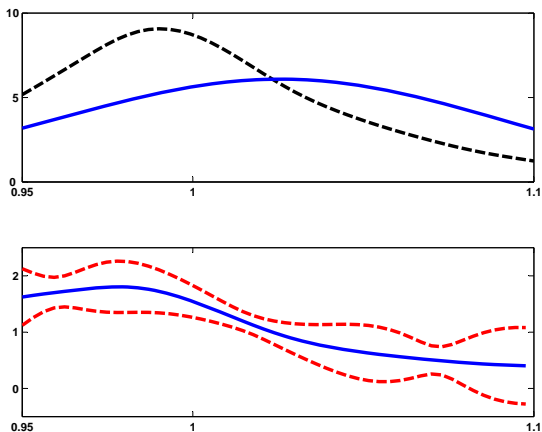


Figure 3: Upper panel:  $\hat{q}$  (black) and  $\hat{p}$  (blue); lower panel: EPK and 95% uniform confidence interval on 20080228 (Härdle, Okhrin and Wang (2010))



## Background

### Shape Invariant Modeling

- Gasser et al. (1984) Zürich Longitudinal Studies on Growth
- Härdle and Marron (1990) Automobile side impact data

### Self Modeling

- Lawton et al. (1972)



## EPK paradox

Empirical pricing kernels are not monotone decreasing in returns

- How to model the changes in the EPK curves based on a common (reference) feature?
- How can EPK deformation relate to patterns in risk perception?
- How does risk perception relate to the business cycle?



## Outline

1. Motivation ✓
2. Empirical Pricing Kernel
3. Shape Invariant Modeling
4. Pricing Kernel and Risk Aversion
5. Conclusions
6. Selected Bibliography



## The Financial Market I

In an arbitrage-free market, the European call price is given by

$$C_t(K, \tau, r, S_t) = e^{-r\tau} \int_0^\infty (S_T - K)^+ q(S_T | \tau, r, S_t) dS_T$$

- $S_t$  the underlying asset price at time  $t$
- $K$  the strike price
- $\tau$  the time to maturity
- $T = t + \tau$  the expiration date
- $r$  constant risk free interest



## The Financial Market II

The call price

$$\begin{aligned}C_t(S_T) &= e^{-r\tau} E^Q \{ (S_T - K)^+ | S_t \} \\ &= e^{-r\tau} E^P \{ (S_T - K)^+ \mathcal{K}_t(S_T) | S_t \}\end{aligned}$$

with  $\mathcal{K}_t(S_T)$  the pricing kernel at time  $t$ , s.t. conditional risk neutral distribution:

$$Q(S_T \leq x | S_t) \stackrel{\text{def}}{=} \int_{-\infty}^x \mathcal{K}_t(\cdot) dP(S_T | S_t)$$

where  $P(S_T | S_t)$  is the conditional distribution of  $S_T$  under  $S_t$



## Data

- ▣ **Source:** Research Data Center (RDC)  
<http://sfb649.wiwi.hu-berlin.de>
- ▣ Datastream DAX 30 Price Index;  
2 years worth of monthly returns in a sliding window
- ▣ EUREX European Option Data; tick observations;  
38 days (once/month) of cross-sectional data: 200304:200605



## Estimation of PK

- EPK: ratio of 2 estimated densities

$$\hat{\kappa}_t(s_t, S_T) = \frac{\hat{q}_t(S_T)}{\hat{p}_t(S_T)}$$

- $\hat{q}_t(S_T)$  by Rookley (1997) method based on the results of Breeden and Litzenberger (1978)

$$q_t(S_T) = e^{r\tau} \frac{\partial^2 C_t(\cdot)}{\partial K^2} \Big|_{K=S_T}$$

- $\hat{p}_t(S_T)$  by kernel density method





## Estimation of RND

Rookley method: for fixed one month maturity estimate a smooth call price function with respect to the moneyness  $K/S_t$

- ▣ implied volatility  $\sigma_{IV}$  substitute the call price
- ▣  $\hat{\sigma}_{IV}$ ,  $\hat{\sigma}'_{IV}$ ,  $\hat{\sigma}''_{IV}$  improve efficiency
- ▣ local polynomial smoothing of degree 3
- ▣ quartic kernel
- ▣ little sensitivity to the bandwidth choice



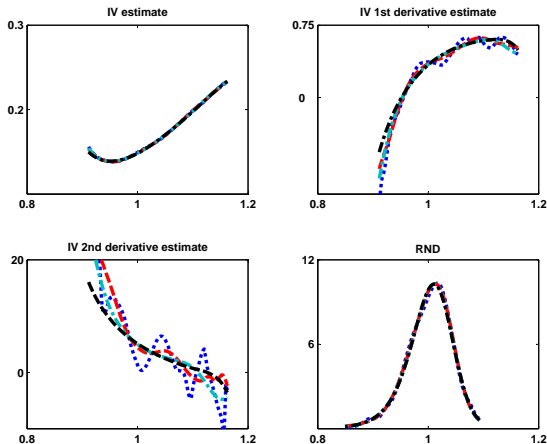


Figure 4:  $\hat{\sigma}_{IV}$ ,  $\hat{\sigma}'_{IV}$ ,  $\hat{\sigma}''_{IV}$  vs.  $K/S_t$ , and  $\hat{q}$  vs.  $S_T/S_t$  for varying bandwidths  $h = (0.05, 0.10, 0.15, 0.20)^\top$  for maturity  $\tau=0.083$  (4w), on 20060118



## Estimation of PDF

- nonparametric kernel density based on overlapping monthly historical returns (2 years)
- quartic kernel
- bandwidth choice: unimodal densities for all periods
- peak varies with the bandwidth and window length



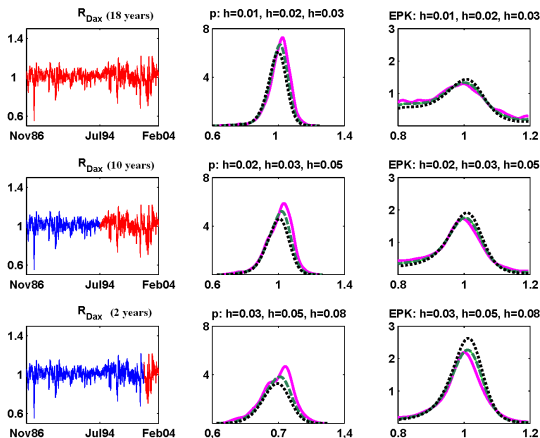


Figure 5: Returns window length (left),  $p$  estimates for increasing bandwidth  $h$  (magenta, green, black) (middle) and the corresponding EPK-s (right) on 20040216



## Shape Invariant Model (SIM)

- $Y_{tj}$  is a noisy sample of  $T$  curves at design points  $u_j$ , with  $j \in \{1, \dots, n\}$ ,  $n = 101$

$$Y_{tj} = \mathcal{K}_t(u_j) + \varepsilon_{tj}, \quad \text{with} \quad \varepsilon_{tj} \sim (0, \sigma_t^2) \quad (1)$$

- The smooth curves are of the form

$$\mathcal{K}_t(u) = \theta_{t1} \mathcal{K}_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right) + \theta_{t4} \quad (2)$$

- $\mathcal{K}_0$  is a reference curve and  $\theta = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4})^\top$  are horizontal and vertical deviation parameters



## Estimation of SIM I

- Synchronisation

$$\mathcal{K}_t(\theta_{t2}u + \theta_{t3}) = \theta_{t1}\mathcal{K}_0(u) + \theta_{t4}, \quad \theta_{t1} > 0, \quad \theta_{t2} > 0$$

- Normalizing conditions

$$T^{-1} \sum_{t=1}^T \theta_{t1} = T^{-1} \sum_{t=1}^T \theta_{t2} = 1, \quad T^{-1} \sum_{t=1}^T \theta_{t3} = T^{-1} \sum_{t=1}^T \theta_{t4} = 0$$

- Common feature (reference) curve

$$\mathcal{K}_0(u) = T^{-1} \sum_{t=1}^T \mathcal{K}_t(\theta_{t2}u + \theta_{t3})$$



## Common Shape

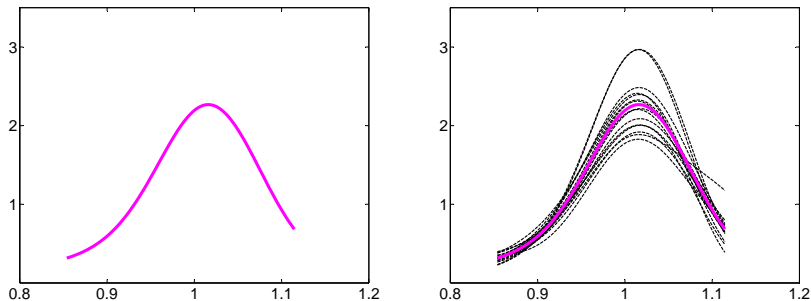


Figure 6: Common shape function  $\hat{\mathcal{K}}_0$  (left) for the pricing kernels between 200304:200605 and transformed curves  $\hat{\mathcal{K}}_t(\theta_{t2}u + \theta_{t3})$  on the common domain (right)



## Estimation of SIM II

- Peak identification

$$0 = \kappa'_t(u) = \frac{\theta_{t1}}{\theta_{t2}} \kappa'_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right)$$

- Inflection point

$$0 = \kappa''_t(u) = \frac{\theta_{t1}}{\theta_{t2}^2} \kappa''_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right),$$

- The solutions also satisfy

$$u_t = \theta_{t2} u_0 + \theta_{t3}$$

- This gives the starting values of  $\theta_{t3}$  and  $\theta_{t2}$





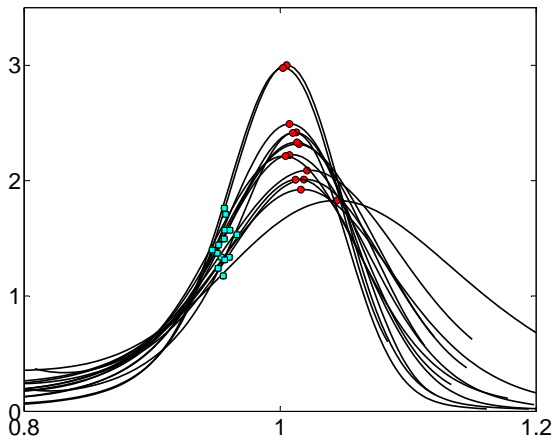


Figure 7: Landmark identification. Pricing kernels with fixed maturity 1 month between 200304:200605



## Estimation of SIM III

- Estimation procedure

$$\min_{\theta} \int \{\hat{\mathcal{K}}_t(\theta_2 u + \theta_3) - \theta_1 \hat{\mathcal{K}}_0(u) - \theta_4\}^2 w(u) du, \quad (3)$$

where  $\hat{\mathcal{K}}_i$  are nonparametric estimates of the curves and the common region is defined for some  $a \geq \inf(u_{t1})$  and  $b \leq \sup(u_{tn})$

$$w(u) = \prod_t 1_{[a,b]} \{(u - \theta_{t3})/\theta_{t2}\}$$



## The Algorithm

- Iterative scheme based on (3)
- Given prior estimates for  $(\theta_{t2}, \theta_{t3})$  and  $\hat{\mathcal{K}}_0$

$$\min_{\theta} \sum_j \{ \hat{\mathcal{K}}_t(\theta_{t2}u_j + \theta_{t3}) - \theta_{t1}\hat{\mathcal{K}}_0(u_j) - \theta_{t4} \}^2 w(u_j) \quad (4)$$

- Update  $(\theta_{t2}, \theta_{t3})$  and  $\hat{\mathcal{K}}_0$
- Convergence is reached fast



## Asymptotics: EPK

$$\begin{aligned}\hat{\mathcal{K}}(u) - \mathcal{K}(u) &= \frac{\hat{q}(u)}{\hat{p}(u)} - \frac{q(u)}{p(u)} \\ &\simeq \frac{\hat{q}(u) - q(u)}{p(u)} - \frac{q(u)}{p(u)} \frac{\hat{p}(u) - p(u)}{p(u)}\end{aligned}$$

### Bias and variance

$$\begin{aligned}\mathbb{E} \left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} &\simeq \mathcal{O}(h_q^2) + \mathcal{O}(h_p^2) \\ \text{Var} \left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} &\simeq \mathcal{O}(Mh_q^5)^{-1} + \mathcal{O}(mh_p)^{-1}\end{aligned}$$

with  $M$  sample size for  $q$ ,  $m$  sample size for  $p$ ,  
 $h_q$  and  $h_p$  the corresponding bandwidths.



## Asymptotics: SIM Parameters

- based on standard non-linear least squares methods

$$\hat{\theta}_t \approx N(\theta_t, \Sigma_t)$$

$$\hat{\Sigma}_t = \hat{\sigma}_t^2 \left[ n^{-1} \sum_{j=1}^n \left\{ \nabla_{\theta} \tilde{\mathcal{K}}_t(u_j; \hat{\theta}) \right\} \left\{ \nabla_{\theta} \tilde{\mathcal{K}}_t(u_j; \hat{\theta}) \right\}^{\top} \right]^{-1},$$

where  $\nabla_{\theta} \mathcal{K}(u; \theta)$  is the first derivative of the function  $\mathcal{K}(u; \theta)$  and

$$\hat{\sigma}_t^2 = n^{-1} \sum_{j=1}^n (\hat{\varepsilon}_{tj} - \bar{\varepsilon}_t)^2$$

with  $\hat{\varepsilon}_{tj} = Y_{tj} - \tilde{\mathcal{K}}_t(u_j)$  and  $\tilde{\mathcal{K}}$  is the SIM estimate, and  $\bar{\varepsilon}_t = n^{-1} \sum_{j=1}^n \hat{\varepsilon}_{tj}$



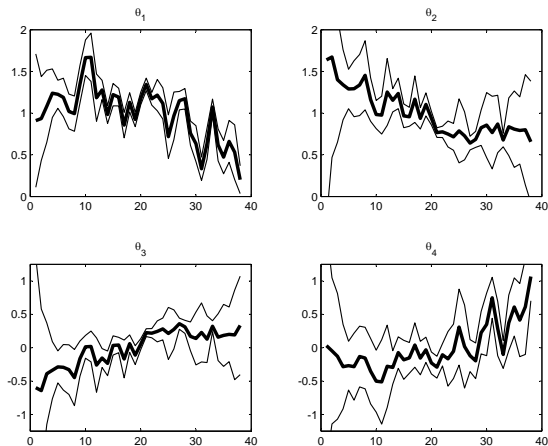


Figure 8: Parameter estimates of the SIM and their confidence intervals at 95% confidence level for the EPK 200304:200605



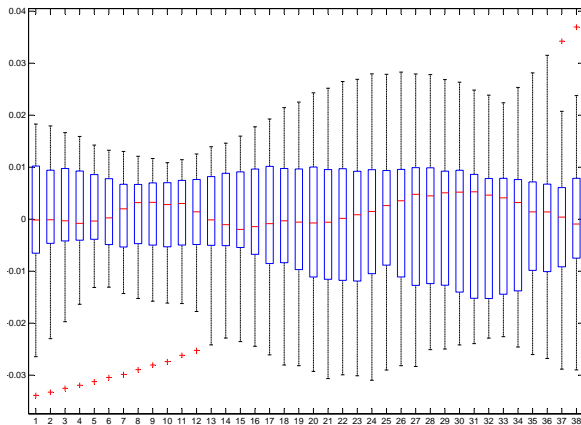


Figure 9: Boxplots of residuals  $\hat{\varepsilon}_{tj} = Y_{tj} - \tilde{\mathcal{K}}_t(u_j)$  for each SIM estimate between 200304:200605



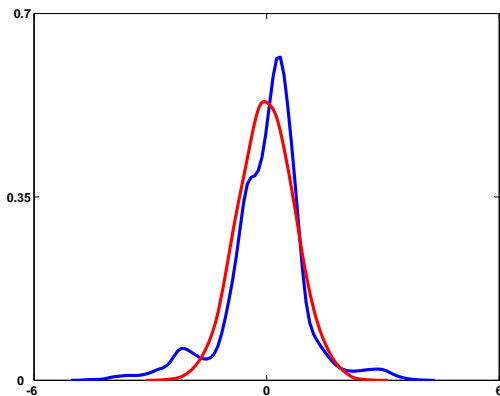


Figure 10: Kernel density estimator for the standardized residuals  $(\hat{\varepsilon}_{tj} - \bar{\varepsilon}_t)/\hat{\sigma}_t$  (blue) of all SIM estimates between 200304:200605 against a simulated normal density (red) with  $\hat{\sigma} = IQR/1.34$





## EPK paradox (Hens, 2010)

- ▣ risk-seeking behaviour
- ▣ incomplete markets
- ▣ heterogeneous beliefs



## ARA and SIM

- risk preferences based on the first derivative of PK  
e.g. Arrow-Pratt measure of **absolute risk aversion** (ARA)
- under SIM specifications the ARA measure is given by:

$$ARA_t(u) = \frac{-\frac{\theta_{t1}}{\theta_{t2}} \mathcal{K}'_0\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right)}{\theta_{t1} \mathcal{K}_0\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right) + \theta_{t4}}$$

- relate time dynamic of ARA to those of EPK via SIM parameters



## ARA Paradoxa

- Standard economic preferences: **declining risk averseness**  
i.e. positive, monotonically decreasing  $ARA(S_T)$
- Black Scholes model:  $ARA(S_T) = \frac{\gamma}{S_T}$ , for  $\gamma > 0$
- Empirical results show that agents
  - ▶ are locally risk proclive:  $ARA(S_T) < 0$ , i.e. EPK paradox
  - ▶ have increasing absolute risk aversion  $ARA'(S_T) > 0$



## The Effect of $\theta_1$ on $\mathcal{K}_0$ and $ARA_0$

- no effect on ARA for values of  $\theta_4$  around 0

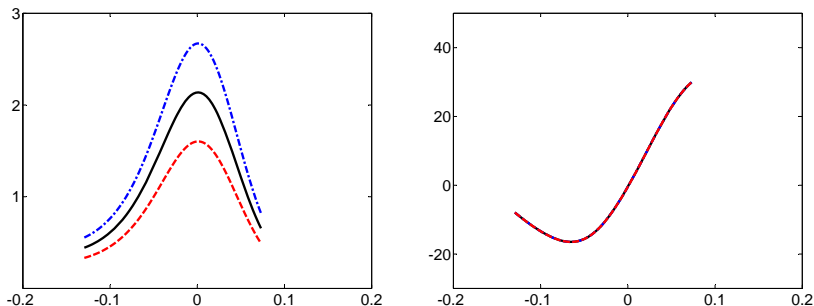


Figure 11: EPK (left) and ARA (right)  $\theta_1 = 0.75$  (red),  $\theta_1 = 1.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)^T$  (black)



## The Effect of $\theta_2$ on $\mathcal{K}_0$ and $ARA_0$

- local risk proclivity decreases with  $\theta_2$

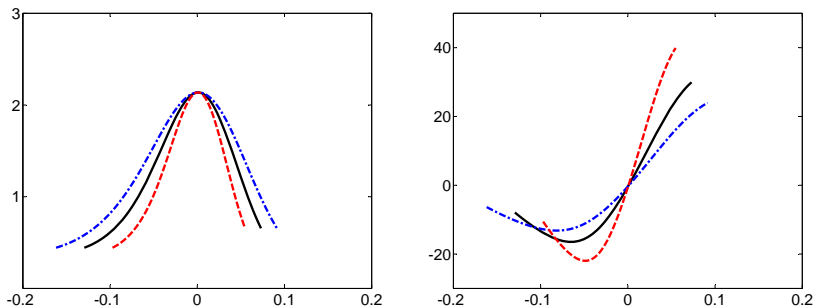


Figure 12: EPK (left) and ARA (right)  $\theta_2 = 0.75$  (red),  $\theta_2 = 1.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)^\top$  (black)



## The Effect of $\theta_3$ on $\mathcal{K}_0$ and $ARA_0$

- shift effect of the risk proclivity domain

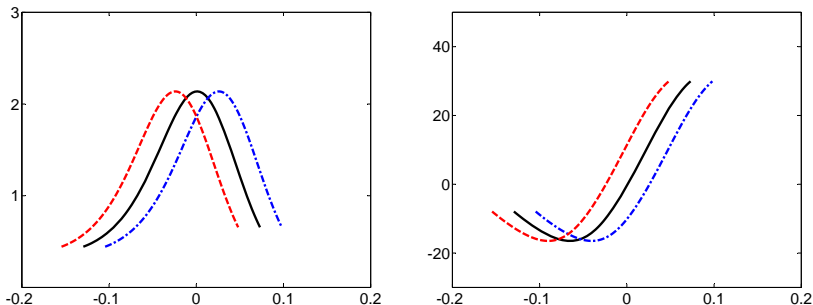


Figure 13: EPK (left) and ARA (right)  $\theta_3 = -0.025$  (red),  $\theta_3 = 0.025$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)^\top$  (black)



## The Effect of $\theta_4$ on $\mathcal{K}_0$ and $ARA_0$

- local risk proclivity decreases with  $\theta_4$

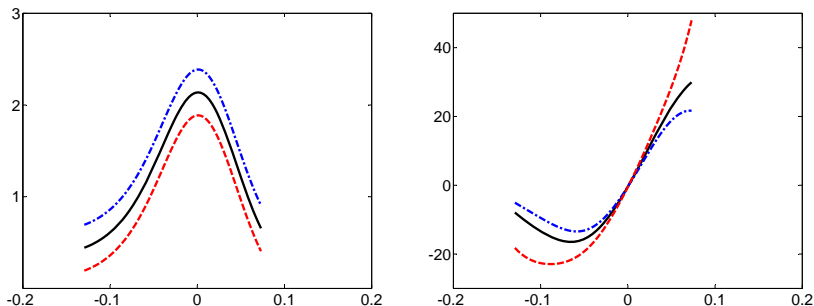


Figure 14: EPK (left) and ARA (right)  $\theta_4 = -0.25$  (red),  $\theta_4 = 0.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)^\top$  (black)



## Risk Aversion and Business Cycle Indicators

- **Data:** Daily observations. German market
- Credit spread ( $CD$ ): 5Y Corporate - Gov. bond yield
- Short term interest rate ( $IR$ ): 3M Gov. bond yield
- Datastream DAX 30 stock index ( $I_{DAX}$ )
- $P_x, P_y$  EPK peak coordinates





## Business Cycle Indicators (BCI)

How are they related to the market conditions?

Indicator	Expansion	Recession
$CS$	↓	↑
$IR$	↓	↑
$I_{DAX}$	↑	↓

Table 1: Behaviour of the economic indicators under the BC phases



	$\Delta\theta_1$	$\Delta\theta_2$	$\Delta\theta_3$	$\Delta\theta_4$	$\Delta P_x$	$\Delta P_y$
$\Delta\theta_1$	1.00					
$\Delta\theta_2$	-0.71	1.00				
$\Delta\theta_3$	-0.71	-0.99	1.00			
$\Delta\theta_4$	-0.93	0.45	0.45	1.00		
$\Delta P_x$	-0.27	0.41	-0.38	0.1	1.00	
$\Delta P_y$	0.96	-0.83	0.83	-0.82	-0.31	1.00
$\Delta CS$	-0.30	0.19	-0.19	0.31	0.13	-0.28
$\Delta IR$	0.01	0.10	-0.08	-0.02	0.53	-0.00
$R_{DAX}$	0.68	-0.57	0.56	-0.59	-0.52	0.69

Table 2: Correlation coefficients between SIM parameters, EPK peak coordinates and BC indicators (significance at 1%, 5%, 10%, )



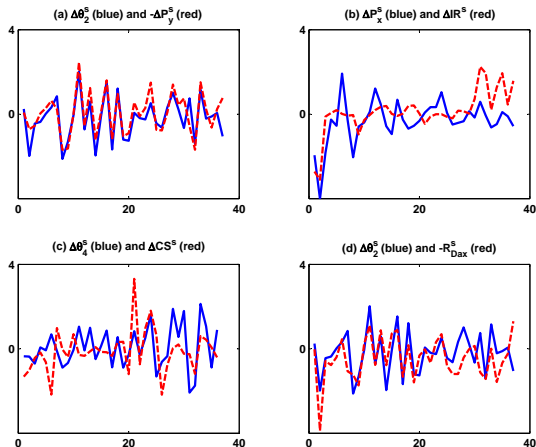


Figure 15: Standardized time series of SIM parameters, EPK peak coordinates and the BCI for 200304:200605



# SIM parameters, EPK peak coordinates, BCI

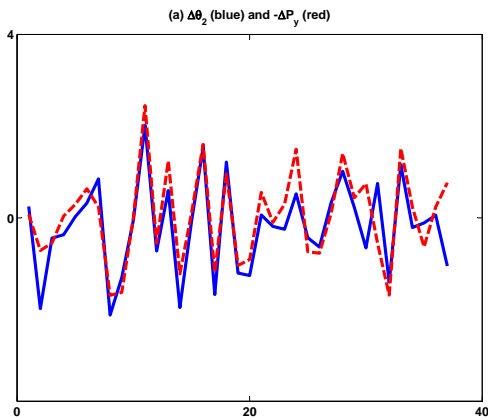


Figure 15: (a): peak height  $P_y$  and peak spread  $\theta_2$  changes are synchronized



# SIM parameters, EPK peak coordinates, BCI

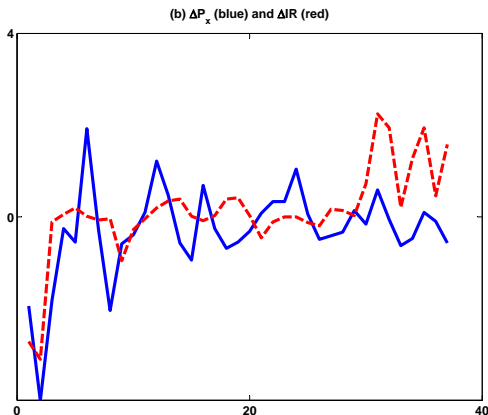


Figure 15: (b): peak location changes  $\Delta P_x$  and  $\Delta IR$  are positively correlated



# SIM parameters, EPK peak coordinates, BCI

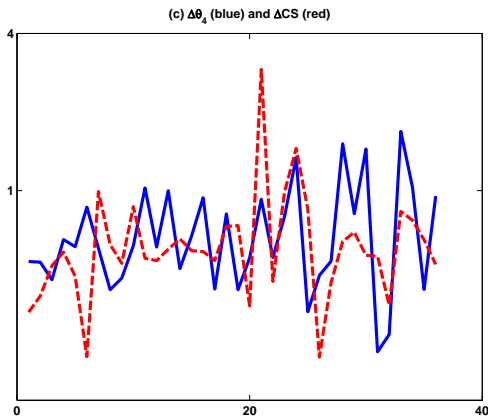


Figure 15: (c):  $\Delta\theta_4$  goes up (risk proclivity decreases) when  $\Delta CS$  increases

EPK Shape Invariant Modeling



# SIM parameters, EPK peak coordinates, BCI

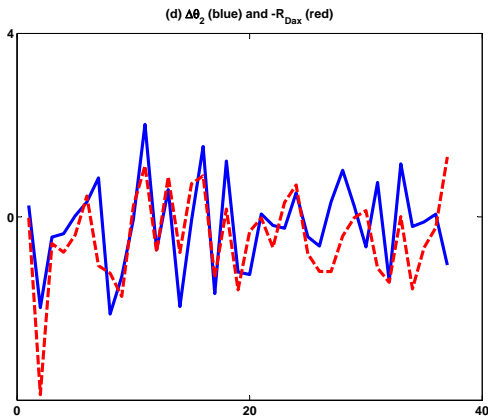


Figure 15: (d):  $\Delta\theta_2$  goes up (risk proclivity decreases) when  $R_{DAX}$  decreases



□ Expanding economy

- higher duration, higher peak height, higher risk proclivity

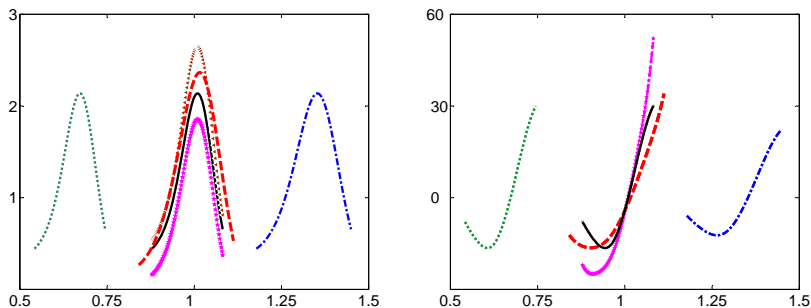


Figure 16: Left:  $\hat{\mathcal{K}}_t$  (red) and its decomposition in marginal effects:  $(\hat{\theta}_1, 1, 0, 0)^\top$  (brown),  $(1, \hat{\theta}_2, 0, 0)^\top$  (blue),  $(1, 1, \hat{\theta}_3, 0)^\top$  (green),  $(1, 1, 0, \hat{\theta}_4)^\top$  (magenta) compared to the baseline model  $\hat{\mathcal{K}}_0$  (black); right:  $ARA_t$  on 20030716





### Declining economy

- lower duration, lower peak height, lower risk proclivity

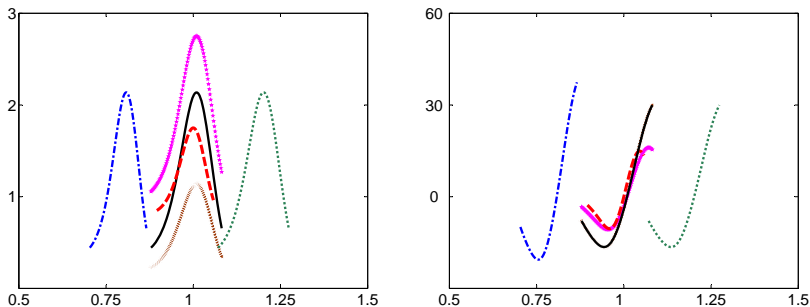


Figure 17: Left:  $\hat{\mathcal{K}}_t$  (red) and its decomposition in marginal effects:  $(\hat{\theta}_1, 1, 0, 0)^\top$  (brown),  $(1, \hat{\theta}_2, 0, 0)^\top$  (blue),  $(1, 1, \hat{\theta}_3, 0)^\top$  (green),  $(1, 1, 0, \hat{\theta}_4)^\top$  (magenta) compared to the baseline model  $\hat{\mathcal{K}}_0$  (black); right:  $ARA_t$  on 20060419



## Conclusions

- Changes in the EPK curve
  - ▶ SIM vs. series methods
- Assessing risk perception
  - ▶  $\theta_2$  and  $\theta_4$  control the shape of ARA
- Risk perception and business cycle
  - ▶ local risk proclivity is pro-cyclical



# Shape Invariant Modeling and Risk Patterns

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## Rookley (1997)

Let  $C_{it}$  be the price of the  $i^{th}$  option at time  $t$  and  $K_{it}$  its strike price, and define the rescaled call option  $c = C/S_t$  in terms of moneyness  $M = S_t/K$  s.t.

$$\begin{aligned}c_{it} &= c\{M_{it}; \sigma(M_{it})\} = \Phi(d_1) - \frac{e^{-r\tau}\Phi(d_2)}{M_{it}} \\d_1 &= \frac{\log(M_{it}) + \{r_t + \frac{1}{2}\sigma(M_{it})^2\}\tau}{\sigma(M_{it})\sqrt{\tau}} \\d_2 &= d_1 - \sigma(M_{it})\sqrt{\tau}\end{aligned}$$



The RND is then

$$q(\cdot) = e^{r\tau} \frac{\partial^2 C}{\partial K^2} = e^{r\tau} S \frac{\partial^2 c}{\partial K^2}$$

with

$$\frac{\partial^2 c}{\partial K^2} = \frac{d^2 c}{dM^2} \left( \frac{M}{K} \right)^2 + 2 \frac{dc}{dM} \frac{M}{K^2}$$

and

$$\begin{aligned} \frac{d^2 c}{dM^2} = & \Phi'(d_1) \left\{ \frac{d^2 d_1}{dM^2} - d_1 \left( \frac{dd_1}{dM} \right)^2 \right\} \\ & - \frac{e^{-r\tau} \Phi'(d_2)}{M} \left\{ \frac{d^2 d_2}{dM^2} - \frac{2}{M} \frac{dd_2}{dM} - d_2 \left( \frac{dd_2}{dM} \right)^2 \right\} \\ & - \frac{2e^{-r\tau} \Phi(d_2)}{M^3} \end{aligned}$$



$$\begin{aligned}\frac{d^2 d_1}{dM^2} = & -\frac{1}{M\sigma(M)\sqrt{\tau}} \left\{ \frac{1}{M} + \frac{\sigma'(M)}{\sigma(M)} \right\} \\ & + \sigma''(M) \left\{ \frac{\sqrt{\tau}}{2} - \frac{\log(M) + r\tau}{\sigma(M)^2 \sqrt{\tau}} \right\} \\ & + \sigma'(M) \left\{ 2\sigma'(M) \frac{\log(M) + r\tau}{\sigma(M)^3 \sqrt{\tau}} \right. \\ & \left. - \frac{1}{M\sigma(M)^2 \sqrt{\tau}} \right\}\end{aligned}$$



$$\begin{aligned}\frac{d^2 d_2}{dM^2} = & -\frac{1}{M\sigma(M)\sqrt{\tau}} \left\{ \frac{1}{M} + \frac{\sigma'(M)}{\sigma(M)} \right\} \\ & -\sigma''(M) \left\{ \frac{\sqrt{\tau}}{2} + \frac{\log(M) + r\tau}{\sigma(M)^2\sqrt{\tau}} \right\} \\ & +\sigma'(M) \left\{ 2\sigma'(M) \frac{\log(M) + r\tau}{\sigma(M)^3\sqrt{\tau}} \right. \\ & \left. -\frac{1}{M\sigma(M)^2\sqrt{\tau}} \right\}\end{aligned}$$

