Nonparametric Estimation of Risk-Neutral Densities

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Financial Market

Arbitrage-free market, European call price

$$C_t(X) = e^{-r_t \tau} \int_0^\infty (S_T - X)^+ q_t(S_T)\, dS_T. \tag{1}$$

- S_T the underlying asset price at time T
- X the strike price
- $\tau = T - t$
- r_t deterministic risk free interest rate
- risk neutral measure Q_t, $dQ_t(S_T) = q_t(S_T) dS_T$

Financial Market

Physical measure P_t, $dP_t(S_T) = p_t(S_T)dS_T$

$$\begin{aligned} C_t(X) &= e^{-r_t\tau}\int_0^\infty (S_T - X)^+ \frac{q_t(S_T)}{p_t(S_T)} p_t(S_T)\, dS_T \\ &= e^{-r_t\tau}\int_0^\infty (S_T - X)^+ m_t(S_T) p_t(S_T)\, dS_T \end{aligned} \tag{2}$$

- $m_t(S_T)$ pricing kernel for discounting payoffs

Estimation of Risk Neutral Density (RND)

- second derivative: Breeden and Litzenberger (1978)

$$q_t(S_T) = e^{r_t \tau} \left\{ \frac{\partial^2 C_t(X)}{\partial X^2} \right\}_{X=S_T}. \tag{3}$$

- pricing kernel (PK)

$$q_t(S_T) = m_t(S_T) p_t(S_T). \tag{4}$$

- parametric stock price dynamic specifications are questionable (e.g. Black-Scholes model, Heston model)
  $\longrightarrow$ employ nonparametric methods

Nonparametric RND Estimation

- second derivative estimation of C can be done via kernel, local polynomial (Aït-Sahalia and Lo, 2000); splines (Shimko 1993); basis expansion (Jarrow and Rudd; 1982)

New approach: $q = mp$; m series expansion

- How does this approach compare with other standard?
- Can it be implemented in a dynamic context?

Outline

The Stochastic Model

Call data $\{X_i, Y_i\}_{i=1}^{n}$

$$Y_i = C(X_i) + \varepsilon_i, \quad \mathsf{E}[\varepsilon_i | X_i] = 0 \tag{5}$$

with

$$C(X) = e^{-r\tau} \int_0^\infty (S_T - X)^+ q(S_T)\, dS_T. \tag{6}$$

- estimate RND $q(S_T)$

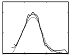
Local Polynomial Regression

Kernel ($\mathcal{K}$) based method

$$\mathcal{K}(u) : \mathbb{R} \to \mathbb{R} \quad \text{s.t.} \quad \int \mathcal{K}(u)du = 1 \quad \text{, and} \quad \mathcal{K}_h(u) = \frac{1}{h}\mathcal{K}\left(\frac{u}{h}\right)$$

Taylor expansion for C

$$\begin{aligned} C(X_i) &= \sum_{j=0}^{p} \frac{C^{(j)}(X)}{j!}(X_i - X)^j + \mathcal{O}\{(X_i - X)^{p+1}\} \qquad (7) \\ &= \sum_{j=0}^{p} \beta_j (X_i - X)^j + \mathcal{O}\{(X_i - X)^{p+1}\} \end{aligned}$$



Estimation of q

Minimize a locally weighted least square regression

$$\min_{\beta} \sum_{i=1}^{n} \left\{ Y_i - \sum_{j=0}^{p} \beta_j(X)(X - X_i)^j \right\}^2 \mathcal{K}_h(X - X_i). \tag{8}$$

- $j!\hat{\beta}_j(z)$ estimates $C^{(j)}(z)$

$$\hat{q}(S_T) = 2e^{r\tau}\hat{\beta}_2(X)_{|X=S_T}$$

Selection of Smoothing Parameter

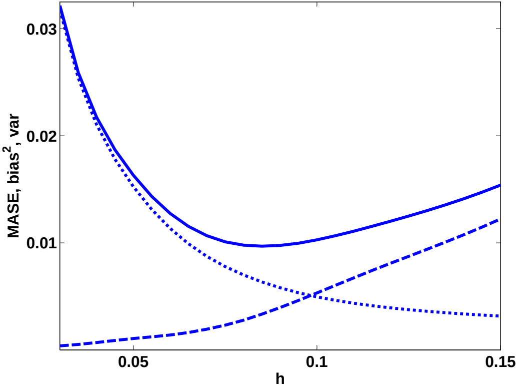


Figure 1: MASE (solid line), squared bias (dashed line) and variance part (dotted line) for simulated data, weights $w(x) = \mathrm{I}(x \in [0.05, 0.95])$

Some Theory

Weighted integrated mean square error (WIMSE)

$$\mathrm{WIMSE}(h, j) = \int \mathrm{E}[\hat{C}^{(j)}(X) - C^{(j)}(X)]^2 \psi(X) dX \tag{9}$$

where $\psi(X)$ is a weight function.

For a second order kernel

$$h^{\star} = \begin{cases} cn^{-1/(2p+2)} & \text{for } p - j \text{ odd} \\ c' n^{-1/(2p+4)} & \text{for } p - j \text{ even} \end{cases} \tag{10}$$

for c and c' constants.

Cross-validation method

Minimize the cross-validation criteria

$$\mathrm{CV}(h) = \sum_{i=1}^{n}\sum_{j\neq i}^{n}\Big\{Y_i - \widehat{C}_{h,-i}(X_i) - \widehat{C}^{(1)}_{h,-i}(X_i)(X_j - X_i) - \frac{1}{2}\widehat{C}^{(2)}_{h,-i}(X_i)(X_j - X_i)^2\Big\}^2 M(X_i) \tag{11}$$

where $0 \leq M(X_i) \leq 1$ is a weight function and $(\widehat{C}_{h,-i}, \widehat{C}^{(1)}_{h,-i}, \widehat{C}^{(2)}_{h,-i})$ the leave-one-out local cubic regression estimate.

Statistical properties

When $p - j$ is odd, the bias is

$$\mathsf{E}\left[\hat{C}^{(j)}(X)\right] - C^{(j)}(X) = h^{p-j+1} c_{1,j,p} \left\{ \frac{\omega^{(p+1)}(X)}{(p+1)!} \right\} + \mathcal{O}(h^{p-l+1}).$$

When $p - j$ is even, the bias is

$$\begin{aligned} \mathsf{E}\left[\hat{C}^{(j)}(X)\right] - C^{(j)}(X) &= h^{p-j+2} c_{2,j,p} \left\{ \frac{\omega^{(p+2)}(X)}{(p+2)!} \right\} \int u^{p+2} K(u) du \\ &+ h^{p-j+2} c_{3,j,p} \left\{ \frac{\omega^{(p+1)}(X) f^{(1)}(X)}{f(X)(p+1)!} \right\} \end{aligned}$$

where $\omega(X) = \left\{ \hat{C}(X) - C(X) \right\} f(X)$, $f(X)$ pdf of X.

Statistical properties

In either case

$$\mathrm{Var}\left\{\hat{C}^{(j)}(X)\right\} = \left\{\frac{c_{4,j,p}\sigma^2(X)}{nh^{2j+1}}\right\} + \mathcal{O}\left\{(nh^{2j+1})^{-1}\right\}.$$

with $c_{a,j,p}$ for $a = 1, 2, 3, 4$ constants (Ruppert and Wand, 1994).

Assuming that $p - j$ is odd:

$$\begin{aligned} \mathrm{MSE}(X, h, j) &= \mathrm{E}\left[\hat{C}^{(j)}(X) - C^{(j)}(X)\right]^2 \\ &= \mathcal{O}\{\underbrace{h^{2(p-j+1)}}_{bias^2} + \underbrace{(nh^{2j+1})^{-1}}_{var}\}. \end{aligned} \tag{12}$$

Application: Data

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de

- Datastream DAX 30 Price Index;
  5000 overlapping monthly returns

- EUREX European Option Data; tick observations;
  $\tau = 1$ month (21 days) on 20040121

Application

- Smoothing in call option space
- Smoothing in implied volatility space

$$\begin{aligned}\hat{C}(X) &= C_{BS}\{X; \hat{\sigma}(X)\} \\ &= \tau S_t \Phi(y + \hat{\sigma}\sqrt{\tau}) - e^{-r\tau} X \Phi(y)\end{aligned}$$

where Φ is the $\mathrm{N}(0,1)$ cdf and

$$y = \frac{\log(\frac{S_t}{K}) + (r - \frac{1}{2}\hat{\sigma}^2)\tau}{\hat{\sigma}\sqrt{\tau}}.$$

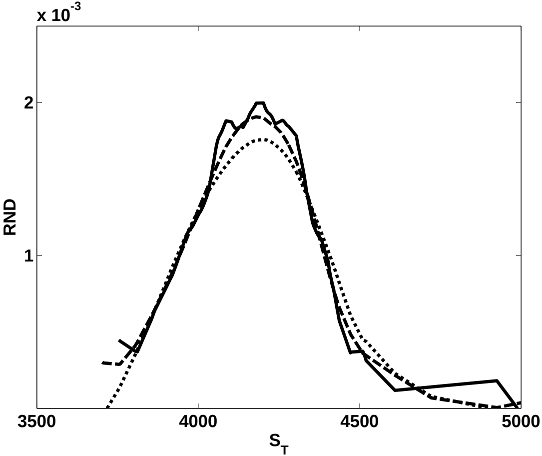
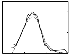


Figure 2: $\hat{q}(S_T)$ by local polynomial smoother for the optimally chosen bandwidth $h = 114.34$ by cross-validation (solid line) and oversmoothing bandwidths $h = 227.59$ (dashed line) and $h = 434.49$ (dotted line)



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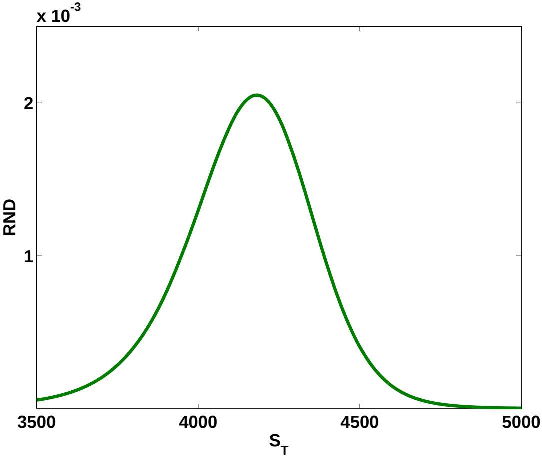
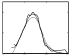


Figure 3: $\hat{q}(S_T)$ by Rookley method with oversmoothing bandwidth $h = 372.42$



The Stochastic Model

Call data $\{Y_i, X_i\}_{i=1}^n$

$$Y_i = C(X_i) + \varepsilon_i, \quad \mathsf{E}[\varepsilon_i | X_i] = 0 \tag{13}$$

with

$$C(X) = e^{-r\tau} \int_0^\infty (S_T - X)^+ m(S_T) p(S_T)\, dS_T$$

- estimate $m(S_T)$ and $p(S_T)$

Estimation of PK by Series Method

Fourier series expansion

$$m(S_T) = \sum_{l=1}^{L} \alpha_l g_l(S_T). \tag{14}$$

where $\{g_l\}_{l=1}^{L}$ are known **orthogonal** basis functions and $\alpha = (\alpha_1, \cdots, \alpha_L)^\top$ is unknown coefficient vector.

$$\begin{aligned} C(X) &= e^{-r\tau} \int_0^\infty (S_T - X)^+ \sum_{l=1}^{L} \alpha_l g_l(S_T) p(S_T) dS_T \qquad (15) \\ &= \sum_{l=1}^{L} \alpha_l \left\{ e^{-r\tau} \int_0^\infty (S_T - X)^+ g_l(S_T) p(S_T) dS_T \right\} \end{aligned}$$

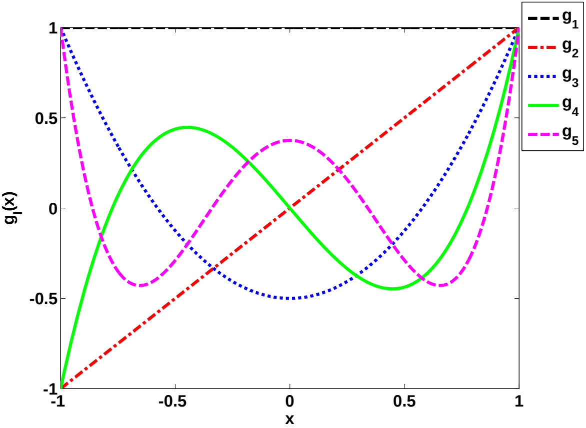


Figure 4: First five terms of the Legendre polynomials

Estimate $\alpha = (\alpha_1, ..., \alpha_L)^\top$ by LS

$$\check{\alpha} = \arg\min_{\alpha} \sum_{i=1}^{n} \left\{ Y_i - \sum_{l=1}^{L} \alpha_l \psi_{il} \right\}^2 \tag{16}$$

where

$$\psi_{il} = \psi_l(X_i) = e^{-r\tau} \int_0^\infty (S_T - X_i)^+ g_l(S_T) p(S_T) dS_T. \tag{17}$$

$\check{\alpha}$ unfeasible; ψ-s unknown (via p).

Estimate ψ_{il}

$$\hat{\psi}_{il} = e^{-r\tau} J^{-1} \sum_{s=1}^{J} (S_T^s - X_i)^+ g_l(S_T^s). \tag{18}$$

or

$$\hat{\psi}_{il} = e^{-r\tau} \int_0^\infty (S_T - X_i)^+ g_l(S_T) \hat{p}(S_T) dS_T. \tag{19}$$

where $\{S_T^s\}_{s=1}^J$ are simulated stock prices and $\hat{p}$ is a KDE of p.

Feasible estimator for α is

$$\hat{\alpha} = (\hat{\Psi}^\top \hat{\Psi})^- \hat{\Psi}^\top Y, \tag{20}$$

where $\hat{\Psi}_{(n\times L)} = (\hat{\psi}_{il})$ and $Y = (Y_1, \cdots, Y_n)^\top$,

$$\hat{C}(X) = \hat{\psi}^L(X)^\top \hat{\alpha}, \tag{21}$$

where $\hat{\psi}^L(X) = (\hat{\psi}_1(X), \ldots, \hat{\psi}_L(X))^\top$ and

$$\hat{m}(S_T) = g^L(S_T)^\top \hat{\alpha}, \tag{22}$$

where $g^L(S_T) = (g_1(S_T), \ldots, g_L(S_T))^\top$.

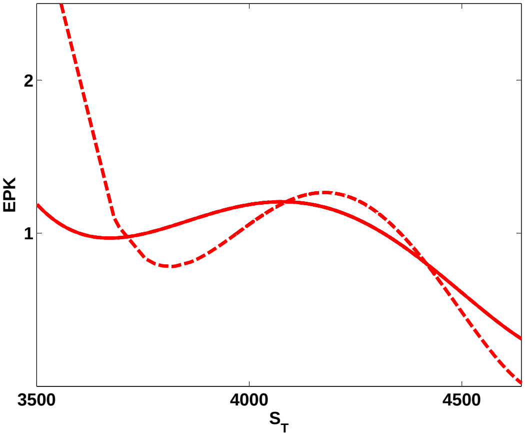


Figure 5: $\hat{m}(S_T)$ by Legendre basis expansion with $L = 5$ based on approximation (18) (solid line) and (19) of Ψ (dashed line)

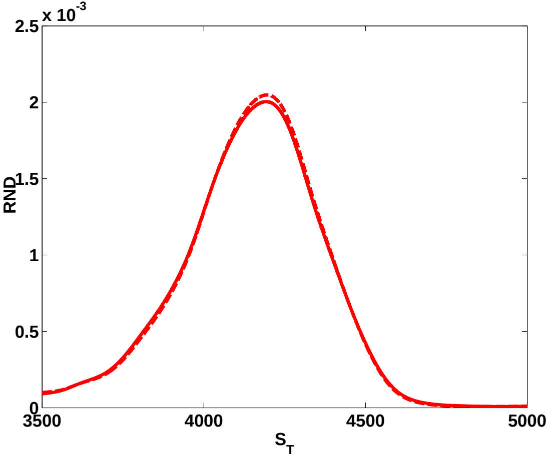


Figure 6: $\hat{q}(S_T)$ by Legendre basis expansion with $L = 5$ based on approximation (18) (solid line) and (19) of Ψ (dashed line)

Choice of the Tuning Parameter L

Optimal selection L: the resulting MISE equals the smallest possible integrated square error Li and Racine (2007)

- Mallows's $C(L)$ (or C_p), Mallows (1973)

$$C(L) = n^{-1} \sum_{i=1}^{n} \left\{ Y_i - \hat{\psi}^L(X_i)^\top \hat{\alpha} \right\}^2 + 2\sigma^2 (L/n)$$

where σ^2 is the variance of ε_i. One can estimate σ^2 by

$$\hat{\sigma}^2 = n^{-1} \sum_{i=1}^{n} \hat{\varepsilon}_i^2, \quad \text{with} \quad \hat{\varepsilon}_i = Y_i - \hat{C}(X_i).$$

- Generalized cross-validation, Craven and Wahba (1979)

$$CV^G(L) = \frac{n^{-1}\sum_{i=1}^{n}\left\{Y_i - \hat{\psi}^L(X_i)^\top \hat{\alpha}\right\}^2}{\{1-(L/n)\}^2}.$$

- Leave one out cross-validation, Stone (1974)

$$CV(L) = \sum_{i=1}^{n}\left\{Y_i - \psi^L_{-i}(X_i)^\top \hat{\alpha}_{-i}\right\}^2$$

where $\hat{\alpha}_{-i}$ is the leave one estimate of α obtained by removing (X_i, Y_i) from the sample.

Estimation of the PDF of S_T

- simulate $S_T^s = S_t e^{r_T^s}$, $s = 1 \dots J$ based on
  - simple historical returns $r_T^s = r_{t-s} = \log(S_{t-s}/S_{t-(s+1)})$
  - weighted historical returns $r_T^s = r_{t-s}\frac{\sigma_t^r}{\sigma_{t-s}^r}$ with $\{\sigma_{t-s}^r\}_{s=0}^J$ from a GARCH(1,1) specification model
- KDE for p

$$\widehat{p}_h(u) = \frac{1}{Jh}\sum_{s=1}^{J}\mathcal{K}\left(\frac{S_T^s - u}{h}\right) \tag{23}$$

with h selected by Silverman's rule-of-thumb.

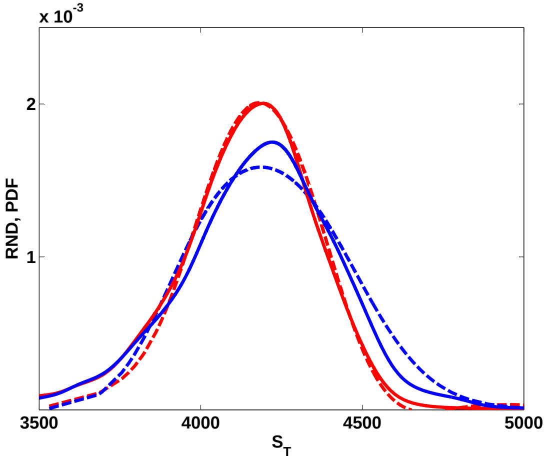


Figure 7: $\hat{q}$ with $\hat{m}$ by Legendre basis expansion with $L = 5$, $\hat{\psi}$ as in (19) (red) and $\hat{p}$ based on log-returns (solid line) and weighted log-returns (dashed line)

Statistical Properties

$$
\begin{aligned}
\int_0^\infty \{\hat{q}(u) & - q(u)\}^2 du = \int_0^\infty \{\hat{m}(u)\hat{p}(u) - m(u)p(u)\}^2 du \\
& = \int_0^\infty [\hat{m}(u)\{\hat{p}(u) - p(u)\}]^2 du \\
& + \int_0^\infty [p(u)\{\hat{m}(u) - m(u)\}]^2 du \\
& + \int_0^\infty 2\hat{m}(u)\{\hat{p}(u) - p(u)\}p(u)\{\hat{m}(u) - m(u)\} du \\
& = \mathcal{O}_p(n^{-4/5} + L/n + L^{-2\nu})
\end{aligned}
$$

for m that is ν-times continuously differentiable.

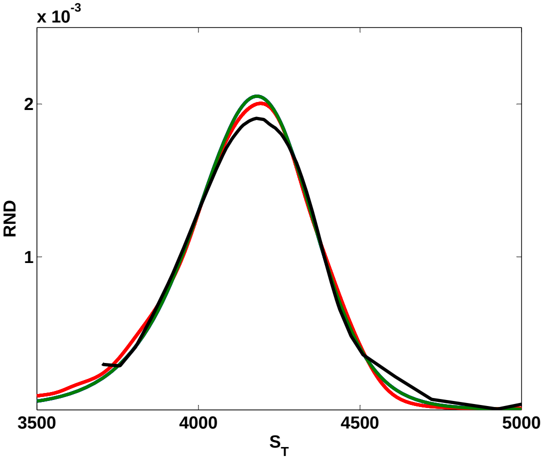


Figure 8: $\hat{q}(S_T)$ by local polynomial regression with $h = 227.59$ in call space (black), by Rookley method $h = 372.42$ in IV space (green), indirect estimation of the pricing kernel as Legendre basis expansion with $L = 5$ (red)

Conclusions

Estimator of q via

- Kernel methods
  - $\hat{q}$ by smoothing in call price space is highly sensitive to the bandwidth and wiggly in the tail
  - smoothing in implied volatility space yields a stable $\hat{q}$
- Pricing kernel (basis expansion)
  - EPK highly volatile w.r.t. basis and $\hat{p}$
  - stable $\hat{q}$ in the central region
  - preferred when performing constrained estimation
  - α_t can be interpreted dynamically

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Rookley (1997)

Return

Let C_{it} be the price of the i^{th} option at time t and K_{it} its strike price, and define the rescaled call option $c = C/S_t$ in terms of moneyness $M = S_t/K$ s.t.

$$
\begin{aligned}
c_{it} &= c\{M_{it}; \sigma(M_{it})\} = \Phi(d_1) - \frac{e^{-r\tau}\Phi(d_2)}{M_{it}} \\
d_1 &= \frac{\log(M_{it}) + \left\{r_t + \frac{1}{2}\sigma(M_{it})^2\right\}\tau}{\sigma(M_{it})\sqrt{\tau}} \\
d_2 &= d_1 - \sigma(M_{it})\sqrt{\tau}
\end{aligned}
$$

The RND is then

$$q(\cdot) = e^{r\tau}\frac{\partial^2 C}{\partial K^2} = e^{r\tau} S \frac{\partial^2 c}{\partial K^2}$$

with

$$\frac{\partial^2 c}{\partial K^2} = \frac{\mathrm{d}^2 c}{\mathrm{d}M^2}\left(\frac{M}{K}\right)^2 + 2\frac{\mathrm{d}c}{\mathrm{d}M}\frac{M}{K^2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2 c}{\mathrm{d}M^2} \quad = \quad & \Phi'(d_1)\left\{\frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} - d_1\left(\frac{\mathrm{d}d_1}{\mathrm{d}M}\right)^2\right\} \\
& -\frac{e^{-r\tau}\Phi'(d_2)}{M}\left\{\frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} - \frac{2}{M}\frac{\mathrm{d}d_2}{\mathrm{d}M} - d_2\left(\frac{\mathrm{d}d_2}{\mathrm{d}M}\right)^2\right\} \\
& -\frac{2e^{-r\tau}\Phi(d_2)}{M^3}
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} \quad = \quad & -\frac{1}{M\sigma(M)\sqrt{\tau}}\left\{\frac{1}{M}+\frac{\sigma'(M)}{\sigma(M)}\right\} \\
& +\sigma''(M)\left\{\frac{\sqrt{\tau}}{2}-\frac{\log(M)+r\tau}{\sigma(M)^2\sqrt{\tau}}\right\} \\
& +\sigma'(M)\left\{2\sigma'(M)\frac{\log(M)+r\tau}{\sigma(M)^3\sqrt{\tau}}\right. \\
& \left. -\frac{1}{M\sigma(M)^2\sqrt{\tau}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} = & -\frac{1}{M\sigma(M)\sqrt{\tau}}\left\{\frac{1}{M}+\frac{\sigma'(M)}{\sigma(M)}\right\} \\
& -\sigma''(M)\left\{\frac{\sqrt{\tau}}{2}+\frac{\log(M)+r\tau}{\sigma(M)^2\sqrt{\tau}}\right\} \\
& +\sigma'(M)\left\{2\sigma'(M)\frac{\log(M)+r\tau}{\sigma(M)^3\sqrt{\tau}}\right. \\
& \left. -\frac{1}{M\sigma(M)^2\sqrt{\tau}}\right\}
\end{aligned}
$$

Series Functions: Legendre polynomials

- First two polynomials

$$g_1(x) = 1$$
$$g_2(x) = 1 - x$$

- Recurrence relation for $l = 2, \cdots, L$

$$g_{l+1}(x) = \frac{1}{l}\left\{(2l - 1 - x)g_{l-1}(x) - (l - 1)g_{l-2}(x)\right\}.$$