Functional Principal Component Analysis for Derivatives of Multivariate Curves

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Derivatives of Multidimensional Functions

- $X(t) : [0,1]^g \rightarrow \mathbb{R}$ random function in $L^2([0,1]^g)$
- $\mathsf{E}\,[X(t)] = 0, \forall t \in [0,1]^g$ and $\int_{[0,1]^g} \mathsf{E}\left[X(t)^2\right] dt < \infty$

$$X^{(d)}(t) \stackrel{\text{def}}{=} \frac{\partial^{|d|}}{\partial t^d} X(t) = \frac{\partial^{d_1}}{\partial t_1^{d_1}} \cdots \frac{\partial^{d_g}}{\partial t_g^{d_g}} X(t_1, \ldots, t_g) \qquad (1)$$

$d = (d_1, ..., d_g)^\top$, $|d| = \sum_{j=1}^{g} |d_j|$, $d_j \in \mathsf{N}$ partial derivative of function X in the spatial direction $j = 1, \ldots, g$

- $X_i(t)$, $i = 1, \ldots, N$ sample counterpart

Objectives

1. estimate $X_i^{(d)}$ from a sample of discretely observed noisy curves
2. represent variability of $X_i^{(d)}$ in a low dimensional function space

Contribution

Functional **P**rincipal **C**omponent **A**nalysis (FPCA)

1. FPCA of derivatives $\left\{X_i^{(d)}\right\}_{i=1,\dots N}$
2. FPCA of curves $\{X_i\}_{i=1,\dots N}$ + derivatives of eigenfunctions

For noisy and discretely observed curves develop estimators and derive statistical properties for $g > 1$

Empirical findings

- state price density (SPD): volatility, skewness and tail factors
- term structure components

Outline

1. Motivation ✓
2. Theoretical Framework
3. Estimation Methodology
4. Asymptotic Properties
5. Application to SPDs
6. Conclusions

Karhunen-Loève Decomposition

Covariance function of $X^{(\nu)}$, $\nu = (\nu_1, \ldots, \nu_g)^\top, \;\; \nu_j \in \mathbb{N}$

$$\sigma^{(\nu)}(t, v) \stackrel{\text{def}}{=} \mathsf{E}[X^{(\nu)}(t) X^{(\nu)}(v)] \tag{2}$$

Eigenvalues $\lambda_1^{(\nu)} \geq \lambda_2^{(\nu)} \geq \ldots$ and functional PC $\varphi_r^{(\nu)}$

$$\int_{[0,1]^g} \sigma^{(\nu)}(t, v) \varphi_r^{(\nu)}(v) dv = \lambda_r^{(\nu)} \varphi_r^{(\nu)}(t) \tag{3}$$

$X^{(\nu)}$ admits decomposition

$$X^{(\nu)}(t) = \sum_{r=1}^{\infty} \delta_r^{(\nu)} \varphi_r^{(\nu)}(t), \tag{4}$$

$\delta_r^{(\nu)} = \int_{[0,1]^g} X^{(\nu)}(t) \varphi_r^{(\nu)}(t) dt$, $\mathsf{E}(\{\delta_r^{(\nu)}\}^2) = \lambda_r^{(\nu)}$, $\mathsf{E}(\delta_r^{(\nu)} \delta_s^{(\nu)}) = 0$ for $r \neq s$, $r, s = 1, \ldots, \infty$

Two Approaches to Derivatives Using FPCA

1. For $\nu = (d_1, \ldots, d_g)^\top$

$$X^{(d)}(t) = \sum_{r=1}^{\infty} \delta_r^{(d)} \varphi_r^{(d)}(t)$$

2. For $\nu = (0, \ldots, 0)^\top$, $\gamma_r := \varphi_r^{(0)}$, $\lambda_r := \lambda_r^{(0)}$ and $\delta_r := \delta_r^{(0)}$

$$X(t) = \sum_{r=1}^{\infty} \delta_r \gamma_r(t)$$

$$X^{(d)}(t) = \sum_{r=1}^{\infty} \delta_r \gamma_r^{(d)}(t)$$

$$\int_{[0,1]^g} \frac{\partial^{|d|}}{\partial v^d} \left(\sigma(t,v)\gamma_r(v)\right) dv = \lambda_r \gamma_r^{(d)}(t). \tag{5}$$

The Empirical Model

Noisy and discrete observations

$$Y_i(t_{ik}) = X_i(t_{ik}) + \varepsilon_{ik} = \sum_{r=1}^{\infty} \delta_{ri}\gamma_r(t_{ik}) + \varepsilon_{ik}, \qquad (6)$$

$t_i = (t_{i1}, \ldots, t_{iT_i})^\top$, $t_{ik} \in [0,1]^g$, $k = 1, \ldots, T_i, i = 1, \ldots, N$
ε_{ik} i.i.d., $\mathrm{E}[\varepsilon_{ik}] = 0$, $\mathrm{Var}(\varepsilon_{ik}) = \sigma^2_{i\varepsilon}$, ε_{ik} independent of $X_i(t_{ik})$.

Dual Method

Given $\{X_i\}_{i=1}^N$, dual covariance is $N \times N$ matrix $M^{(\nu)}$

$$M_{ij}^{(\nu)} = \int_{[0,1]^g} X_i^{(\nu)}(t) X_j^{(\nu)}(t) dt$$

Eigenvectors $p_r^{(\nu)}$ and eigenvalues $l_r^{(\nu)}$ of $M^{(\nu)}$

- For $\nu = d$

$$\hat{\varphi}_r^{(d)}(t) = \frac{1}{\sqrt{l_r^{(d)}}} \sum_{i=1}^N p_{ir}^{(d)} X_i^{(d)}(t) \; , \; \hat{\lambda}_r^{(d)} = \frac{l_r^{(d)}}{N} \text{ and } \hat{\delta}_{ri}^{(d)} = \sqrt{l_r^{(d)}} p_{ir}^{(d)}$$

- For $\nu = \vec{0}$

$$\hat{\gamma}_r^{(d)}(t) = \frac{1}{\sqrt{l_r}} \sum_{i=1}^N p_{ir} X_i^{(d)}(t) \; , \; \hat{\lambda}_r^{(0)} = \frac{l_r^{(0)}}{N} \text{ and } \hat{\delta}_{ri}^{(0)} = \sqrt{l_r^{(0)}} p_{ir}^{(0)}$$

Multivariate Local Polynomial Estimator

Let $k = (k_1, \dots, k_g)^\top$, $k_l \in \mathbb{N}$ and $|k| \stackrel{\text{def}}{=} \sum_{j=1}^{g} |k_j|$

$$\min_{\beta(t)} \sum_{l=1}^{T} \left[Y(t_l) - \sum_{0 \le |k| \le \rho} \beta_k(t)(t_l - t)^k \right]^2 K_B(t_l - t)$$

K_B non-negative, symmetric and bounded multivariate kernel fct; B a $g \times g$ diagonal bandwidth matrix $diag(B) = b = (b_1, \dots, b_g)^\top$

$$\hat{X}_b^{(\nu)}(t) = \nu! \hat{\beta}_\nu(t) = \nu! \sum_{l=1}^{T} W_\nu^T \left((t_l - t) \circ b^{-1} \right) Y(t_l). \qquad (7)$$

weight function W_ν^T, $a \circ b \stackrel{\text{def}}{=} (a_1 b_1, \dots, a_g b_g)^\top$,
$c! \stackrel{\text{def}}{=} c_1! \times \dots \times c_g!$, for any vectors a, $b \in \mathbb{R}^g$ and $c \in \mathbb{N}^g$

Quadratic Integral Estimation

$\theta_{\nu,\rho}$ estimator of the squared integrated functions $\int_{[0,1]^g} X^{(\nu)}(t)^2 dt$

$$\int_{[0,1]^g} \nu!^2 \sum_{k=1}^{T}\sum_{l=1}^{T} W_\nu^T\left((t_k - t)\circ b^{-1}\right) W_\nu^T\left((t_l - t)\circ b^{-1}\right) Y(t_l)Y(t_k)dt$$
$$-\nu!^2\hat{\sigma}_\varepsilon^2 \int_{[0,1]^g} \sum_{k=1}^{T} W_\nu^T\left((t_k - t)\circ b^{-1}\right)^2 dt$$

$\hat{\sigma}_\varepsilon^2$ is a consistent estimator of σ_ε^2. The second term is introduced to cancel the bias in $\mathrm{E}\left[Y^2(t_k)\right] = X(t_k)^2 + \sigma_\varepsilon^2$.

Empirical Dual Matrix

Replace the integrals in (10) with the Riemann sums

$$\hat{M}_{ij}^{(\nu)} = \begin{cases} \nu!^2 \sum_{k=1}^{T_i} \sum_{l=1}^{T_j} w_\nu^T(t_{ik}, t_{jl}, b) Y_j(t_{jl}) Y_i(t_{ik}) & \text{if } i \neq j \\ \nu!^2 \Big(\sum_{k=1}^{T_i} \sum_{l=1}^{T_i} w_\nu^T(t_{ik}, t_{il}, b) Y_i(t_{il}) Y_i(t_{ik}) & \\ -\hat{\sigma}_{i\varepsilon}^2 \sum_{k=1}^{T_i} w_\nu^T(t_{ik}, t_{ik}, b) \Big) & \text{if } i = j \end{cases}$$

$$w_\nu^T(t_{ik}, t_{jl}, b) := T^{-1} \sum_{m=1}^{T} W_\nu^T\left((t_{ik} - t_m) \circ b^{-1}\right) W_\nu^T\left((t_{jl} - t_m) \circ b^{-1}\right)$$

The estimator for $M^{(0)}$ is given by setting $\nu = (0, \ldots, 0)^\top$ and the estimator for $M^{(d)}$ by $\nu = d$.

Estimation of Basis Functions

Denote $\hat{\lambda}^{(d)}_{r,T}$, $\hat{\delta}^{(d)}_{r,T}$, $\hat{\varphi}^{(d)}_{r,T}$ and $\hat{\gamma}^{(d)}_{r,T}$ the empirical counterparts of $\hat{\lambda}^{(d)}_{r}$, $\hat{\delta}^{(d)}_{r}$, $\hat{\varphi}^{(d)}_{r}$ and $\hat{\gamma}^{(d)}_{r}$ with

$$\hat{\gamma}^{(d)}_{r,T}(t) = \frac{1}{\sqrt{\hat{l}_r}} \sum_{i=1}^{N} \hat{p}_{ir,T} \hat{X}^{(d)}_{i,h}(t);\ \hat{\varphi}^{(d)}_{r,T}(t) = \frac{1}{\sqrt{\hat{l}^{(d)}_r}} \sum_{i=1}^{N} \hat{p}^{(d)}_{ir,T} \hat{X}^{(d)}_{i,h}(t)$$

with local polynomial estimator with bandwidth h

$$\hat{X}^{(d)}_{i,h}(t) = d!\hat{\beta}_d(t) = d! \sum_{l=1}^{T} W^T_d \left((t_l - t) \circ h^{-1}\right) Y_i(t_l). \qquad (8)$$

Optimal Choice of L

For the constant $C_{NT} = \min(\sqrt{N}, \sqrt{T})$ and $L^{\max} < \min(N, T)$

$$PC^{(\nu)}(k^*) = \min_{k \in \mathbb{N}, k \leq L_{\max}} \left[\left(\sum_{r=k+1}^{N} \hat{\lambda}_r^{(\nu)} \right) + k \left(\sum_{r=L^{\max}}^{N} \hat{\lambda}_r^{(\nu)} \right) \left(\frac{\log(C_{NT}^2)}{C_{NT}^2} \right) \right]$$

or

$$IC^{(\nu)}(k^*) = \min_{k \in \mathbb{N}, k \leq L} \left[\log \left(\frac{1}{N} \sum_{r=k+1}^{N} \hat{\lambda}_r^{(\nu)} \right) + k \left(\frac{\log(C_{NT}^2)}{C_{NT}^2} \right) \right].$$

Here using $\nu = (0, \ldots, 0)^\top$ will give L while using $\nu = d$ will give the factors L_d.

Lemma 1

Lemma

Under Assumptions 7.1- 7.4, X is $m \geq 2|\nu|$ times continuously differentiable, the local polynomial regression is of order ρ with $|\nu| \leq \rho < m$ and $|\hat{\sigma}^2_\varepsilon - \sigma^2_\varepsilon| = \mathcal{O}_P(T^{-1/2})$. As $T \to \infty$ and $\max(b)^{\rho+1} b^{-\nu} \to 0$, $\frac{\log(T)}{Tb_1 \times \cdots \times b_g} \to 0$ *as* $Tb_1 \times \cdots \times b_g b^{4\nu} \to \infty$

$$
\begin{aligned}
\mathrm{E}_{t,Y}\left[\theta_{\nu,\rho}\right] - \int_{[0,1]^g} X^{(\nu)}(t)^2 dt \\
= \mathcal{O}_p\left(\max(b)^{\rho+1} b^{-\nu} + \frac{1}{T^{3/2}(b^{2\nu} b_1 \times \cdots \times b_g)} \right)
\end{aligned}
$$

$$
\mathrm{Var}_{t,Y}(\theta_{\nu,\rho}) = \mathcal{O}_p\left(\frac{1}{T^2 b_1 \times \cdots \times b_g b^{4\nu}} + \frac{1}{T} \right),
$$

$\mathrm{E}_{t,Y}$ and $\mathrm{Var}_{t,Y}$ conditional expectation and variance given t, Y.

Proposition 4.1 and Remark 4.2

Proposition

Under the requirements of Lemma 1

$$|M_{ij}^{(\nu)} - \hat{M}_{ij}^{(\nu)}| = \mathcal{O}_P\left(\max(b)^{\rho+1} b^{-\nu} + \left(\frac{1}{T^2 b_1 \times \cdots \times b_g b^{4\nu}} + \frac{1}{T}\right)^{1/2}\right)$$

Remark

Under the assumptions of Lemma 1 and using Proposition 4.1 we can estimate $M^{(\nu)}$ *such that if* $m > \rho \geq \frac{g}{2} - 1 + 3\sum_{l=1}^{g} \nu_l$, *bandwidth* $b = T^{-\alpha}$ *with* $\frac{1}{2(\rho+1-\sum_{l=1}^{g}\nu_l)} \leq \alpha \leq \frac{1}{g+4\sum_{l=1}^{g}\nu_l}$ *then* $|M_{ij}^{(\nu)} - \hat{M}_{ij}^{(\nu)}| = \mathcal{O}_P(1/\sqrt{T})$.

Basis Properties

For $\max(h)^{p+1} h^{-d} \to 0$, $\left(\max(h)^{p+1} T h^{-d}\right)^{-1} \to 0$ as $T \to \infty$ and p chosen such that $p - |d|$ is odd

$$\mathrm{E}_{t,Y}\left[\frac{1}{\sqrt{l_r^{(\nu)}}} \sum_{i=1}^{N} p_{ir}^{(\nu)} \left(X_i^{(d)}(t) - \hat{X}_{i,h}^{(d)}(t)\right)\right] = \mathcal{O}_p(\max(h)^{p+1} h^{-d})$$

$$\mathrm{Var}_{t,Y}\left(\frac{1}{\sqrt{l_r^{(\nu)}}} \sum_{i=1}^{N} p_{ir}^{(\nu)} \hat{X}_{i,h}^{(d)}(t)\right) = \mathcal{O}_p\left(\frac{1}{N T h_1 \times \cdots \times h_g h^{2d}}\right).$$

Proposition 4.3

Proposition

Under the requirements of Lemma 1, Assumptions 7.6 and 7.7, Remark 4.2, and for $\inf_{s \neq r} |\lambda_r - \lambda_s| > 0$, $r, s = 1, \ldots, N$ *and* $\max(h)^{p+1} h^{-d} \to 0$ *with* $NTh_1 \ldots h_g h^{2d} \to \infty$ *as* $T, N \to \infty$ *we obtain*

a) $|\gamma_r^{(d)}(t) - \hat{\gamma}_{r,T}^{(d)}(t)| = \mathcal{O}_p\left(\max(h)^{p+1} h^{-d}\right) + \mathcal{O}_p\left((NTh_1 \times \cdots \times h_g h^{2d})^{-1/2}\right)$

b) $|\hat{\varphi}_r^{(d)}(t) - \hat{\varphi}_{r,T}^{(d)}(t)| = \mathcal{O}_p\left(\max(h)^{p+1} h^{-d}\right) + \mathcal{O}_p\left((NTh_1 \times \cdots \times h_g h^{2d})^{-1/2}\right)$

As a consequence, the resulting global optimal bandwidth is given by $h_{r,opt} = \mathcal{O}_p\left((NT)^{-1/(g+2p+2)}\right)$.

Properties under a Factor Structure

If a true factor model with L components is assumed, the basis representation to reconstruct $X^{(d)}$ is arbitrary, in the sense that

$$X^{(d)}(t) = \sum_{r=1}^{L} \delta_r \gamma_r^{(d)}(t) = \sum_{r=1}^{L_d} \delta_r^{(d)} \varphi_r^{(d)}(t). \tag{9}$$

where L is an upper bound for L_d.

$$\hat{X}_{i,FPCA_1}^{(d)}(t) \stackrel{\text{def}}{=} \sum_{r=1}^{L} \hat{\delta}_{ir,T} \hat{\gamma}_{r,T}^{(d)}(t) \approx \hat{X}_{i,FPCA_2}^{(d)}(t) \stackrel{\text{def}}{=} \sum_{r=1}^{L_d} \hat{\delta}_{ir,T}^{(d)} \hat{\varphi}_{r,T}^{(d)}(t). \tag{10}$$

Proposition 4.4

Proposition

Assume that a factor model with L factors holds for X. For $NT^{-1} \to 0$, together with the requirements of Proposition 4.3, the true curves can be reconstructed

a) $|X_i^{(d)}(t) - \hat{X}_{i,FPCA_1}^{(d)}(t)| = \mathcal{O}_p\left(T^{-1/2} + \max(h)^{p+1}h^{-d} + (NTh_1 \times \cdots \times h_g h^{2d})^{-1/2}\right)$

b) $|X_i^{(d)}(t) - \hat{X}_{i,FPCA_2}^{(d)}(t)| = \mathcal{O}_p\left(T^{-1/2} + \max(h)^{p+1}h^{-d} + (NTh_1 \times \cdots \times h_g h^{2d})^{-1/2}\right).$

Application to SPDs: Modeling Framework

Let $C : \mathbb{R}^2_{\geq 0} \rightarrow \mathbb{R}$ denote the price function of a European call option with strike price k and maturity τ such that

$$C(k, \tau) = \exp\left(-r_\tau \tau\right) \int_0^\infty (s_\tau - k)^+ q(s_\tau, \tau)\, ds_\tau, \tag{11}$$

where r_τ is the annualized risk free interest rate for maturity τ, s_τ the unknown price of the underlying asset at maturity, k the strike price and q the state price density of s_τ. One can show that

$$q(s_\tau, \tau) = \exp\left(r_\tau \tau\right) \left.\frac{\partial^2 C(k, \tau)}{\partial k^2}\right|_{k = s_\tau}. \tag{12}$$

Let s_0 be the asset price at the moment of pricing and assume it to be fixed. Then by the no-arbitrage condition, the forward price for maturity τ is

$$F_\tau = \int_0^\infty s_\tau q(s_\tau, \tau) ds_\tau = s_0 \exp(r_\tau \tau). \tag{13}$$

Suppose that the call price is homogeneous of degree one in the strike price. Then

$$C(k, \tau) = F_\tau C(k/F_\tau, \tau). \tag{14}$$

If we denote $m = k/F_\tau$ the moneyness, it is easy to verify that

$$\frac{\partial^2 C(k, \tau)}{\partial k^2} = \frac{1}{F_\tau} \frac{\partial^2 C(m, \tau)}{\partial m^2}. \tag{15}$$

Then one can show that for $d = (2, 0)^\top$,
$C^{(d)}(m, \tau)|_{m = s_\tau / F_\tau} = q(s_\tau / s_0, \tau) = s_0 q(s_\tau, \tau)$.

Data Description

- **Source**: Research Data Center (RDC)
- DAX 30 Index opening price
- EUREX European Call Option: settlement prices
  daily observations, time window length: 2002 - 2012
- LIBOR rates
- VDAX implied volatility index for DAX 30 underlying

Selection of L

$r,\ L_{\max}$	1	2	3	4	5	6	7	8
$\hat{\lambda}_{r,T} \times 10^6$	133.29	18.90	2.69	1.62	0.49	0.34	0.26	0.09
$\hat{\lambda}_{r,T}/\hat{\lambda}_{r+1,T}$	7.05	7.01	1.66	3.28	1.44	1.31	2.83	1.18
$k^*(PC^{(0)})$	-	-	-	-	-	-	7	8
$k^*(IC^{(0)})$	-	-	-	-	-	-	7	-

Table 1: Estimated eigenvalues and eigenvalue ratios. Number of factors by $PC^{(0)}$ and $IC^{(0)}$ criteria

Term Effect and Jumps

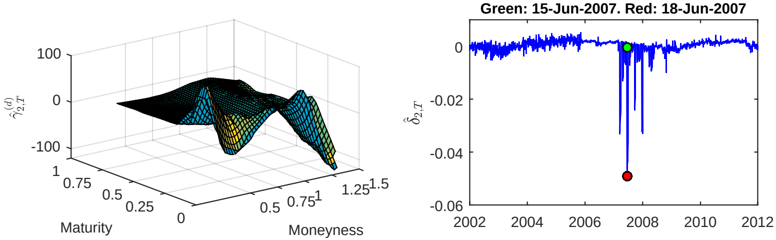
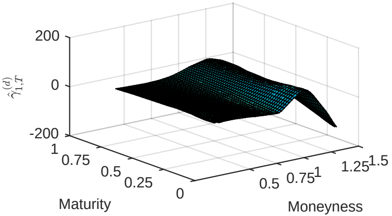


Figure 1: Estimated components $\hat{\gamma}^{(d)}_{2,T}$ and its loadings obtained by the decomposition of the dual covariance matrix $\hat{M}^{(0)}$. The effect of expiration date on the level of estimated loadings $\hat{\delta}_{2,T}$

Estimated Variance Factor



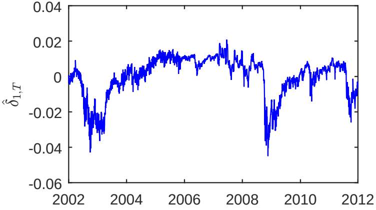


Figure 2: Estimated components $\hat{\gamma}^{(d)}_{1,T}$ and its loadings obtained by the decomposition of the dual covariance matrix $\hat{M}^{(0)}$

Estimated Skewness Factor

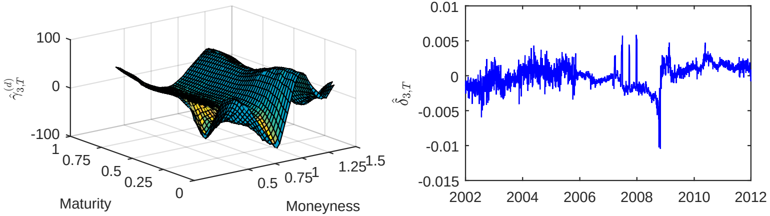


Figure 3: Estimated components $\hat{\gamma}^{(d)}_{3,T}$ and its loadings obtained by the decomposition of the dual covariance matrix $\hat{M}^{(0)}$

Estimated Tail Factor or Volatility of Volatility

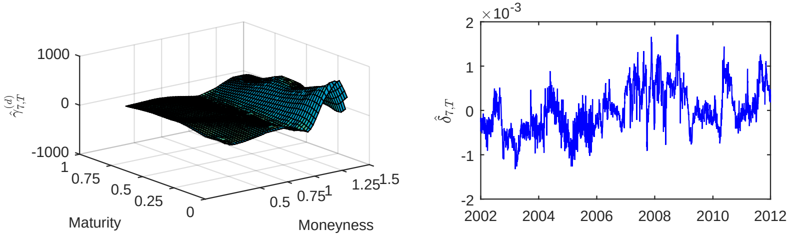


Figure 4: Estimated components $\hat{\gamma}^{(d)}_{7,T}$ and its loadings obtained by the decomposition of the dual covariance matrix $\hat{M}^{(0)}$

AR(1) Parameters Dynamics

$$\hat{\delta}_{ir,T} = b_r \hat{\delta}_{i-1r,T} + e_{ir}, \quad \mathrm{Var}(e_{ir}) = \sigma^2_{er}, \quad r = 1, 2, 3 \qquad (16)$$

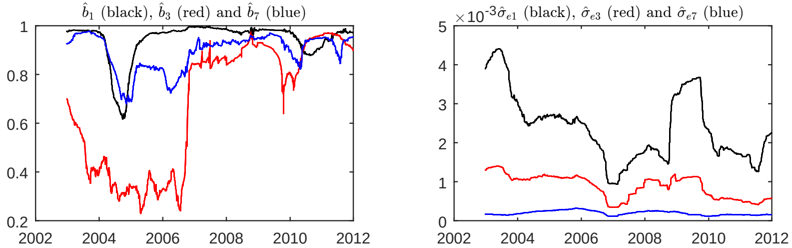


Figure 5: AR(1) coefficients and error standard deviations estimated daily with a moving window of 250 observations

Error Covariance Dynamics

$$\hat{\delta}_{ir,T} = b_r \hat{\delta}_{i-1r,T} + e_{ir}, \quad \mathrm{Var}(e_{ir}) = \sigma^2_{er}, \quad r = 1, 2, 3 \qquad (17)$$

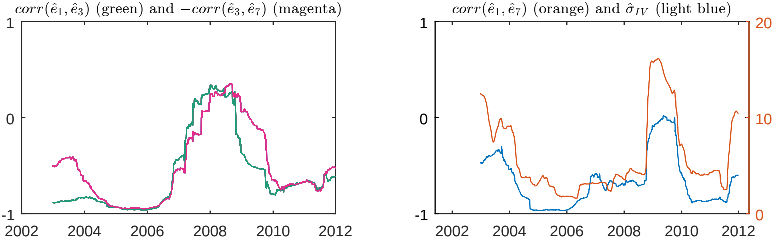


Figure 6: Pairwise correlations between residuals of univariate AR(1) regressions, and time varying standard deviation of the VDAX estimated daily with a moving window of 250 observations

Conclusions

- When an underlying factor model is assumed, estimating curve derivatives from observed discrete and noisy data using a low-dimensional representation, $M^(0)$ method performs better both asymptotically and in finite sample.

- We identify three main components, which can be interpreted as volatility, skewness and tail factors. We also find evidence for term structure variation.

- We find evidence for time varying leverage effect and sign reversal.

Assumptions 7.1 -7.3

Assumption

The random grid $t_{i1}, \dots, t_{iT_i}$, $t_{ij} \in [0,1]^g$ *has a common bounded and continuously differentiable density* f *with support* $\operatorname{supp}(f) = [0,1]^g$ *and the integrand* $u \in \operatorname{supp}(f)$ *and* $\inf_u f(u) > 0$.

Assumption

$\mathsf{E}(\varepsilon_{ik}) = 0$, $\operatorname{Var}(\varepsilon_{ik}) = \sigma^2_{i\varepsilon} > 0$ *and* ε_{ik} *are independent of* X_i, *and* $\mathsf{E}\left[\varepsilon^4_{ik}\right] < \infty, \forall i, k$.

Assumption

Kernel K *is bounded and has compact support on* $[-1,1]^g$ *such that for* $u \in \mathbb{R}^g$ $\int uu^T K(u)du = \mu(K)I$ *where* $\mu(K) \neq 0$ *is a scalar and* I *is the* $g \times g$ *identity matrix.*

Assumptions 7.4 - 7.6

Assumption

$\rho - \sum_{l=1}^{g} d_l$ and $p - \sum_{l=1}^{g} d_l$ are odd.

Assumption

$|\hat{\sigma}_{i\varepsilon}^2 - \sigma_{i\varepsilon}^2| = \mathcal{O}_P(T^{-1/2})$

Assumption

$$\sup_{r\in\mathbb{N}} \sup_{t\in[0,1]^g} |\varphi_r^{(d)}(t)| < \infty\ , \ \sup_{r\in\mathbb{N}} \sup_{t\in[0,1]^g} |\gamma_r^{(d)}(t)| < \infty$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \mathsf{E}\left[\left(\delta_{ri}^{(\nu)}\right)^2 \left(\delta_{si}^{(\nu)}\right)^2\right] < \infty\ , \ \sum_{q=1}^{\infty}\sum_{s=1}^{\infty} \mathsf{E}\left[\left(\delta_{ri}^{(\nu)}\right)^2 \delta_{si}^{(\nu)}\delta_{qi}^{(\nu)}\right] < \infty$$

for all $r \in \mathbb{N}$.

Assumptions 7.6

Assumption

The eigenvalues are distinguishable such that for any T and N and fixed $r \in 1, \ldots, L$ there exists $0 < C_{1,r} < \infty$, $0 < C_{2,r} \leq C_{3,r} < \infty$ such that

$$NC_{2,r} \leq l_r^{(\nu)} \leq NC_{3,r}$$

$$\min_{s=1,\ldots,N; s \neq r} |l_r^{(\nu)} - l_s^{(\nu)}| \geq NC_{1,r}.$$