# Pricing Chinese Rain

## a Multi-Site, Multi-Period Equilibrium Model

Wolfgang Karl Härdle
Maria Osipenko

Ladislaus von Bortkiewicz
Chair of Statistics
C.A.S.E. Centre for Applied Statistics and Economics
School of Business and Economics
and Faculty of Agriculture and Horticulture
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de

# No rain, no grain...

# Weather risks

- source of uncertainty in crop production
- livestock farms and demand for food products affected
- weather derivatives (WD) are financial instruments that permit the trade with weather risks

→ crop insurance issuer can transfer weather risks on financial markets

→ make crop insurance affordable for farmers (China)

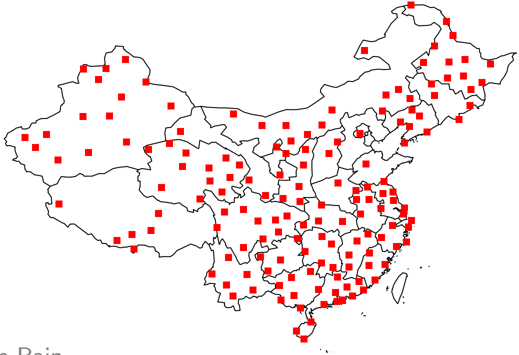
# Rain does not fall on one roof alone...

- Agriculture
- Other industries - tourism, entertainment, food retail
- Diversification of financial portfolio (Perez-Gonzalez & Yun, 2010)

雨

# Rainfall Data

- Daily rainfall data (from RDC)
- 29 Provinces, 105 stations in China
- from 19510101 to 20091130



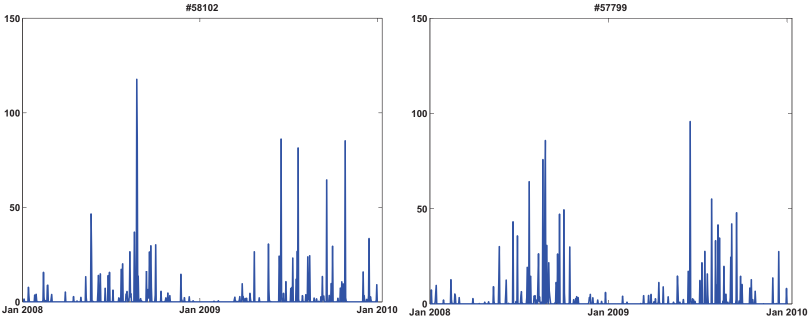



Figure 1: Daily precipitation amount in 0.1 mm for Anhui (left) and Jiangxi (right).

# Pricing rainfall

- development of appropriate pricing approach
- statistical modelling of relevant weather variables
- quantification of the relationship between weather variables and production

# Outline

1. Motivation ✓
2. Pricing Model
3. Statistical model for rainfall
4. Income-rainfall relationship
5. Simulation results
6. Outlook

# Definitions

Given

- Set of geographical sites $\mathcal{S}$
- planing periods $t = 0, 1, \ldots, T$
- set of agents $J$ contains buyers (crop insurance) and an investor

Portfolios: $\alpha_{j,t} = (\alpha_{j,t,s_1}, \ldots, \alpha_{j,t,s_n})^\top, \; s_i \in \mathcal{S}, \; i \leq n$ **weather bonds** and $\beta_{j,t}$ **risk free assets** $B_t$.

Price of the $s$th weather bond $W_{t,s}$, $s \in \mathcal{S}$, at $t = 0, \ldots, T$ **positive** random variable on $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P}\}$.

# Agents on the Market

Buyer(Crop insurer) $j$

- rainfall exposed income $I_j$
- portfolio: WDs + Bond
- exponential utility with risk aversion $a_j$

Investor $m$

- specializes on issue of WDs
- portfolio: WDs + Bond
- exponential utility with risk aversion $a_m$

# Buyer's optimization problem

Profit of Buyer $j$

$$\Pi_{j,T} = I_j\{(W_{T,s})_{s\in\mathcal{S}_j}, P_{j,T}\} + \sum_{s\in\mathcal{S}_j} \alpha_{j,T,s} W_{T,s} + \beta_{j,T} B_T$$
$$= I_j\{(W_{T,s})_{s\in\mathcal{S}_j}, P_{j,T}\} + V_{j,T}$$

with $I_j\{(W_{T,s})_{s\in\mathcal{S}_j}, P_{j,T}\}$ a function of weather events $(W_{T,s})_{s\in\mathcal{S}_j}$, production price $P_{j,T}$ and $\beta_{j,T}B_T$, $\alpha_{j,T}W_{T,s}$ payoffs of bond and WD on station $s \in \mathcal{S}_j$ (set of stations Buyer $j$ depend on).
Utility maximization

$$\max_{\{\alpha_{j,t+1,s}\}_{s\in\mathcal{S}_j}} \mathsf{E}_t \left\{ U_j \left( \Pi_{j,T} \right) \right\}$$
$$\text{s.t.} \quad \sum_{s\in\mathcal{S}_j} \alpha_{j,t+1,s} W_{t,s} + \beta_{j,t+1} B_t - V_{j,t,s} = 0.$$

# Investor's optimization problem

Profit of investor $m$

$$\begin{aligned} \Pi_{m,T} &= -\sum_{s\in\mathcal{S}} \alpha_{m,T,s} W_{T,s} + \beta_{m,T} B_T \\ &= V_{m,T} \end{aligned}$$

with $\sum_{s\in\mathcal{S}} \alpha_{m,T,s} W_{T,s}$, $\beta_{m,T} B_T$ payoffs of WD and bond, $\mathcal{S}$ set of all traded stations.

Utility maximization

$$\max_{\{\alpha_{m,t+1}\}_{s\in\mathcal{S}}} \mathsf{E}_t \left\{ U_m \left( \Pi_{m,T} \right) \right\}$$

$$\text{s.t.} \quad \sum_{s\in\mathcal{S}} \alpha_{m,t+1,s} W_{t,s} - \beta_{m,t+1} B_t + V_{m,t} = 0.$$

# Solution via dynamic programming

| time | state variables | control variable |
|---|---|---|
| 0 | $(W_{0,s})_{s\in S}, (V_{0,k})_{k=j,m}$ | $(\alpha_{1,j,s})_{s\in\mathcal{S}_j}, (\alpha_{1,m,s})_{s\in\mathcal{S}}$ |
| ... | | |
| T-1 | $(W_{T-1,s})_{s\in\mathcal{S}}, (V_{T-1,k})_{k=j,m}$ | $(\alpha_{T,j,s})_{s\in\mathcal{S}_j}, (\alpha_{T,m,s})_{s\in\mathcal{S}}$ |
| T | $(W_{T,s})_{s\in\mathcal{S}}, \{I_j(W_{T,s}, P_T)\}_{s\in\mathcal{S}_j}$ | – |

- start in $T-1$ and maximize the expected utility of $T$ choosing $(\alpha_{kTs})_{s\in\mathcal{S},\ k=j,m}$
- under utility indifference derive demand/supply functions for $T-1$,
- move to the next period and the maximize the corresponding expectation, continue to the present period.

# Buyer's Inverse Demand

$$
\begin{aligned}
W_{T-1s'} &= \frac{1}{a_j R \alpha_{jTs'}} \log \frac{\mathrm{E}_{T-1}\left[\exp\left\{-a_j(I_j + \sum_{s\in\mathcal{S}_j, s\neq s'} \alpha_{jTs} W_{Ts})\right\}\right]}{\mathrm{E}_{T-1}\left[\exp\left\{-a_j(I_j + \sum_{s\in\mathcal{S}_j} \alpha_{jTs} W_{Ts})\right\}\right]} \\
\Theta_{jT-1} &= \exp\{a_j \sum_{s\in\mathcal{S}_j} \alpha_{jTs} W_{T-1s} R\} \\
&\quad \mathrm{E}_{T-1}\big[\exp\{-a_j(I_j + \sum_{s\in\mathcal{S}_j} \alpha_{jTs} W_{Ts})\}\big],
\end{aligned}
$$

# Buyer's Inverse Demand

$$
\begin{aligned}
W_{ts'} &= \frac{1}{a_j \alpha_{jt+1s'} R^{T-t}} \\
&\quad \log \frac{\mathrm{E}_t\{\exp(-a_j \sum_{s \in \mathcal{S}_j, s \neq s'} \alpha_{jt+1s} W_{t+1s} R^{T-(t+1)}) \Theta_{jt+1}\}}{\mathrm{E}_t\{\exp(-a_j \sum_{s \in \mathcal{S}_j} \alpha_{jt+1s} W_{t+1s} R^{T-(t+1)}) \Theta_{jt+1}\}}, \\
\Theta_{jt} &= \exp(a_j R^{T-t} \sum_{s \in \mathcal{S}_j} \alpha_{jt+1s} W_{ts}) \\
&\quad \mathrm{E}_t\{\exp(-a_j R^{T-(t+1)} \sum_{s \in \mathcal{S}_j} \alpha_{jt+1s} W_{t+1s}) \Theta_{jt+1}\}, \\
\text{with } R &= 1 + r, \; 0 \leq t < T - 1.
\end{aligned}
$$

# Investor's Inverse Supply

$$
\begin{aligned}
W_{T-1s'} &= \frac{1}{a_m R \alpha_{mTs'}} \log \frac{\mathrm{E}_{T-1}\left\{\exp\left(a_m \sum_{s\in\mathcal{S}} \alpha_{mTs} W_{Ts}\right)\right\}}{\mathrm{E}_{T-1}\left\{\exp\left(a_m \sum_{s\neq s' s\in\mathcal{S}} \alpha_{mTs} W_{Ts}\right)\right\}}, \\
W_{ts'} &= \frac{1}{a_m \alpha_{mt+1s'} R^{T-t}} \\
&\quad \log \frac{\mathrm{E}_t\left\{\exp\left(a_m \sum_{s\in\mathcal{S}} \alpha_{mt+1s} W_{t+1s} R^{T-(t+1)}\right)\Theta_{m,t+1}\right\}}{\mathrm{E}_t\left\{\exp\left(a_m \sum_{s\neq s' s\in\mathcal{S}} \alpha_{mt+1s} W_{t+1s} R^{T-(t+1)}\right)\Theta_{m,t+1}\right\}}, \\
\text{with } R &= 1+r,\ 0 \le t < T-1, \Theta_{m,T} = 1, \\
\Theta_{m,t} &= \exp(-a_m R^{T-t} \sum_{s\in\mathcal{S}} \alpha_{mt+1s} W_{ts}) \\
&\quad \mathrm{E}_t\{\exp(a_m R^{T-(t+1)} \sum_{s\in\mathcal{S}} \alpha_{mt+1s} W_{t+1s})\Theta_{mt+1}\}
\end{aligned}
$$

# Investor: single site vs multi-site

Proposition

In a single period model if $W_{T,s'}$ and $(W_{T,s})_{s\in\mathcal{S}\setminus\{s'\}}$ are positive (negative) associated,
then for $a_m > 0$ and given $(\alpha_{m,T,s})_{s\in\mathcal{S}\setminus\{s'\}}$ of the same sign,
investors supply for weather bond in $s'$ $W_{T-1,s'}(\alpha_{m,T,s'})$ shifts upwards (downwards) in comparison to the single-site case.

continue to 5.2

# Buyer: single site vs multi-site

## Proposition

If

$$\mathrm{Cov}[U_j(\alpha_{j,T,s'}W_{T,s'}), U_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}] \geq (\leq)$$

$$\frac{\mathrm{Cov}\{U_j(I_j), U_j(\alpha_{j,T,s'}W_{T,s'})\}\mathrm{Cov}[U_j(I_j), U_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}]}{\mathrm{E}\{U_j(I_j)\}^2}$$

$$-\frac{\mathrm{E}[\bar{U}_j(I_j)\bar{U}_j(\alpha_{j,T,s'}W_{T,s'})\bar{U}_j\{(W_{T,s}\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}\}]}{\mathrm{E}\{U_j(I_j)\}} \tag{1}$$

then for $a_j > 0, j \in J$ and given $(\alpha_{j,T,s})_{s\in\mathcal{S}_j\setminus\{s'\}}$ of the same sign buyers demand for WD in $s'$ shifts downwards (upwards) compared to the single-site case. continue to 5.2

# Single site vs multi-site

- investor: + (-) **dependencies** in underlying weather risks $\rightarrow$ $\downarrow$ **($\uparrow$) supply** due to higher (lower) risks she bears.

- buyer: $\uparrow\downarrow$ **demand** depending on the sign of (1). This condition can be checked for a concrete application.

# Market Clearance

$$\sum_{j \in \mathcal{J}} \alpha^*_{j,t,s} = \alpha^*_{m,t,s}, \quad \text{for} \quad 0 \le t \le T$$

equilibrium prices $(W^*_{t,s})^{t=1,\dots,T}_{s \in \mathcal{S}}$
equilibrium quantities $(\alpha^*_{k,t,s})^{t=1,\dots,T}_{s \in \mathcal{S}}$ with $k = \{j, m\}$
which clear the market for set of buyers $j \in \mathcal{J}$, and set of stations $s \in \mathcal{S}$.

# A multi-site rainfall model

Wilks (1998)
Rainfall amount $R_{s',t}$ at time $t$ in station $s'$:

$$R_{s',t} = r_{s',t} X_{s',t}, \tag{2}$$

where

- $X_{s',t}$ rainfall occurrence at $t$ in $s'$

$$X_t = \begin{cases} 1 \text{ (wet, } \geq \ X_{min}), \\ 0 \text{ (dry, } < \ X_{min}), \end{cases}$$

- $r_{s',t}$ is positive rainfall amount.

# Spatial dependence of $\{X_{s,t}\}_{s\in\mathcal{S},t=1,\dots,T}$

Threshold probability

$$p_{crit,s',t} = \begin{cases} p_{01,s',t} \text{ if } X_{s',t-1} = 0, \\ p_{11,s',t} \text{ if } X_{s',t-1} = 1, \end{cases},$$

where

$$p_{01,s',t} = P(X_{s',t} = 1 | X_{s',t-1} = 0),$$

$$p_{11,s',t} = P(X_{s',t} = 1 | X_{s',t-1} = 1).$$

# Spatial dependence of $\{X_{s,t}\}_{s\in\mathcal{S},t=1,\ldots,T}$

$X_{s',t}$ generated as

$$X_{s',t} = \begin{cases} 1 \text{ if } w_{s',t} \leq \Phi^{-1}(p_{crit,s',t}), \\ 0 \text{ if } w_{s',t} > \Phi^{-1}(p_{crit,s',t}), \end{cases}$$

$\Phi(\cdot)$ cdf of standard normal distribution, $\{w_{s,t}\}_{s\in\mathcal{S}} \sim N(0_{|\mathcal{S}|}, \Sigma)$, with $\Sigma_{s,s'} = \mathrm{Corr}(w_{s,t}, w_{s',t})$ such that the empirical correlations $\mathrm{Corr}(X_{s,t}, X_{s',t})$ of the rainfall occurrences are mimicked in the generated rainfall occurrence series. continue to 3.8

# Spatial dependence of $\{r_{s,t}\}_{s\in\mathcal{S},t=1,\dots,T}$

Rainfall amount generated as

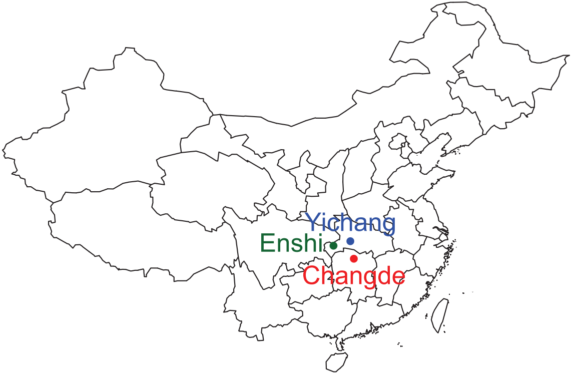
$$r_{s,t} = r_{min} - \beta_{s,t} \log \Phi(v_{s,t}) \tag{3}$$

where

$$\beta_{s,t} = \begin{cases} \beta_{1,s,t} \text{ if } \Phi(w_{s,t})/p_{s,crit} \leq \alpha_{s,t}, \\ \beta_{2,s,t} \text{ if } \Phi(w_{s,t})/p_{s,crit} > \alpha_{s,t}, \end{cases} , \tag{4}$$

and $v_{s,t}$ are normal covariates correlated such that the generated rainfall time series mimic the empirical correlation in the rainfall data. continue to 3.10

# Stations




continue to simulation

# Empirical rainfall I

Test the order of Markov chain using BIC (Katz, 1983):

| Order/BIC | Changde | Enshi | Yichang |
|---|---|---|---|
| 0 | 70.83 | 60.02 | 19.86 |
| 1 | 53.21 | 43.21 | 4.531 |
| 2 | 53.47 | 44.69 | 9.032 |
| 3 | 65.64 | 59.72 | 33.38 |

Table 1: BIC criterion for different orders of Markov models for rainfall occurrences.

# Empirical rainfall II

| Parameter | Changde | Enshi | Yichang |
|---|---|---|---|
| $\hat{p}_{01,\cdot,t\in\text{May}}$ | 0.38 | 0.27 | 0.17 |
| $\hat{p}_{11,\cdot,t\in\text{May}}$ | 0.60 | 0.53 | 0.65 |

Table 2: Transitional probabilities to wet states for rainfall occurrences in May.

# Empirical rainfall III

The estimated correlations of wet day occurrences in May ("wet" is $> 0.1$ mm precipitation) $\widehat{\mathrm{Corr}}(X_{s,t}, X_{s',t})$ (black) and $\mathrm{Corr}(w_{\cdot,t}, w_{s',t})$ (red) what is $w_{s',t}$:

| | Changde | Enshi | Yichang |
|---|---|---|---|
| Changde | - | 0.42 0.65 | -0.01 0 |
| Enshi | - | - | -0.04 0 |
| Yichang | - | - | - |

Table 3: Parameters for the generation of the rainfall occurrences in May.

# Empirical rainfall IV

The multi-site rainfall amount $r_{s,t}|X_{s,t} = 1$ follows a mixture of two exponential distributions with mixing parameter $\alpha_{s,t}$ and means $\beta_{1,s,t}, \beta_{2,s,t}$ with pdf

$$
\begin{aligned}
f_t(r_{s,t} = r|X_{s,t} = 1, \beta_{1,s,t}, \beta_{2,s,t}, \alpha_{s,t}) = {} & \alpha_{s,t}/\beta_{1,s,t} \exp\left(-r/\beta_{1,s,t}\right) \\
& + \left(1 - \alpha_{s,t}\right)/\beta_{2,s,t} \exp\left(-r/\beta_{2,s,t}\right)
\end{aligned}
$$

| Parameter | Changde | Enshi | Yichang |
|---|---|---|---|
| $\alpha_{\cdot,t\in\text{May}}$ | 0.73 | 0.60 | 0.67 |
| $\beta_{1,\cdot,t\in\text{May}}$ | 16.02 | 13.84 | 8.99 |
| $\beta_{2,\cdot,t\in\text{May}}$ | 0.73 | 0.85 | 0.90 |

Table 4: Estimated parameters of the mixture of exponential distributions.

# Empirical rainfall V

The estimated rainfall amount correlations $\widehat{\text{Corr}}(R_{s,t}, R_{s',t})$ (black) and $\text{Corr}(v_{\cdot,t}, v_{s',t})$ (red) what is $v_{s',t}$:

| | Changde | Enshi | Yichang |
|---|---|---|---|
| Changde | - | 0.26 0.31 | -0.01 0 |
| Enshi | - | - | -0.02 0 |
| Yichang | - | - | - |

Table 5: Parameters for the generation of the rainfall amounts in May.

# Income-Rainfall Relationship

Indices: cumulative rainfall (RX) and wet day index (WX).

- $RX_{\tau_1,\tau_2,s} = \sum_{t=\tau_1}^{\tau_2} R_{ts}$ total rainfall in $[\tau_1, \tau_2]$.
  - ▶ important for planting and nutrition season
  - ▶ positive correlation with crop volumes
  - → price RX futures for May
- $WX_{\tau_1,\tau_2,s} = \sum_{t=\tau_1}^{\tau_2} X_{ts}$ number of wet days over $[\tau_1, \tau_2]$
  - ▶ important for harvesting, excess rainfall damage
  - ▶ crop volume distribution is better if $WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}$
  - → price call options on WX futures for August with $WX_{crit}$=5 mm and $K$= 5 days.

# Income-Rainfall Relationship

- WX: $\forall j \in \mathcal{J}$ ▸ go to simulation

$$I_j = \begin{cases} \mathcal{N}(\mu^+, \sigma^+), & \text{if } \forall\ s\ WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}, \\ \mathcal{N}(\mu^0, \sigma^0), & \text{if } \exists\ s\ WX_{\tau_1,\tau_2,s\in\mathcal{S}_j} < WX_{crit}, \\ \mathcal{N}(\mu^-, \sigma^-), & \text{otherwise}, \end{cases}$$

- RX: insurers income $I_j \sim \mathcal{N}(\mu^+, \sigma^+)\ \forall j \in \mathcal{J}$

| | Changde | Enshi | Yichang |
|---|---|---|---|
| $I_1$ | $\rho_{11} = 0.5$ | $\rho_{12} = 0.5$ | $\rho_{13} = 0.0$ |
| $I_2$ | $\rho_{21} = 0.5$ | $\rho_{22} = 0.0$ | $\rho_{23} = 0.5$ |

Table 6: $\rho$-values used for simulation.

- set $\mu^+ = 500$, $\mu^0 = 100$, $\mu^- = 50$ and $\sigma^+ = \sigma^0 = \sigma^- = 100$.

# Stylized Economy

- 2 representative crop insurance companies, 1 representative investor
- 3 traded stations in China go to map
- $r_t = r = 5\%$ p.a.,
- profit $\Pi(W_T, P_T)$, with $P_T$ constant. go to table

# Single Period: Investor's Supply and Insurers' Demand

Occurrences of wet days in Changde and Enshi are positive correlated

$\rightarrow$ payoffs of WX calls are positive associated,

$\rightarrow$ investor's supply $\downarrow$ ▸ show Prop. 1

In (1) ▸ show (1) evaluated for $0 < \alpha_{jTs} \leq 100$ LHS<RHS

$\rightarrow$ buyer's demand $\uparrow$

# Single Period: Investor's Supply and Insurers' Demand

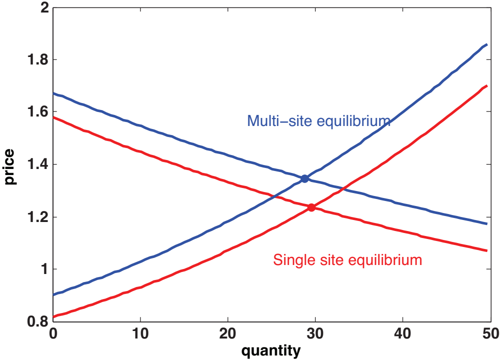



Figure 2: Supply/demand for WX call on Changde, $K$=5.

# Single Period WX call trading: Prices

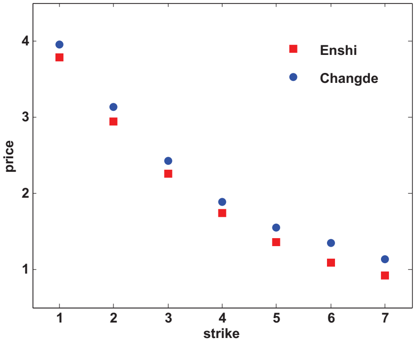



Figure 3: Prices of call options for different strikes $K$ in a single-period WX call trading.

# Two-Period vs Single Period RX future trading: Equilibrium Prices

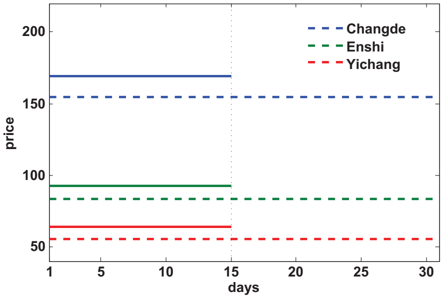



Figure 4: Single period (dashed) and two-period (solid) equilibrium prices for RX futures in May.

# Two-Period RX future trading: Insurers' Income

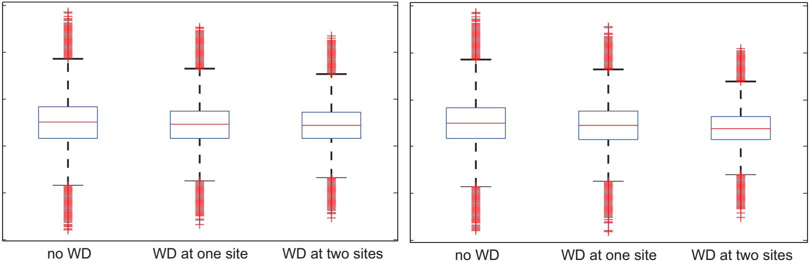



Figure 5: Income distribution of insurer 1 (left) and insurer 2 (right) at single and multiple sites two-period RX futures trading. Note: improvement of insurer 2 is better since payoffs of her RX futures (Changde and Yichang) are uncorrelated, for insurer 1 (Changde and Enshi) they are positive correlated.

# Summary

- pricing of rainfall WD in a multi-site, multi-period setting
- agents trade with multiple sites simultaneously
- Insurer is better off with WD in terms of her utility

# Literature

P. Benitez, T. Kuosmanen, R. Olschewski and G.van Kooten
*Conservation payments under risk: a stochastic dominance approach*
American Journal of Agricultural Economics 88(1): 1-15, 2006.

D. Cox and V. Isham
*A simple spatial-temporal model of rainfall*
Proceedings of the Royal Society of London 415(1849): 317-328, 1988.

H. Föllmer, A. Schied
*Stochastic Finance*
de Gruyter, Berlin, 2002

# Literature


R.W. Katz
*An application of chain-dependent processes to meteorology*
Journal of Applied Probability 14(3): 598-603, 1977.

T. Kim, H. Ahn, G. Chung and C. Yoo
*Stochastic multi-site generation of daily rainfall occurrence in south Florida*
Stochastic Environmental Research and Risk Assessment 22: 705-717, 2008.

Y. Lee and S. Oren
*A multi-period equilibrium pricing model of weather derivatives*
Energy Syst 1: 3-30, 2010.

# Literature

M. Odening, O. Mußhoff, W. Xu
*Analysis of rainfall derivatives using daily precipitation models: opportunities and pitfalls*
Journal of Agricultural and Resource Economics 29(3): 387-403, 2004.

F. Perez-Gonzalez and H. Yun
*Risk management and firm value: evidence from weather derivatives*
AFA 2010 Atlanta Meetings Paper, 2010.

R. Stern and R. Coe
*A model fitting analysis of rainfall data*
Jour. Roy. Stat. Soc. 147: 1-34, 1984.

# Literature

D.S. Wilks
*Multisite generalization of a daily stochastic precipitation generation model*
Journal of Hydrology 210: 178-191, 1998.

D. Vedenov and B. Barnett
*Efficiency of weather derivatives as primary crop insurance instruments*
Agricultural Finance Review 67 (1): 135-156, 2007.

# Pricing Chinese Rain

## a Multi-Site, Multi-Period Equilibrium Model

Wolfgang Karl Härdle
Maria Osipenko

Ladislaus von Bortkiewicz
Chair of Statistics
C.A.S.E. Centre for Applied Statistics
and Economics
School of Business and Economics
and Faculty of Agriculture and
Horticulture
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de

雨