TEDAS - Tail Event Driven ASset allocation

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S&P 500 Stocks

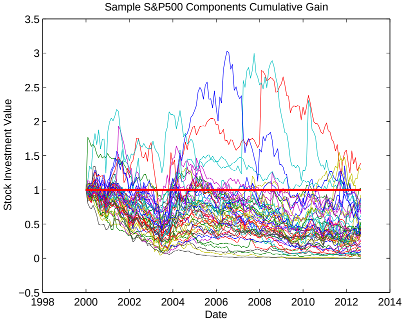


Figure 1: 50 random S&P 500 Sample Components' Cumulative Return: **94%** of stocks lost the value of the initial investment (thick red line)

Core & Satellites

Hedge funds and S&P500

- diversification - reduction of the portfolio risk
- construction - a more diverse universe of assets
- allocation - a higher risk-adjusted return.

Hedge Funds

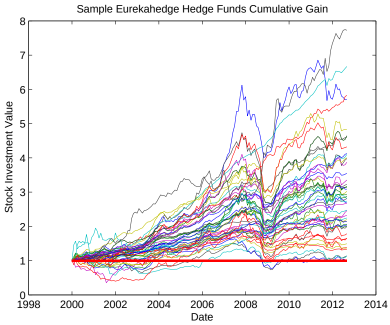


Figure 2: 50 Eurekahedge Hedge Funds Indices' Cumulative Return: **0%** of funds lost the value of the initial investment (thick red line)

Diversification

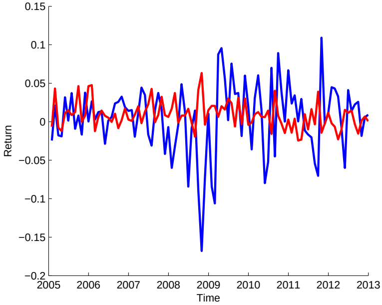


Figure 3: S&P 500 and Eurekahedge North America Macro Hedge Fund Index monthly returns in 20050131-20121231

Traditional Assets/Hedge Funds

Hedge Funds	US	UK	SW	GER	JAP
Conv. arb.	0.10	0.08	0.07	0.10	−0.02
Dedic. sh. bias	−0.77	−0.53	−0.33	−0.46	−0.48
Fix. inc. arb.	0.10	0.13	0.01	0.08	−0.10
Glob. macro	0.30	0.19	0.10	0.27	−0.11
Man. fut.	−0.10	0.02	−0.09	−0.03	0.03

Table 1: Correlation statistics for traditional asset class and hedge funds' indices; based on monthly data Jan. 1994 - Aug. 2001; table from Lhabitant (2002, p.164)

TLND Hedge Funds' Strategies

More

Tail Risk

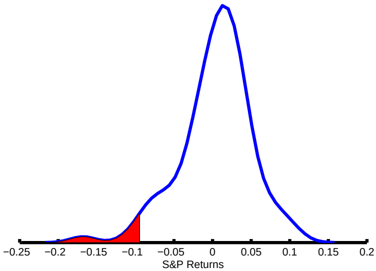


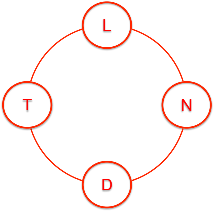
Figure 4: Estimated density of S&P 500 returns

The TLND challenge

- **T**ail dependence
- **L**arge universe: $p > n$
- **N**on normality
- **D**ynamics

TEDAS Objectives

- Hedge tail events
  - Quantile regression
  - Variable selection in high dimensions
- Improve Asset Allocation
  - Higher-order moments' optimization
  - Modelling of moments' dynamics



Outline

1. Motivation ✓
2. TEDAS framework
3. Empirical Application
4. Conclusions
5. Technical Details

Tail Events

- $Y \in \mathbb{R}^n$ core log-returns; $X \in \mathbb{R}^{n \times p}$ satellites' log-returns, $p > n$
- $$q_\tau(x) \stackrel{\text{def}}{=} F^{-1}_{Y|x}(\tau) = x^\top \beta(\tau),$$
  $$\beta(\tau) = \arg \min_{\beta \in \mathbb{R}^p} \mathsf{E}_{Y|X}\, \rho_\tau\{Y - X^\top \beta\},$$
  $$\rho_\tau(u) = u\{\tau - \mathbf{I}(u < 0)\}$$
- L_1 penalty $\lambda_n \|\hat{\omega}^\top \beta\|_1$ to nullify "excessive" coefficients; λ_n and $\hat{\omega}$ controlling penalization; constraining $\beta \leq 0$ yields ALQR ▸ Details

$$\hat{\beta}^{\text{adapt}}_{\tau,\lambda_n} = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda_n \|\hat{\omega}^\top \beta\|_1 \qquad (1)$$

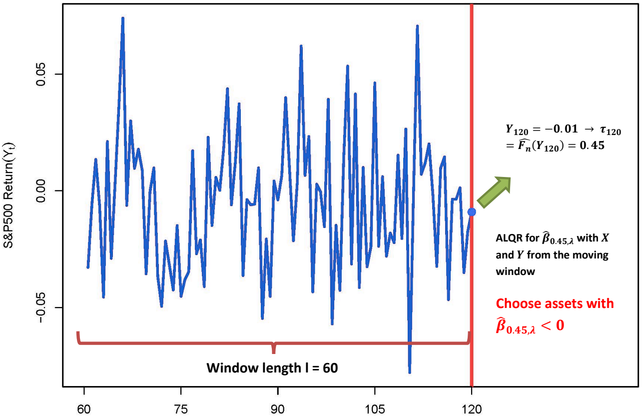
TEDAS Step 1

Initial wealth $W_0 = \$1$, $t = l, \ldots, n$; $l = 60$ length of the moving window

- Portfolio constituents' selection

    1. determine core asset return Y_t, set $\tau_t = \widehat{F}_n(Y_t)$ ▸ Notation
    2. ALQR for $\hat{\beta}_{\tau_t,\lambda_n}$ using the observations $X \in \mathbb{R}^{t-l+1,\ldots,t \times p}$, $Y \in \mathbb{R}^{t-l+1,\ldots,t}$
    3. if $Y_t < 0$, choose X_j, $j = 1, \ldots, k < p$ with $\hat{\beta}_{\tau_t,\lambda_n} < 0$; if $Y_t > 0$, choose X_j, $j = 1, \ldots, k < p$ with $\hat{\beta}_{\tau_t,\lambda_n} > 0$

TEDAS Step 1


S&P500 Return(Y_t)
0.05
0.00
−0.05
60
75
90
105
120
Window length l = 60
$Y_{120} = -0.01 \rightarrow \tau_{120} = \widehat{F_n}(Y_{120}) = 0.45$
ALQR for $\widehat{\beta}_{0.45,\lambda}$ with X and Y from the moving window
Choose assets with $\widehat{\beta}_{0.45,\lambda} < 0$

TEDAS Step 2

- Portfolio selection

  1. apply TEDAS Gestalt to X_j, obtain $\widehat{w}_t \in \mathbb{R}^k$
  2. determine the realized portfolio wealth for $t+1$, $\widehat{X}_{t+1} \stackrel{\text{def}}{=} (X_{t+1,1}, \ldots, X_{t+1,k})^\top$: $W_{t+1} = W_t(1 + \widehat{w}_t^\top \widehat{X}_{t+1})$

TEDAS Example

1. Suppose $t = 86$ (Feb. 2007), $W_{86} = \$1.125$, accumulated wealth $W_{86} = \$1.429$, $Y_{86} = -1.85\% < 0$
2. $\widehat{F}_n(Y_{86}) = 0.25$, so estimate $\hat{\beta}_{0.25} < 0$
3. ALQR on $X \in \mathbb{R}^{60\times163}$, $Y \in \mathbb{R}^{60}$ yields $\hat{\beta}_{0.25} = (-0.77, -1.12, -0.41)^\top$, *Latin American Arbitrage*, *North America Macro*, *Emerging Markets CTA/Managed Futures*
4. TEDAS CF-CVaR optimization Details yields $\widehat{w}_{86} = (0.22, 0.16, 0.62)^\top$; $\widehat{X}_{87} = (0.38\%, 0.45\%, 0.76\%)^\top$, $W_{87} = W_{86}(1 + \widehat{w}_{86}^\top \widehat{X}_{87}) = \$1.438$ (return of 0.62%) while $Y_{87} = -1.53\%$

Optimization strategies

TEDAS strategy	Model	Weights' optimization
TEDAS 1	NA	Equal weights
TEDAS 2	Time-Varying ▸ Details	Cornish-Fisher-CVaR optimization ▸ Details
TEDAS 3	conditional distribution	Expected utility optimization ▸ Details

Hedge funds

- Monthly data
  - Core asset (Y): S&P 500, Nikkei225, DAX 30, FTSE 100
  - Satellite assets (X): 164 Eurekahedge hedge funds indices
- Span: 20000131-20131031 (166 months)
- Source: Bloomberg

Benchmark Strategies

1. RR: dynamic risk-return optimization ▸ Details
2. PESS: tail risk optimization ▸ Details
3. S&P500 Buy-and-Hold

TEDAS with Y= S&P 500

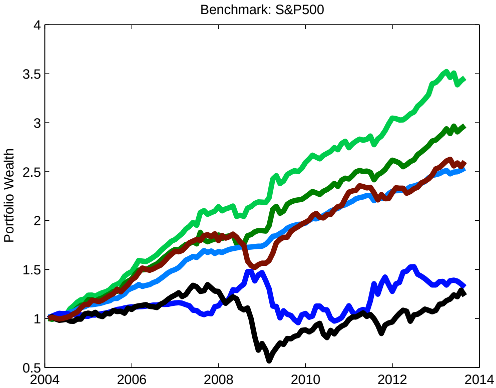


Figure 5: Cumulative portfolio wealth comparison: TEDAS 1, TEDAS 3, TEDAS 2, RR, PESS, S&P 500 buy & hold; X = hedge funds' indices' returns matrix TEDASstrategies2

TEDAS with Y= Nikkei 225

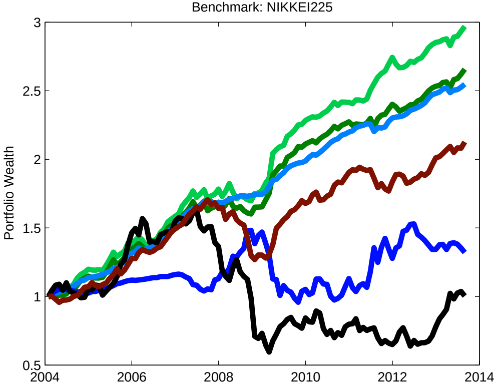


Figure 6: Cumulative portfolio wealth comparison: TEDAS 1, TEDAS 3, TEDAS 2, RR, PESS, Nikkei 225 buy & hold; X = hedge funds' indices' returns matrix TEDASstrategies2

TEDAS with Y= FTSE 100

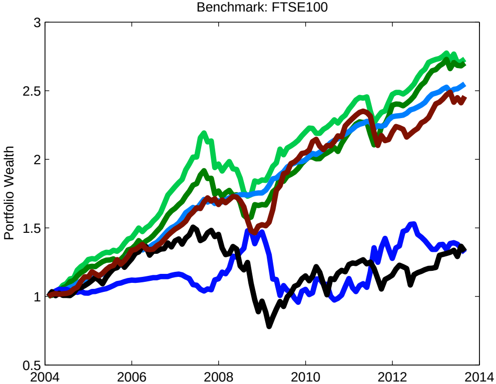


Figure 7: Cumulative portfolio wealth comparison: TEDAS 1, TEDAS 3, TEDAS 2, RR, PESS, FTSE100 buy & hold; X = hedge funds' indices' returns matrix `TEDASstrategies2`

TEDAS with Y= DAX30

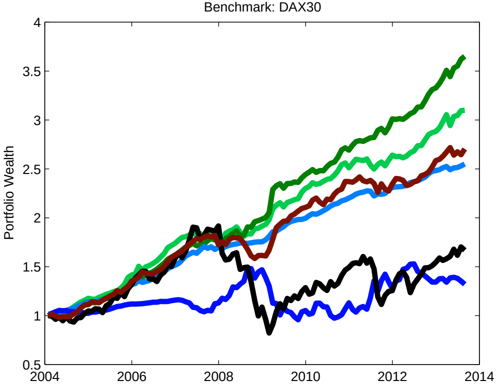


Figure 8: Cumulative portfolio wealth comparison:TEDAS 1, TEDAS 3, TEDAS 2, RR, PESS, DAX30 buy & hold; X = hedge funds' indices' returns matrix TEDASstrategies2

Histograms of $\hat{q}$

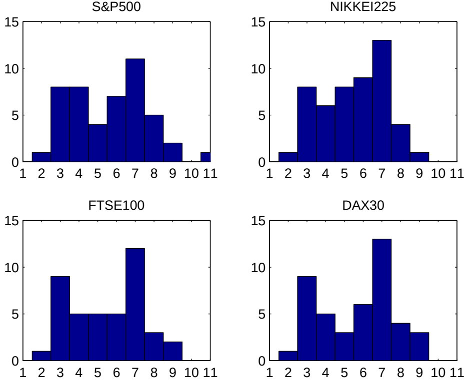


Figure 9: Frequency of the number of selected variables for 4 different Y

Selected Hedge Funds: S&P 500

Table 2: The selected hedge funds for S&P 500 benchmark

Top 5 influential hedge funds	Frequency
Latin American Onshore Fixed Income Hedge Fund Index	19
Emerging Markets Dual Approach Absolute Return Fund Index	18
Large North American Hedge Fund Index	15
Europe Macro Hedge Fund Index	15
Asia Macro Hedge Fund Index	14

Selected Hedge Funds: Nikkei 225

Table 3: The selected hedge funds for Nikkei 225 benchmark

Top 5 influential hedge funds	Frequency
Small Latin American Hedge Fund Index	22
Taiwan Hedge Fund Index	20
North America Top-Down Absolute Return Fund Index	16
Large North American Hedge Fund Index	15
Latin American Onshore Fixed Income Hedge Fund Index	15

Selected Hedge Funds: FTSE 100

Table 4: The selected hedge funds for FTSE 100 benchmark

Top 5 influential hedge funds	Frequency
Small Latin American Hedge Fund Index	15
Europe Macro Hedge Fund Index	15
Latin American Onshore Fixed Income Hedge Fund Index	14
Asia Pacific Top-Down Absolute Return Fund Index	14
Latin American Fixed Income Hedge Fund Index	13

Selected Hedge Funds: DAX 30

Table 5: The selected hedge funds for DAX 30 benchmark

Top 5 influential hedge funds	Frequency
Emerging Markets Dual Approach Absolute Return Fund Index	21
North America Macro Hedge Fund Index	19
Taiwan Hedge Fund Index	16
Asia CTA Hedge Fund Index	14
Europe Macro Hedge Fund Index	14

Dynamic Moment Parameters

Table 6: Parameter estimates in (14) and (15) at $t = 160$

	s_1	s_2	s_3	s_4	s_5	s_6
ω	0.12***	0.33***	0.40***	0.61***	0.60***	0.47***
α_1	0.72***	0.29***	0.47***	0.30***	0.33***	0.45***
β_1	0.13***	0.35***	0.12***	0.05**	0.07***	0.06
a_0	0.05	−0.02	−0.27	0.08***	0.11	−0.08
a_1^-	−0.33*	0.08	−0.55	−0.13***	−0.13	−0.69**
a_1^+	−0.43	0.22***	1.17	−0.46***	−0.33	−0.47***
a_2	1.45***	2.16***	0.96	2.14***	1.77***	1.10**
b_0	−5.03***	−1.96*	−5.53	−3.65***	−4.16**	−1.89***
b_1^-	0.70**	0.21	0.88	1.56	−1.53	0.51***
b_1^+	0.79***	−1.40***	0.87	0.89	−1.87	−0.79***
b_2	−11.67	−0.69	0.12	−0.64***	0.38	1.11***
$Factor_{LL}$	−193.87	−201.40	−215.14	−216.53	−218.57	−225.10
$Model_{LL}$	2394					

parameter estimates under the *NIG* distribution, for the log-returns of Eurekahedge 6 hedge funds (160 data monthly returns, 01.01.2000 - 30.04.2013)
the conditional variance of factors follows a GARCH(1,1) model
the conditional dynamics of skew and shape parameters is bounded by a logistic transformation
the $*$, $**$ and $***$ denote significance at the 10%, 5% and 1% levels, respectively

Evolution of ξ_t and ν_t

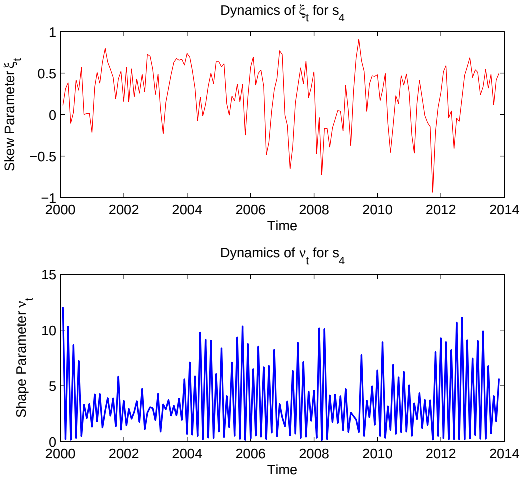


Figure 10: Evolution of the skew and shape parameters for s_4 in Table 6

Conditional Skewness and Kurtosis

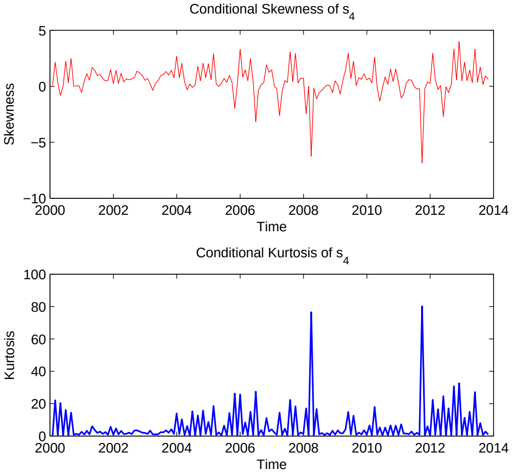


Figure 11: Evolution of the conditional skewness and kurtosis for s_4

Choice of τ

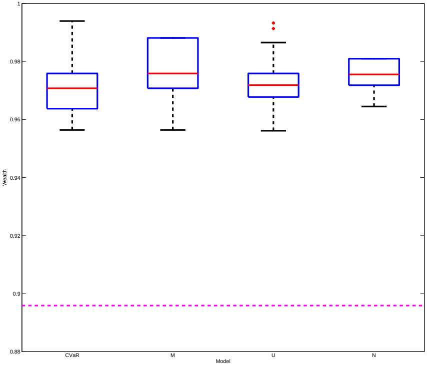


Figure 12: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; DAX30 portfolio value (magenta); negative tail event, 5 assets

Choice of τ

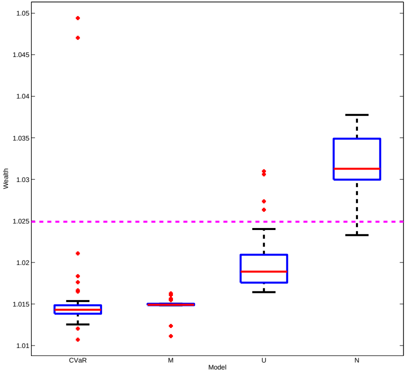


Figure 13: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; DAX30 portfolio value (magenta); positive tail event, 11 assets

Choice of τ

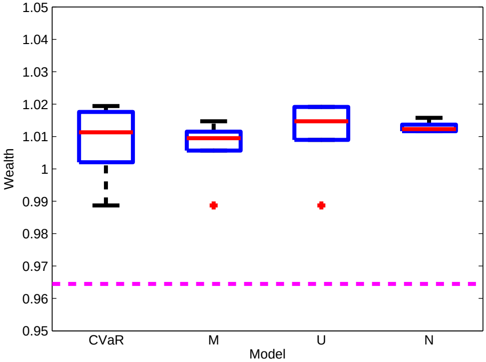


Figure 14: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; FTSE100 portfolio value (magenta); negative tail event, 5 assets

Choice of τ

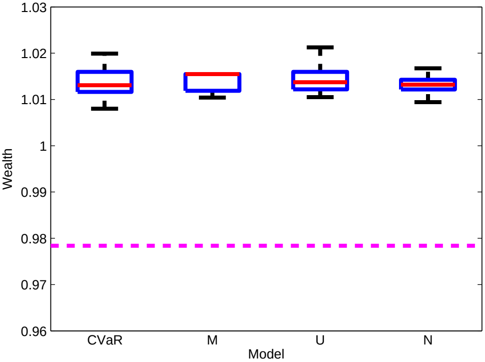


Figure 15: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; FTSE100 portfolio value (magenta); negative tail event, 11 assets

Choice of τ

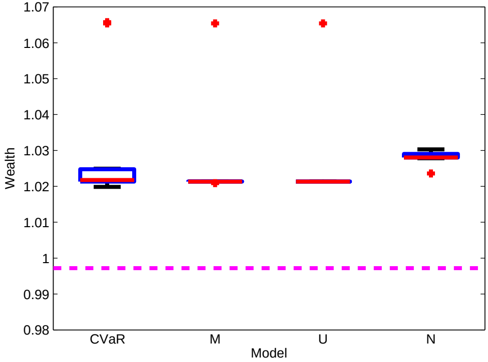


Figure 16: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; S&P500 portfolio value (magenta); negative tail event, 3 assets

Choice of τ

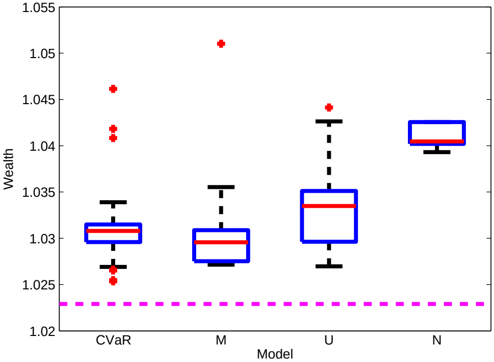
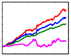


Figure 17: Next-day portfolio value for τ-values within 0.1 from τ_t under different models; S&P500 portfolio value (magenta); positive tail event, 9 assets

Conclusions: TLND challenge

- Lasso quantile regression captures **T**ail events
- Lasso asset selection resolves **L**arge dimensionality problem
- Higher-moment optimization for **N**on-normality
- **D**ynamic portfolio optimization through conditional distribution modelling
- TEDAS: out-of-sample performance superior to benchmark strategies



TEDAS - Tail Event Driven Asset Allocation

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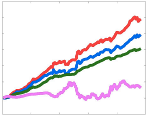

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Notation

- $\hat{q}_\tau \stackrel{\text{def}}{=} \widehat{F}_n^{-1}(\tau)$, with

$$\widehat{F}_n(Y_t) \stackrel{\text{def}}{=} \int_{-\infty}^{Y_t} \hat{f}_n(u)\,du = \frac{1}{n}\sum_{i=1}^{n} H\left(\frac{Y_t - Y_i}{h}\right), \qquad (2)$$

  where $\hat{f}_n(Y_t) \stackrel{\text{def}}{=} (1/nh)\sum_{i=1}^{n} K\{(Y_t - Y_i)/h\}$,
  $H(x) = \int_{-\infty}^{x} K(u)du$, $K(\cdot) = \varphi(\cdot)$;
  Silverman (1986) rule-of-thumb:

$$h = 1.06 s n^{-1/5}\ ,\ s \text{ sample standard deviation of } Y$$

- $\hat{\beta}_{\tau,\lambda_n}$ are the estimated non-zero ALQR coefficients

Back to "TEDAS Step 1"

Lasso Shrinkage

Linear model: $Y = X\beta + \varepsilon$; $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times p}$, $\beta \in \mathbb{R}^p$, $\{\varepsilon_i\}_{i=1}^n$ i.i.d., independent of $\{X_i; i = 1, ..., n\}$

The optimization problem for the lasso estimator:

$$\begin{aligned} \hat{\beta}^{\text{lasso}} = & \arg \min_{\beta \in \mathbb{R}^p} f(\beta) \\ & \text{subject to} \quad g(\beta) \geq 0 \end{aligned} \tag{3}$$

where

$$f(\beta) = \frac{1}{2} (y - X\beta)^\top (y - X\beta)$$
$$g(\beta) = t - \|\beta\|_1$$

where t is the size constraint on $\|\beta\|_1$ Back to "Tail Events"

Lasso Duality

If (1) is convex programming problem, then the Lagrangian is

$$L(\beta, \lambda) = f(\beta) - \lambda g(\beta).$$

and the primal-dual relationship is

$$\underbrace{\underset{\beta}{\text{minimize}} \ \sup_{\lambda \geq 0} \ L(\beta, \lambda)}_{\text{primal}} \geq \underbrace{\underset{\lambda \geq 0}{\text{maximize}} \ \inf_{\beta} \ L(\beta, \lambda)}_{\text{dual}}$$

Then the dual function $L^*(\lambda) = \inf_{\beta} \ L(\beta, \lambda)$ is

$$L^*(\lambda) = \frac{1}{2} y^\top y - \frac{1}{2} \hat{\beta}^\top X^\top X \hat{\beta} - t \frac{(y - X\hat{\beta})^\top X \hat{\beta}}{\|\hat{\beta}\|_1}$$

with $(y - X\hat{\beta})^\top X \hat{\beta} / \|\hat{\beta}\|_1 = \lambda$ Back to "Tail Events"

Paths of Lasso Coefficients

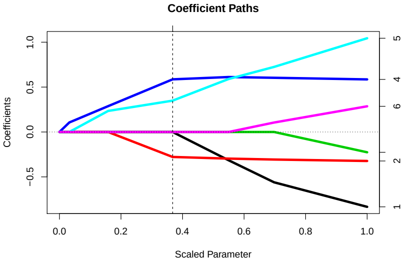
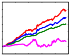


Figure 18: Lasso shrinkage of coefficients in the hedge funds dataset example (6 covariates were chosen for illustration); each curve represents a coefficient as a function of the scaled parameter $\hat{s} = t/\|\beta\|_1$; the dashed line represents the model selected by the BIC information criterion ($\hat{s} = 3.7$)

Back to "Tail Events"



Example of Lasso Geometry

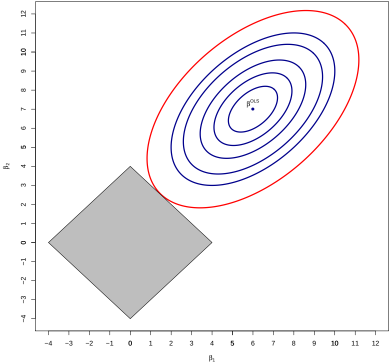


Figure 19: Contour plot of the residual sum of squares objective function centered at the OLS estimate $\widehat{\beta}^{ols} = (6, 7)$ and the constraint region $\sum |\beta_j| \leq t$ MVAlassocontour

Quantile Regression

The loss $\rho_\tau(u) = u\{\tau - \mathbf{I}(u < 0)\}$ gives the (conditional) quantiles $F^{-1}_{y|x}(\tau) \stackrel{\text{def}}{=} q_\tau(x)$.

Minimize

$$\hat{\beta}_\tau = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta).$$

Re-write:

$$\underset{(\xi,\zeta)\in\mathbb{R}^{2n}_{+}}{\text{minimize}} \quad \left\{ \tau 1_n^\top \xi + (1-\tau) 1_n^\top \zeta | X\beta + \xi - \zeta = Y \right\}$$

with ξ, ζ are vectors of "slack" variables Back to "Tail Events"

Non-Positive (NP) Lasso-Penalized QR

The lasso-penalized QR problem with an additional non-positivity constraint takes the following form:

$$
\begin{aligned}
\underset{(\xi,\zeta,\eta,\tilde{\beta})\in\mathbb{R}_+^{2n+p}\times\mathbb{R}^p}{\text{minimize}} \quad & \tau 1_n^\top \xi + (1-\tau)1_n^\top \zeta + \lambda 1_n^\top \eta \\
\text{subject to} \quad & \xi - \zeta = Y + X\tilde{\beta}, \\
& \xi \geq 0, \\
& \zeta \geq 0, \\
& \eta \geq \tilde{\beta}, \\
& \eta \geq -\tilde{\beta}, \\
& \tilde{\beta} \geq 0, \qquad \tilde{\beta} \overset{\text{def}}{=} -\beta
\end{aligned}
\tag{4}
$$

Back to "Tail Events"

Solution

Transform into matrix (I_p is $p \times p$ identity matrix; $E_{p\times n} = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$):

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b, \;\; Bx \le 0 \end{array}$$

where $A = \begin{pmatrix} I_n & -I_n & 0 & X \end{pmatrix}$, $b = Y$, $x = \begin{pmatrix} \xi & \zeta & \eta & \beta \end{pmatrix}^\top$,

$$c = \begin{pmatrix} \tau 1_n \\ (1-\tau)1_n \\ \lambda 1_p \\ 0 1_p \end{pmatrix}, \quad B = \begin{pmatrix} -E_{p\times n} & 0 & 0 & 0 \\ 0 & -E_{p\times n} & 0 & 0 \\ 0 & 0 & -I_p & I_p \\ 0 & 0 & -I_p & -I_p \\ 0 & 0 & 0 & I_p \end{pmatrix}$$

Back to "Tail Events"

Solution - Continued

The previous problem may be reformulated into *standard form*

$$
\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & Cx = d, \\
& x + s = u, \;\; x \geq 0, s \geq 0
\end{aligned}
$$

and the dual problem is:

$$
\begin{aligned}
\text{maximize} \quad & d^\top y - u^\top w \\
\text{subject to} \quad & C^\top y - w + z = c, \;\; z \geq 0, w \geq 0
\end{aligned}
$$

Back to "Tail Events"

Solution - Continued

The KKT conditions for this linear program are

$$F(x, y, z, s, w) = \left\{ \begin{array}{c} Cx - d \\ x + s - u \\ C^\top y - w + z - c \\ x \circ z \\ s \circ w \end{array} \right\} = 0,$$

with $y \geq 0$, $z \geq 0$ dual slacks, $s \geq 0$ primal slacks, $w \geq 0$ dual variables.

This can be solved by a primal-dual path following algorithm based on the *Newton method*

Back to "Tail Events"

Adaptive Lasso Procedure

Lasso estimates $\hat{\beta}$ can be inconsistent (Zou, 2006) in some scenarios.

Lasso soft-threshold function gives biased results

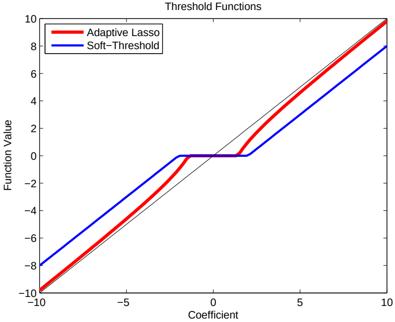


Figure 20: Threshold functions for simple and adaptive Lasso

Adaptive Lasso Procedure

The adaptive Lasso (Zou, 2006) yields a sparser solution and is less biased.

L_1 - penalty replaced by a re-weighted version; $\hat{\omega}_j = 1/|\hat{\beta}_j^{\text{init}}|^{\gamma}$, $\gamma = 1$, $\hat{\beta}^{\text{init}}$ is from (3)

The adaptive lasso estimates are given by:

$$\hat{\beta}_{\lambda}^{\text{adapt}} = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} (Y_i - X_i^{\top}\beta)^2 + \lambda \|\hat{\omega}^{\top}\beta\|_1$$

(Bühlmann, van de Geer, 2011): $\hat{\beta}_j^{\text{init}} = 0$, then $\hat{\beta}_j^{\text{adapt}} = 0$

Back to "Tail Events"

Simple and Adaptive Lasso Penalized QR

Simple lasso-penalized QR optimization problem is:

$$\hat{\beta}_{\tau,\lambda} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda \|\beta\|_1 \tag{5}$$

Adaptive lasso-penalized QR model uses the re-weighted penalty:

$$\hat{\beta}_{\tau,\lambda}^{\text{adapt}} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - X_i^\top \beta) + \lambda \|\hat{\omega}^\top \beta\|_1 \tag{6}$$

Adaptive lasso-penalized QR procedure can ensure oracle properties for the estimator Details

Back to "Tail Events"

Algorithm for Adaptive Lasso Penalized QR

The optimization for the adaptive lasso quantile regression can be re-formulated as a lasso problem:

- the covariates are rescaled: $\tilde{X} = (X_1 \circ \hat{\beta}_1^{\text{init}}, \dots, X_p \circ \hat{\beta}_p^{\text{init}})$;
- the lasso problem (5) is solved:

$$\hat{\hat{\beta}}_{\tau,\lambda} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho_\tau(Y_i - \tilde{X}_i^\top \beta) + \lambda \|\beta\|_1$$

- the coefficients are re-weighted as $\hat{\beta}_{\tau,\lambda}^{\text{adapt}} = \hat{\hat{\beta}}_{\tau,\lambda} \circ \hat{\beta}^{\text{init}}$

Back to "Tail Events"

Oracle properties

In the linear model, let $Y = X\beta + \varepsilon = X^1\beta^1 + X^2\beta^2 + \varepsilon$, where $X = (X^1, X^2)$, $X^1 \in \mathbb{R}^{n\times q}$, $X^2 \in \mathbb{R}^{n\times(p-q)}$; β^1_q are true nonzero coefficients, $\beta^2_{p-q} = 0$ are noise coefficients; $q = \|\beta\|_0$.

Also assume that $\lambda q/\sqrt{n} \to 0$ and $\lambda/\{\sqrt{q}\log(n \vee p)\} \to \infty$ and certain regularity conditions are satisfied Details

Oracle properties

Then the adaptive L_1 QR estimator has the "oracle" properties (Zheng et al., 2013):

1. Variable selection consistency:

$$\mathsf{P}(\beta^2 = 0) \geq 1 - 6\exp\left\{-\frac{\log(n \vee p)}{4}\right\}.$$

2. Estimation consistency: $\|\beta - \hat{\beta}\| = \mathcal{O}_p(\sqrt{q/n})$

3. Asymptotic normality: $u_q^2 \stackrel{\text{def}}{=} \alpha^\top \Sigma_{11}\alpha$, $\forall \alpha \in \mathbb{R}^q$, $\|\alpha\| < \infty$,

$$n^{1/2} u_q^{-1} \alpha^\top (\beta^1 - \hat{\beta}^1) \xrightarrow{\mathcal{L}} \mathsf{N}\left\{0, \frac{(1-\tau)\tau}{f^2(\gamma^*)}\right\}$$

where γ^* is the τth quantile and f is the pdf of ε

Back to "Tail Events"

Selected Hedge Funds' Strategies

1. *Convertible arbitrage* hedge funds focus on the mispricing of convertible bonds. A typical position involves a long position in the convertible bond and a short position in the underlying asset.

2. *Fixed income arbitrage* hedge funds tend to profit from price anomalies between related securities and/or bet on the evolution of interest rates spreads. Typical trading strategies are butterfly-like structures, cash/futures basis trading strategies or relative swap spread trades.

3. *Event-driven* hedge funds focus on price movements generated by an anticipated corporate event, such as a merger, an acquisition, a bankruptcy, etc. Back

Selected Hedge Funds' Strategies

4. *Long/short equity* hedge funds represent the original hedge fund model. They invest in equities both on the long and the short sides, and generally have a small net long exposure. They are genuinely opportunistic strategies and could be classified as "double alpha, low beta" funds.

5. *Market neutral* hedge funds seek to neutralize certain market risks by taking offsetting long and short positions in instruments with actual or theoretical relationships. Most of them are in fact long/short equity hedge funds.

6. *Dedicated short bias* hedge funds are essentially long/short equity hedge funds, that maintain a consistent net short exposure, therefore attempting to capture profits when the market declines. Back

Selected Hedge Funds' Strategies

7. *Emerging market* hedge funds invest in equities and fixed-income securities of emerging markets around the world.
8. *Global macro* hedge funds take very large directional bets on overall market directions that reflect their forecasts of major economic trends and/or events.
9. *Managed futures* hedge funds implement discretionary or systematic trading in listed financial, commodity and currency futures around the world. The managers of these funds are known as commodity trading advisors (CTAs).
10. *Multi-strategy* hedge funds regroup managers acting in several of the above-mentioned strategies. Return

Traditional Assets/Hedge Fund Indices

Table 7: Correlation statistics for MSCI and hedge funds' indices returns

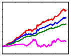
Hedge Fund Indices	MSCI Indices								
	WRD	EUR	US	UK	FR	SW	GER	JAP	PAC
Asia CTA	−0.01	0.02	−0.02	−0.06	0.01	−0.09	0.04	−0.03	0.02
Asia Distressed Debt	0.30	0.30	0.24	0.31	0.31	0.26	0.27	0.26	0.34
Asia Macro	−0.01	−0.01	−0.04	0.01	−0.02	0.07	−0.03	0.06	0.06
Global CTA FoF	0.02	0.08	−0.08	0.09	0.10	0.09	0.07	0.06	0.10
Global Event Driven FoF	0.65	0.59	0.58	0.66	0.59	0.50	0.57	0.47	0.67
Global Macro FoF	0.19	0.22	0.07	0.24	0.22	0.18	0.20	0.23	0.31
CTA/Managed Futures	−0.04	0.02	−0.13	0.03	0.03	0.07	−0.01	0.04	0.05
Event Driven	0.82	0.75	0.75	0.78	0.75	0.64	0.75	0.62	0.83
Fixed Income	0.70	0.65	0.63	0.70	0.65	0.56	0.62	0.51	0.78
Long Short Equities	0.82	0.78	0.74	0.76	0.77	0.64	0.77	0.64	0.82
Asia inc Japan Distr. Debt	0.30	0.30	0.24	0.31	0.31	0.26	0.27	0.26	0.34
Asia inc Japan Macro	0.34	0.33	0.31	0.27	0.33	0.24	0.35	0.31	0.40

Calculations based on monthly data Jan. 2000 - Jul. 2012

WRD - World, EUR - Eurozone, FR - France, SW - Switzerland, PAC - Pacific ex. Japan

FoF means "fund of funds"

▸ Back



Risk-Return Asset Allocation

Log returns $X_t \in \mathbb{R}^p$:

$$\begin{aligned} \min_{w_t \in \mathbb{R}^p} \quad & \sigma^2_{P,t}(w_t) \stackrel{\text{def}}{=} w_t^\top \Sigma_t w_t \\ \text{s.t.} \quad & \mu_{P,t}(w_t) = r_T, \\ & w_t^\top 1_p = 1, \\ & w_{i,t} \geq 0 \end{aligned} \tag{7}$$

where r_T "target" return, $\Sigma_t \stackrel{\text{def}}{=} \mathsf{E}_{t-1}\{(X_t - \mu)(X_t - \mu)^\top\}$, Σ_t is modeled with a GARCH model ▸ Details ▸ Back to "Benchmark Strategies"

▸ Return to "TEDAS Gestalten"

Tail Risk Asset Allocation

Portfolio returns $X \in \mathbb{R}^{n \times p}$, Bassett et al. (2004)

$$\begin{aligned} \min_{(\beta,\alpha)^\top \in \mathbb{R}^p} \quad & \sum_{t=1}^{n} \rho_\tau \left\{ X_{t1} - \sum_{j=2}^{p} (X_{t1} - X_{tj})\beta_j - \alpha \right\} \\ \text{s.t.} \quad & w^\top \hat{\mu} = r_T, \\ & w^\top 1_p = 1, \end{aligned} \tag{8}$$

where r_T is the "target" return for the portfolio and $w = w(\beta) = (1 - \sum_{j=2}^{p} \beta_j, \beta^\top)^\top$, $\tau \in (0,1)$, $\hat{\mu} \stackrel{\text{def}}{=} \overline{X}$ sample returns' mean ▸ Back to "Benchmark Strategies"

Multi-Moment Utility Optimization

The (dynamic) investment decision: $U(\cdot)$ utility function; $X_t \in \mathbb{R}^p$ log-returns, w_t weights, $\mu_{P,t}(w_t) \stackrel{\text{def}}{=} w_t^\top \mu$, $\mu \stackrel{\text{def}}{=} \mathrm{E}_{t-1}(X_t)$, r_T "target" return:

$$\max_{w_t \in \mathbb{R}^p} \mathrm{E}_{t-1}\left\{U(W_t)\right\}, \quad \text{s.t. } \mu_{P,t}(w_t) = r_T,\ w^\top 1_p = 1, w_{i,t} \geq 0, \tag{9}$$

$$\begin{aligned}\mathrm{E}_{t-1}\left\{U(W_t)\right\} \approx\ & U\{\overline{W}_t\} + \frac{1}{2}U^{(2)}\{\overline{W}_t\}\sigma^2_{W_t} + \\ & + \frac{1}{3!}U^{(3)}\{\overline{W}_t\}S_{W_t} + \frac{1}{4!}U^{(4)}\{\overline{W}_t\}K_{W_t},\end{aligned}$$

where $W_t \stackrel{\text{def}}{=} 1 + w_t^\top X_t$ is the end-of-period t wealth, $\overline{W}_t \stackrel{\text{def}}{=} \mathrm{E}_{t-1}(W_t)$, $\sigma^2_{W_t} \stackrel{\text{def}}{=} \mathrm{E}_{t-1}\left\{(W_t - \overline{W}_t)^2\right\}$, $S_{W_t} \stackrel{\text{def}}{=} \mathrm{E}_{t-1}\left\{(W_t - \overline{W}_t)^3\right\}$, $K_{W_t} \stackrel{\text{def}}{=} \mathrm{E}_{t-1}\left\{(W_t - \overline{W}_t)^4\right\}$; $U^{(n)}(\cdot)$ is the nth derivative of $U(\cdot)$

Return to "TEDAS Gestalten"

Utility Function Example

- CARA utility:

$$U(W) = -\exp(-\eta W),$$

where η coefficient of risk aversion

- then:

$$\begin{aligned} \mathsf{E}_{t-1}\{U(W_t)\} &= \mathsf{E}_{t-1}\{-\exp(-\eta W_t)\} \\ &\approx -\exp(-\eta \overline{W}_t)\left(1 + \frac{\eta^2}{2}\sigma^2_{W_t} - \frac{\eta^3}{3!}S_{W_t} + \frac{\eta^4}{4!}K_{W_t}\right) \end{aligned}$$

Back to "Strategies"

Portfolio Moments

The portfolio moments:

$$
\begin{aligned}
\sigma^2_{W_t} &= w_t^\top M_t^2 w_t \\
S_{W_t} &= w_t^\top M_t^3 (w_t \otimes w_t) \\
K_{W_t} &= w_t^\top M_t^4 (w_t \otimes w_t \otimes w_t),
\end{aligned}
$$

where $\otimes$ Kronecker product,

$$M_t^2 \stackrel{\text{def}}{=} \mathsf{E}_{t-1}(r_t - \mu)^2 \tag{10}$$

$$M_t^3 \stackrel{\text{def}}{=} \mathsf{E}_{t-1}\{(r_t - \mu)(r_t - \mu)^\top \otimes (r_t - \mu)^\top\} \tag{11}$$

$$M_t^4 \stackrel{\text{def}}{=} \mathsf{E}_{t-1}\{(r_t - \mu)(r_t - \mu)^\top \otimes (r_t - \mu)^\top \otimes (r_t - \mu)^\top\}, \tag{12}$$

A dynamic distribution model is used to obtain M_t^2, M_t^3, M_t^4 in (10), (11), (12) Return to "TEDAS Gestalten"

Dynamic Distribution Model

- joint normality is questionable
- possible persistence in the dynamics of moments
- reaction of distribution parameters to past shocks
- computational feasibility

Back to "Strategies"

Descriptive Statistics

Table 8: Monthly returns of 3 Eurekahedge hedge funds' indices

	Japan Multi-Strategy		North America Fixed Income		Europe Arbitrage	
Univariate statistics						
Normality tests						
JB	533.775	(0.000)	294.089	(0.000)	610.407	(0.000)
KS	0.503	(0.000)	0.473	(0.000)	0.485	(0.000)
Omnibus	82.773	(0.000)	43.761	(0.000)	171.079	(0.000)
Dynamic conditional moments' tests						
ARCH	11.227	(0.000)	34.966	(0.000)	26.592	(0.000)
Bera-Lee	48.469	(0.000)	36.475	(0.000)	40.783	(0.000)
Bera-Zuo	203.723	(0.000)	20.149	(0.166)	421.847	(0.000)
Multivariate statistics						
Test						
Omnibus	326.226	(0.000)				
Mardia	301.199	(0.000)				
Henze-Zirkler	9.862	(0.000)				

Standard errors and p-values are given in parentheses.

ARCH, Bera-Lee and Bera-Zuo stand for the test statistics of the ARCH test by Engle (1982) and information matrix tests for testing variation in second, third and fourth conditional moments

Back to "Strategies"

Generalized Hyperbolic (GH) Distribution

A vector X has a multivariate GH distribution if

$$X = \mu + W\delta + \sqrt{W}AZ, \tag{13}$$

where

(i) $Z \sim \mathrm{N}(0, I_k)$

(ii) $A \in \mathbb{R}^{d \times k}$

(iii) μ, $\delta \in \mathbb{R}^d$

(iv) $W \geq 0$, scalar-valued random variable, independent of Z, $W \sim GIG(\lambda, \alpha, \beta)$; GIG is the generalized inverse Gaussian distribution

Back to "Strategies"

Multivariate Affine GH Distribution

- margins of the (MGH) distribution not mutually independent for some choice of $\Sigma = AA^\top$
- MAGH distribution, Schmidt et al. (2006), models margins and dependency independently

$Y \sim MAGH(\lambda, \alpha, \beta, \mu, \Sigma)$ if

(i) $X = (X_1, \dots, X_d)^\top$, $X_i \sim GH(0, 1, \alpha_i, \beta_i)$, $i = 1, \dots, d$

(ii) $Y = AX + \mu$, $AA^\top = \Sigma$ positive definite

Back to "Strategies"

Normal Inverse Gaussian (NIG) Distribution

- obtained from the GH distribution with $\lambda = -0.5$
- "semi-heavy tails" property: fits financial data well

The density is written as:

$$f_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right\} \frac{K_1\left\{\alpha\sqrt{\delta^2 + (x-\mu)^2}\right\}}{\sqrt{\delta^2 + (x-\mu)^2}},$$

where $0 \leq |\beta| \leq \alpha$, $\delta > 0$, K_1 is the modified Bessel function of the third kind and order 1

Location-Scale Property: let $\overline{\alpha} \stackrel{\text{def}}{=} \delta\alpha$ and $\overline{\beta} \stackrel{\text{def}}{=} \delta\beta$, then $X \sim NIG(\overline{\alpha}, \overline{\beta}, \mu, \delta) \Leftrightarrow (X-\mu)/\delta \sim NIG(\overline{\alpha}, \overline{\beta}, 0, 1)$

Back to "Strategies"

Choice of the Matrix A

- assume $X = As$, X random signal generated by another random vector $s = (s_1, \ldots, s_d)$, s_i statistically independent, $i = 1, \ldots, d$ and a mixing matrix A, both unknown
- the *independent component analysis* (ICA) technique separates source signals s from a set of mixed signals X without or with very little aid of information about f or the mixing process A
- ICA estimates A and s by maximizing the nongaussianity of linear combinations of X

Back to "Strategies"

The Model for Portfolio Returns

- assume $\varepsilon_t = As_t$, $\mathsf{E}\,\varepsilon_t = 0$, $\mathsf{E}\,\varepsilon_t \varepsilon_t^\top = I_d$, $\mathsf{E}\,s_t = 0$, $\mathsf{E}\,s_t s_t^\top = I_d$
- define $\mathsf{E}(s_t|\mathcal{F}_t) = 0$, $D_t \stackrel{\text{def}}{=} \mathsf{E}(s_t s_t^\top|\mathcal{F}_t) \stackrel{\text{def}}{=} \operatorname{diag}(d_{1t}, \dots, d_{dt})$
- let $z_{it} \sim NIG(\overline{\alpha}_{it}, \overline{\beta}_{it}, 0, 1)$, then $s_{it} \sim NIG(\overline{\alpha}_{it}/\sqrt{d_{it}}, \overline{\beta}_{it}/\sqrt{d_{it}}, 0, \sqrt{d_{it}})$
- *MANIG*: multivariate affine normal inverse Gaussian distribution
- model for portfolio returns $r_t = m_t + \varepsilon_t$, $r_t|\mathcal{F}_t \sim MANIG(m_t, \Sigma_t, \omega_t)$, where $\omega_t = (\omega_{1t}, \dots, \omega_{dt})^\top$ and $\omega_{it} = (\alpha_{it}, \beta_{it})^\top$, $i = 1, \dots, d$, $\Sigma_t = M_t^2 = AD_tA^\top$, d_{it} can be modeled as GARCH-type processes

Back to "Strategies"

Moment Dynamics

- reparametrize the model to have asymmetry and shape parameters $\xi_{it} = \beta_{it}/\alpha_{it}$, $\nu_{it} = \sqrt{\alpha_{it}^2 - \beta_{it}^2}$
- introduce asymmetric GARCH-like dynamics:

$$\nu_{i,t} = a_{i,0} + a_{i,1}^{-}|s_{i,t-1}|N_{i,t-1} + a_{i,1}^{+}|s_{i,t-1}|P_{i,t-1} + a_{i,2}\nu_{i,t-1} \tag{14}$$

$$\xi_{i,t} = b_{i,0} + b_{i,1}^{-}s_{i,t-1}N_{i,t-1} + b_{i,1}^{+}s_{i,t-1}P_{i,t-1} + b_{i,2}\xi_{i,t-1}, \tag{15}$$

where $N_{i,t} = \mathbf{I}(z_{i,t} \leq 0)$, $P_{i,t} = 1 - N_{i,t}$

Back to "Strategies"

Portfolio Moments

$$M_t^3 = AM_{s_t}^3 (A \otimes A)^\top, \quad M_t^4 = AM_{s_t}^4 (A \otimes A \otimes A)^\top,$$

where

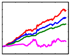
$$M_{s_t}^3 = \mathrm{E}_{t-1}(s_{i,t}s_{j,t}s_{k,t}) = \sum_{r=1}^{p} d_{ir,t}d_{jr,t}d_{kr,t}sk_{rt}^s$$

$$M_{s_t}^4 = \mathrm{E}_{t-1}(s_{i,t}s_{j,t}s_{k,t}s_{l,t})$$

$$= \sum_{r=1}^{p} d_{ir,t}d_{jr,t}d_{kr,t}d_{lr,t}kurt_{rt}^s + \sum_{r=1}^{p}\sum_{s \neq r} \psi_{rs,t},$$

$\psi_{rs,t} = d_{ir,t}d_{jr,t}d_{ks,t}d_{ls,t} + d_{ir,t}d_{js,t}d_{kr,t}d_{ls,t} + d_{is,t}d_{jr,t}d_{kr,t}d_{ls,t}$,
$D_t^{1/2} = (d_{ij,t})_{i,j=1,\ldots,p}$, sk_{it}^s, $kurt_{it}^s$ are obtained with α_{it}, β_{it}

Back to "Example"



Conditional VaR (CVaR) Optimization

Given $\alpha > 0.5$ confidence level,

$$\min_{w_t \in \mathbb{R}^p} \text{CVaR}_\alpha(w_t), \quad \text{s.t. } \mu_{P,t}(w_t) = r_T, w_t^\top 1_p = 1, w_{i,t} \geq 0, \tag{16}$$

$$\text{CVaR}_\alpha(w_t) = -\frac{1}{1-\alpha} q_\alpha^*(w_t)\sigma_{P,t}(w_t), \tag{17}$$

▸ Proof

where (via Cornish-Fisher (CF) expansion):

$$q_{\alpha^*}^*(w_t) = \left\{1 + \frac{S_{P,t}(w_t)}{6} z_{\alpha^*} + \frac{K_{P,t}(w_t)}{24}(z_{\alpha^*}^2 - 1) - \frac{S_{P,t}^2(w_t)}{36}(2z_{\alpha^*}^2 - 1)\right\} \varphi(z_{\alpha^*}), \tag{18}$$

where $\alpha^* \stackrel{\text{def}}{=} 1 - \alpha$ ▸ Return to "TEDAS Gestalten" ▸ Back to "TEDAS Example"

The Orthogonal GARCH Model

- $X_t \in \mathbb{R}^{n \times p}$, $\Gamma_t = B_t \in \mathbb{R}^{p \times p}$ matrix of standardized eigenvectors of $n^{-1} X_t^\top X_t$ ordered according to decreasing magnitude of eigenvalues
- $F_t = P_t \stackrel{\text{def}}{=} X_t \Gamma_t$ PCs of X_t
- factors f, introduce noise u_i, i.e.
  $y_j = b_{j1} f_1 + b_{j2} f_2 + \ldots + b_{jk} f_k + u_i$ or $Y_t = F_t B_t^\top + U_t$
- then $\Sigma_t = \mathrm{Var}(X_t) = \mathrm{Var}(F_t B_t^\top) + \mathrm{Var}(U_t) = B_t \Delta_t B_t^\top + \Omega_t$, $\Delta_t = \mathrm{Var}(F_t)$ diagonal matrix of PC variances at t

Return to "Risk-Return Asset Allocation"

Cornish-Fisher VaR Optimization

Log returns $X_t \in \mathbb{R}^p$:

$$\begin{aligned} \underset{w \in \mathbb{R}^d}{\text{minimize}} \quad & W_t\{-q_\alpha(w_t) \cdot \sigma_p(w_t)\} \\ \text{subject to} \quad & w_t^\top \mu = \mu_p, \;\; w_t^\top 1 = 1, \;\; w_{t,i} \geq 0 \end{aligned}$$

here $W_t \stackrel{\text{def}}{=} W_0 \cdot \prod_{j=1}^{t-1} w_{t-j}^\top (1 + X_{t-j})$, $\tilde{w}$, W_0 initial wealth,
$\sigma_p^2(w) \stackrel{\text{def}}{=} w_t^\top \Sigma_t w_t$,

$$q_\alpha(w) \stackrel{\text{def}}{=} z_\alpha + (z_\alpha^2 - 1)\frac{S_p(w)}{6} + (z_\alpha^3 - 3z_\alpha)\frac{K_p(w)}{24} - (2z_\alpha^3 - 5z_\alpha)\frac{S_p(w)^2}{36},$$

here $S_p(w)$ skewness, $K_p(w)$ kurtosis, z_α is $N(0,1)$ α-quantile If $S_p(w)$, $K_p(w)$ zero, then obtain Markowitz allocation

Return to "TEDAS Gestalten"

Portfolio Skewness and Kurtosis

Skewness S_P and excess kurtosis K_P are given by moment expressions

$$S_P(w) = \frac{1}{\sigma_P^3(w)}(m_3 - 3m_2 m_1 + 2m_1^3)$$

$$K_P(w) = \frac{1}{\sigma_P^4(w)}(m_4 - 4m_3 m_1 + 6m_2 m_1^2 + 3m_1^4) - 3$$

where portfolio non-central moments also depend on w:

$$m_1 = \mu_P(w) \stackrel{\text{def}}{=} w^\top \mu$$

$$m_2 = \sigma_P^2 + m_1^2$$

$$m_3 = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} w_i w_j w_k S_{ijk}$$

Portfolio Skewness and Kurtosis - Continued

$$m_4 = \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d} w_i w_j w_k w_l K_{ijkl},$$

where $\sigma_P^2(w) = w^\top \Sigma w$ and $S_{ijk} = \mathsf{E}(r_i \times r_j \times r_k)$, $K_{ijkl} = \mathsf{E}(r_i \times r_j \times r_k \times r_l)$ can be computed via sample averages from returns data.

S_{ijk}, K_{ijkl} determine the d-dimensional portfolio co-skewness and co-kurtosis tensors

$$S \stackrel{\text{def}}{=} \{S_{ijk}\}_{i,j,k=1,\dots,d} \in \mathbb{R}^{d\times d\times d}$$

$$K \stackrel{\text{def}}{=} \{K_{ijkl}\}_{i,j,k,l=1,\dots,d} \in \mathbb{R}^{d\times d\times d\times d}.$$

Back

Regularity Conditions for Adaptive Lasso QR

A1 Sampling and smoothness: $\forall x$ in the support of X_i, $\forall y \in \mathbb{R}$, $f_{Y_i|X_i}(y|x)$, $f \in \mathcal{C}^k(\mathbb{R})$, $|f_{Y_i|X_i}(y|x)| < \overline{f}$, $|f'_{Y_i|X_i}(y|x)| < \overline{f'}$; $\exists \underline{f}$, such that $f_{Y_i|X_i}(x^\top \beta_\tau | x) > \underline{f} > 0$

A2 Restricted identifiability and nonlinearity: let $\delta \in \mathbb{R}^p$, $T \subset \{0, 1, ..., p\}$, δ_T such that $\delta_{Tj} = \delta_j$ if $j \in T$, $\delta_{Tj} = 0$ if $j \notin T$; $T = \{0, 1, ..., s\}$, $\overline{T}(\delta, m) \subset \{0, 1, ..., p\} \setminus T$, then $\exists m \geq 0$, $c \geq 0$ such that

$$\inf_{\boldsymbol{\delta} \in A, \boldsymbol{\delta} \neq 0} \frac{\delta^\mathsf{T} \operatorname{E}(X_i X_i^\top)\delta}{\|\delta_{T \cup \overline{T}(\delta, m)}\|^2} > 0, \qquad \frac{3\underline{f}^{3/2}}{8\overline{f}'} \inf_{\delta \in A, \delta \neq 0} \frac{\operatorname{E}[|X_i^\mathsf{T}\delta|^2]^{3/2}}{\operatorname{E}[|X_i^\mathsf{T}\delta|^3]} > 0,$$

where $A \stackrel{\text{def}}{=} \{\delta \in \mathbb{R}^p : \|\delta_{T^c}\|_1 \leq c\|\delta_T\|_1, \|\delta_{T^c}\|_0 \leq n\}$

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Regularity Conditions - Continued

A3 Growth rate of covariates:

$$\frac{q^3\{\log(n \vee p)\}^{2+\eta}}{n} \to 0, \eta > 0$$

A4 Moments of covariates: Cramér condition

$$\mathrm{E}[|x_{ij}|^k] \leq 0.5 C_m M^{k-2} k!$$

for some constants C_m, M, $\forall k \geq 2$, $j = 1, ..., p$

A5 Well-separated regression coefficients: $\exists b_0 > 0$, such that $\forall j \leq q$, $|\hat{\beta}_j| > b_0$

Proof of the CF-CVaR Expansion 1

Define the Cornish-Fisher expansion:

$$q_{1-\alpha} \stackrel{\text{def}}{=} z_{1-\alpha} + (z_{1-\alpha}^2 - 1)s + (z_{1-\alpha}^3 - 3z_{1-\alpha})k - (2z_{1-\alpha}^3 - 5z_{1-\alpha})s^2,$$

where $s \stackrel{\text{def}}{=} S/6$, $k \stackrel{\text{def}}{=} K/24$, S and K are skewness and excess kurtosis, respectively; $z_{1-\alpha} \stackrel{\text{def}}{=} \Phi^{-1}(1-\alpha)$.

Re-write:

$$q_{1-\alpha} = a_0 + a_1 z_{1-\alpha} + a_2 z_{1-\alpha}^2 + a_3 z_{1-\alpha}^3, \tag{19}$$

where $a_0 = -s$, $a_1 = 1 - 3k + 5s^2$, $a_2 = s$, $a_3 = k - 2s^2$

Proof of the CF-CVaR Expansion 2

Define the *conditional Value-at-Risk* (CVaR) or *expected shortfall* (ES):

$$\mathrm{CVaR}_\alpha \stackrel{\mathrm{def}}{=} \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_q \mathrm{d}\, q,$$

where $\mathrm{VaR}_q \stackrel{\mathrm{def}}{=} -\Phi^{-1}(\alpha)\sigma\sqrt{T}$

Observe:

$$\begin{aligned}
\mathrm{CVaR}_\alpha &= -\frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(q)\sigma\sqrt{T}\mathrm{d}\, q && (20)\\
&= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_\alpha^1 \Phi^{-1}(q)\mathrm{d}\, q && (21)\\
&= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} u\varphi(u)\mathrm{d}\, u, && (22)
\end{aligned}$$

where (22) follows from the change of variable: $u = z_q = \Phi^{-1}(q)$

Proof of the CF-CVaR Expansion 3

Substitute (19) into (22):

$$\begin{aligned}\mathrm{CVaR}_\alpha &= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3\right)\varphi(z)\mathrm{d}\,z \\ &= a_0 A_0 + a_1 A_1 + a_2 A_2 + a_3 A_3,\end{aligned}$$

$$\begin{aligned}A_0 &= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} \varphi(z)\mathrm{d}\,z = -\sigma\sqrt{T}, \\ A_1 &= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} z\varphi(z)\mathrm{d}\,z = \frac{\sigma\sqrt{T}}{1-\alpha}\varphi(z_\alpha), \\ A_2 &= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} z^2\varphi(z)\mathrm{d}\,z = -\sigma\sqrt{T}\left(\frac{\varphi(z_\alpha)z_\alpha}{1-\alpha}+1\right), \\ A_3 &= -\frac{\sigma\sqrt{T}}{1-\alpha}\int_{-\infty}^{-z_\alpha} z^3\varphi(z)\mathrm{d}\,z = \frac{\sigma\sqrt{T}}{1-\alpha}(z_\alpha^2+2)\varphi(z_\alpha).\end{aligned}$$

Collecting terms and simplifying gives the desired result. □

▸ Return to "Conditional VaR Optimization"

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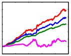

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