Efficient Iterative ML Estimation of High-Parameterized Time Series Models

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VAR

Application: Impulse response analysis.

Example 1

Let x_i denote a $(d \times 1)$ vector of time series variables, $i = 1, \ldots, n$.

$$x_i = \underbrace{\omega}_{(d\times 1)} + \underbrace{A}_{(d\times d)} x_{i-1} + \varepsilon_i,$$

is known as VAR(1). Least squares estimation is based on moment conditions $\mathrm{E}(\varepsilon_i) = 0$ and $\mathrm{E}(\varepsilon_i \varepsilon_i^\top) = \Sigma_\varepsilon$.

VARMA

Application: Forecasting of macroeconomic variables.

Example 2

Let x_i denote a $(d \times 1)$ vector of time series variables, $i = 1, \dots, n$.

$$x_i = \underbrace{\omega}_{(d\times 1)} + \underbrace{A}_{(d\times d)} x_{i-1} + \varepsilon_i + \underbrace{B}_{(d\times d)} \varepsilon_{i-1},$$

is known as VARMA$(1,1)$. Maximum likelihood estimation needs a distribution assumption like $\varepsilon_i \sim \mathrm{F}(0, \Sigma_\varepsilon)$, with $\mathrm{E}(\varepsilon_i \varepsilon_i^\top) = \Sigma_\varepsilon$.

VMEM

Applications: Forecasting of liquidity measures, risk management, volatility contagion.

Example 3

Let $x_i \geq 0$ denote a $(d \times 1)$ vector of time series variables, $i = 1, \ldots, n$.

$$x_i = \mu_i \odot \varepsilon_i,$$
$$\mu_i = \underbrace{\omega}_{(d\times 1)} + \underbrace{A}_{(d\times d)} x_{i-1} + \underbrace{B}_{(d\times d)} \mu_{i-1},$$

is known as VMEM(1,1), where "$\odot$" is the component-wise Hadamard product, and $\varepsilon_{ij} \geq 0$, $j = 1, \ldots, d$. GMM estimation is based on $\varepsilon_i \sim (1_d, \Sigma_\varepsilon)$ with $\mathrm{E}(\varepsilon_i \varepsilon_i^\top) = \Sigma_\varepsilon$.

Copula-GARCH

Applications: VaR-estimation, asset pricing.

Example 4

Let x_i denote a $(d \times 1)$ time series variables, $i = 1, \ldots, n$.

$$
\begin{aligned}
x_i &= \Sigma_i \varepsilon_i, \\
\Sigma_i &= \operatorname{diag}(\sigma_{i1}, \ldots, \sigma_{id}) \\
\sigma_{ij}^2 &= \omega_j + a_j x_{i-1,j}^2 + b_j \sigma_{i-1,j}^2,
\end{aligned}
$$

is known as copula-GARCH$(1,1)$, where
$\varepsilon_i \sim F_{\varepsilon_i}(\varepsilon_{i1}, \ldots, \varepsilon_{id}) = C\{F_{\varepsilon_{i1}}(\varepsilon_{i1}) \ldots, F_{\varepsilon_{id}}(\varepsilon_{id})\}$ with $\mathrm{E}(\varepsilon_i) = 0$.

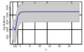
Related to practitioners

- Volatility contagion via connectedness measures

- Asset and option pricing
- Estimation of VaR and ES
- Forecasting of macroeconomic variables
- Modeling of liquidity measures
- ...

Challenges for large d

- *Non*-Gaussian white noise with a *non*-elliptical dependence structure
    - High-dimensional copulae, see Smith et al. (2010, JASA) and Okhrin et al. (2013, JoE).
- Complexity of log-likelihood
    - Iterative maximization of parts of the log-likelihood, see Song et al. (2005, JASA).
    - Decomposition of the parameter space in order to update the estimator.
    - Analytical first-order derivatives of the entire log-likelihood are not required.



Outline

1. Motivation ✓
2. Efficient estimation
3. Simulation I
4. Sparse and efficient estimation
5. Simulation II
6. Iterative Generalized Least Squares Estimation
7. Application
8. Summary

An iterative estimation procedure

- Let $X = (X_1^\top, \dots, X_n^\top)^\top$ be the finite history of the d-dimensional stochastic process $\{X_i\}_{i=1,2,\dots}$.
- Each X_i has conditional density $f_{X_i|\mathcal{F}_{i-1}}(\cdot;\vartheta)$.
- W.l.o.g. decompose $\vartheta = \mathfrak{v}(\vartheta_1, \dots, \vartheta_G) \stackrel{\text{def}}{=} (\vartheta_1^\top, \dots, \vartheta_G^\top)^\top$, s.t.

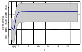
$$
\begin{aligned}
\ell_i(\vartheta) &= \log f_{X_i|\mathcal{F}_{i-1}}(X_{i1}, \dots, X_{id}; \vartheta) \\
&= \sum_{j=1}^{d} \log f_{X_{ij}|\mathcal{F}_{i-1}}(X_{ij}; \vartheta_1, \dots, \vartheta_k) \\
&\quad + \log c_{X_i|\mathcal{F}_{i-1}} \left\{ F_{X_{i1}|\mathcal{F}_{i-1}}(X_{i1}; \vartheta_1, \dots, \vartheta_k), \right. \\
&\quad \left. \dots, F_{X_{id}|\mathcal{F}_{i-1}}(X_{id}; \vartheta_1, \dots, \vartheta_k); \vartheta_{k+1}, \dots, \vartheta_G \right\}.
\end{aligned}
$$

- Construct the log-likelihood

$$\begin{aligned}
\mathcal{L}(\vartheta) &= \sum_{i=1}^{n} \ell_i(\vartheta) \\
&= \sum_{i=1}^{n} \{\ell_i^m(\vartheta_1, \ldots, \vartheta_k) + \ell_i^c(\vartheta)\} \\
&= \mathcal{L}^m(\vartheta_1, \ldots, \vartheta_k) + \mathcal{L}^c(\vartheta).
\end{aligned}$$

- Shorthand notation, e.g.,

$$\dot{\mathcal{L}}(\vartheta_0) = \left. \frac{\partial \mathcal{L}(\vartheta)}{\partial \vartheta} \right|_{\vartheta = \vartheta_0}.$$



Algorithm

Step 1:

$$(1) \quad (\vartheta_{1,n}^1, \ldots, \vartheta_{k,n}^1) = \arg \underset{\vartheta_1, \ldots, \vartheta_k}{\text{zero}} \dot{\mathcal{L}}^m(\vartheta_1, \ldots, \vartheta_k)$$

$$(2) \quad (\vartheta_{k+1,n}^1, \ldots, \vartheta_{G,n}^1) = \arg \underset{\vartheta_{k+1}, \ldots, \vartheta_G}{\text{zero}} \dot{\mathcal{L}}^c(\vartheta_{1,n}^1, \ldots, \vartheta_{k,n}^1, \vartheta_{k+1}, \ldots, \vartheta_G)$$

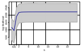
Step $h > 1$:

$$(1) \quad \vartheta_{1,n}^h = \arg \max_{\vartheta_1} \mathcal{L}(\vartheta_1, \vartheta_{2,n}^{h-1}, \ldots, \vartheta_{G,n}^{h-1})$$

$$(2) \quad \vartheta_{2,n}^h = \arg \max_{\vartheta_2} \mathcal{L}(\vartheta_{1,n}^h, \vartheta_2, \vartheta_{3,n}^{h-1}, \ldots, \vartheta_{G,n}^{h-1})$$

$$\vdots$$

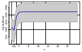
$$(G) \quad \vartheta_{G,n}^h = \arg \max_{\vartheta_G} \mathcal{L}(\vartheta_{1,n}^h, \ldots, \vartheta_{G-1,n}^h, \vartheta_G)$$



Asymptotic properties

Theorem

Let the random variables of the sequence X have an identical conditional density $f_{X_i|\mathcal{F}_{i-1}}(\cdot;\vartheta)$ for which Assumptions 1-2 hold. If $\vartheta_n^1 \xrightarrow{\mathrm{P}} \vartheta_0$, then $\vartheta_n^h \xrightarrow{\mathrm{P}} \vartheta_0$, $\forall\ h = 2, 3, \ldots$. ▸ Assumptions



Theorem

Let the random variables of the sequence X have an identical conditional density $f_{X_i|\mathcal{F}_{i-1}}(\cdot;\vartheta)$ for which Assumptions 1-4 hold. Then,

$$n^{1/2}(\vartheta_n^h - \vartheta_0) \quad \xrightarrow{\mathcal{L}} \mathrm{N}\left\{0, \mathcal{B}_h \Sigma(\vartheta_0) \mathcal{B}_h^\top\right\}.$$

▸ Assumptions *and* ▸ Definitions

Corollary

Under Assumptions 1-4,

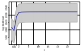
$$\lim_{h\to\infty} n^{1/2}(\vartheta_n^h - \vartheta_0) \xrightarrow{\mathcal{L}} \mathrm{N}\left\{0, \mathcal{J}(\vartheta_0)^{-1}\right\}.$$

Properties under misspecification

Under certain regularity assumptions, see White (1994):

- ϑ_n^h is a consistent estimator for $\vartheta_n^\star$ - the minimizer of the Kullback-Leibler divergence.
- $n^{1/2}(\vartheta_n^h - \vartheta_n^\star)$ converges to a multivariate Gaussian distribution as $n \to \infty$.
- The asymptotic covariance of $\lim_{h\to\infty} n^{1/2}\vartheta_n^h$ collapses to

$$\left\{\mathcal{H}(\vartheta_n^\star)^{-1}\right\} \mathcal{T}_2 \Sigma(\vartheta_n^\star) \mathcal{T}_2^\top \left\{\mathcal{H}(\vartheta_n^\star)^{-1}\right\}^\top.$$



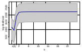
Setup I

Similar to Kascha (2012, Econometric Reviews):

$$x_i = A\,x_{i-1} + \varepsilon_i + B\,\varepsilon_{i-1}.$$

- $d = 5$, $n = 50$, $r = 24$
- Replication: 500
- ε_{ij} follow t_{ν_j} margins coupled with a correlation matrix ϑ_G of a Gaussian copula with $G = 4$.
- Decomposition

$$\begin{aligned} \vartheta_1 &= (\nu_1, \dots, \nu_d)^\top, \\ \vartheta_2 &= \mathrm{vec}(A), \\ \vartheta_3 &= \mathrm{vec}(B). \end{aligned}$$



Results

Figure 1: The updated mean of the centered estimates of degrees of freedom $\nu_{j,n}^{h} - \nu_j$ (solid line) and $\nu_{j,n}^{h-1} - \nu_j$ (dashed line), $j = 1, \ldots, d$.

Figure 2: The updated mean squared error for A_n^h and B_n^h.

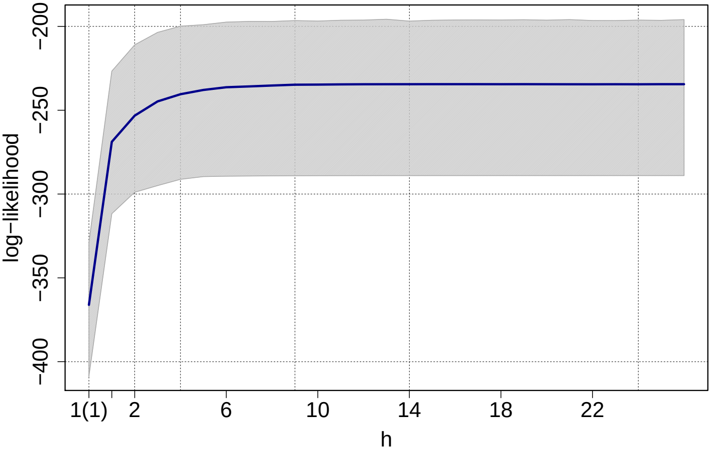


Figure 3: The median of the log-likelihood for each step of the iteration. The gray area contains 95% of the sample.

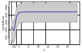
Penalized 2-stage ML estimation

- Curse of dimensionality
  - Need to balance the trade-off between few parameters per ϑ_g, $g = 1, \ldots, G$, and a large G!
  - Parameter shrinkage via nonconcave penalized likelihood, see Fan and Li (2001, JASA).

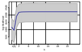
- First derivative of the SCAD penalty

$$p'_{\lambda,a}(x) \quad = \quad \lambda \mathbf{I}(x \leq \lambda) + \max(a\lambda - x, 0) / (a-1) \mathbf{I}(x > \lambda),$$

$a > 2$ and $x > 0$. ▸ Simulation



- Split the parameters into
  - penalized parameters $\vartheta_{p_m} \stackrel{\text{def}}{=} \mathfrak{v}(\vartheta_1, \vartheta_2)$ and $\vartheta_{p_c} \stackrel{\text{def}}{=} \mathfrak{v}(\vartheta_{G-1}, \vartheta_G)$ and
  - non-penalized parameters $\vartheta_m \stackrel{\text{def}}{=} \mathfrak{v}(\vartheta_3, \dots, \vartheta_k)$ and $\vartheta_c \stackrel{\text{def}}{=} \mathfrak{v}(\vartheta_{k+1}, \dots, \vartheta_{G-2})$.
- Introduce
  - meaningful penalization targets $\breve{\vartheta}_1, \breve{\vartheta}_2, \breve{\vartheta}_{G-1}, \breve{\vartheta}_G$ and
  - the modified SCAD-penalty $\breve{p}_{\lambda,a}(\gamma) = p_{\lambda,a}(|\gamma - \breve{\gamma}|)$.
- W.l.o.g. let $\vartheta_{1,0} = \breve{\vartheta}_1$ and $\vartheta_{G,0} = \breve{\vartheta}_G$, so that $f_i(\cdot; \vartheta_0)$ has a less complicated functional form than $f_i(\cdot; \vartheta)$ for $\vartheta \neq \vartheta_0$.



The penalized log-likelihoods are

$$\mathcal{L}^{p_m}(\vartheta_1,\ldots,\vartheta_k) = \mathcal{L}^m(\vartheta_1,\ldots,\vartheta_k) - n\sum_{l=1}^{r_1+r_2} \breve{p}_{\lambda_n^m, a^m}(\vartheta_{l,p_m}),$$

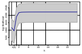
$$\mathcal{L}^{p_c}(\vartheta) = \mathcal{L}^c(\vartheta) - n\sum_{l=1}^{r_{G-1}+r_G} \breve{p}_{\lambda_n^c, a^c}(\vartheta_{l,p_c}).$$

Algorithm

Step 1*:*

$$(1) \quad (\vartheta^1_{1,n},\ldots,\vartheta^1_{k,n}) = \arg\underset{\vartheta_1,\ldots,\vartheta_k}{\text{zero}}\, \dot{\mathcal{L}}^{p_m}(\vartheta_1,\ldots,\vartheta_k)$$

$$(2) \quad (\vartheta^1_{k+1,n},\ldots,\vartheta^1_{G,n}) = \underset{\vartheta_{k+1},\ldots,\vartheta_G}{\arg\text{zero}}\, \dot{\mathcal{L}}^{p_c}(\vartheta^1_{1,n},\ldots,\vartheta^1_{k,n},\vartheta_{k+1},\ldots,\vartheta_G)$$



Theorem

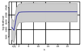
Let the random variables of the sequence X have an identical conditional density $f_i(\cdot;\vartheta)$ for which Assumptions 1-3 and 5 hold and let the penalty fulfill certain regularity conditions. If $\lambda_n^m, \lambda_n^c \to 0$, $n^{1/2}\lambda_n^m \to \infty$ and $n^{1/2}\lambda_n^c \to \infty$ as $n \to \infty$, then,

(a) $\vartheta^1_{1,n} \overset{a.s.}{\to} \breve{\vartheta}_1$ *and* $\vartheta^1_{G,n} \overset{a.s.}{\to} \breve{\vartheta}_G$,

(b) $\vartheta^1_{2,n} + \mathcal{O}(\mathrm{a}_n^m) \overset{\mathrm{P}}{\to} \vartheta_{2,0}$ *and* $\vartheta^1_{G-1,n} + \mathcal{O}(\mathrm{a}_n^c) \overset{\mathrm{P}}{\to} \vartheta_{G-1,0}$, *with* $\mathrm{a}_n^m, \mathrm{a}_n^c \to 0$ *for* $\lambda_n^m, \lambda_n^c \to 0$ *as* $n \to \infty$,

(c) $\vartheta^1_{m,n} \overset{\mathrm{P}}{\to} \vartheta_{m,0}$ *and* $\vartheta^1_{c,n} \overset{\mathrm{P}}{\to} \vartheta_{c,0}$.

▸ Assumptions



Iterative Efficient and Sparse Estimation

Step $h > 1$*:*

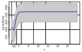
(1) {blank step}

(2) $\vartheta^{h}_{2,n} = \arg\max_{\vartheta_2} \mathcal{L}(\breve{\vartheta}_1, \vartheta_2, \vartheta^{h-1}_{3,n}, \ldots, \vartheta^{h-1}_{G-1,n}, \breve{\vartheta}_G)$

$\vdots$

$(G-1)$ $\vartheta^{h}_{G-1,n} = \arg\max_{\vartheta_{G-1}} \mathcal{L}(\breve{\vartheta}_1, \vartheta^{h}_{2,n}, \ldots, \vartheta^{h}_{G-2,n}, \vartheta_{G-1}, \breve{\vartheta}_G)$

(G) {blank step}



Corollary

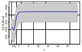
Under the assumptions of Theorem 3. If $\lambda_n^m, \lambda_n^c \to 0$*,* $n^{1/2}\lambda_n^m \to \infty$ *and* $n^{1/2}\lambda_n^c \to \infty$ *as* $n \to \infty$*,* $\tilde{\vartheta}_n^h \xrightarrow{\text{P}} \tilde{\vartheta}_0 \; \forall \, h = 2, 3, \ldots$*, where* $\tilde{\vartheta} = \mathfrak{v}(\vartheta_2, \ldots, \vartheta_{G-1})$*.*

Corollary

Under the assumptions of Theorem 2 and Theorem 3. If $\lambda_n^m, \lambda_n^c \to 0$*,* $n^{1/2}\lambda_n^m \to \infty$ *and* $n^{1/2}\lambda_n^c \to \infty$ *as* $n \to \infty$*, then*

$$n^{1/2}\mathcal{B}_{h,n}^{-1}\left\{(\tilde{\vartheta}_n^h - \tilde{\vartheta}_0) + \widetilde{\Gamma}^{h-1}\mathcal{K}_n \, \mathrm{b}_n\right\} \xrightarrow{\mathcal{L}} \mathrm{N}\left\{0, \Sigma(\tilde{\vartheta}_0)\right\}.$$

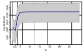
▸ Definitions



Setup II

$$
\begin{aligned}
x_i &= \mu_i \odot \varepsilon_i, \\
\mu_i &= \omega + A\, x_{i-1} + \mathrm{diag}(b_{11}, \ldots, b_{dd})\, \mu_{i-1},
\end{aligned}
$$

- $d = 15$, $n = 500$, $r = 375$.
- Replications: 500.
- Penalized parameters: 210 off-diagonal elements of A.
- $\varepsilon_{ij} \sim$ Weibull(γ_j) are contemporaneous dependent via R-vine, see Kurowicka and Joe (2011).
- Decomposition $\mathfrak{v}(\gamma_j, \omega_j, A_{j\bullet}, b_{jj})$ for $j = 1, \ldots, d$. Application



Results

Figure 4: Comparison of the true matrix A (left) with one updated estimate A_n^h (right).

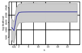


Figure 5: The updated average bias of A_n^h (left) and the corresponding standard deviation (sd) (right).

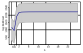
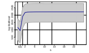


Figure 6: The updated mean of the centered estimates $B_n^h - B$ (solid line) and the corresponding standard deviation (sd) illustrated as grey area.

Figure 7: The updated mean of the centered estimated parameters of the Weibull distributions $\gamma_{j,n}^{h} - \gamma_j$, $j = 1, \ldots, d$ (solid line) and the sd illustrated as grey area.



Numerical criteria

Define for the parameter vector z and its estimate z_n^h

1. the relative absolute error:

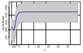
$$\mathrm{RAE}^h \stackrel{\mathrm{def}}{=} \frac{||z - z_n^h||_1}{||z - z_n^1||_1}$$

2. the sign consistency:

$$\mathrm{SC}^h \stackrel{\mathrm{def}}{=} \sum_{k \neq \ell} \mathbf{I}\left\{\mathrm{sign}(A_{k\ell,0}) = \mathrm{sign}(A^h_{k\ell,n})\right\}.$$

h	Parameter	RAE^h		SC^h	
1(1)	$A_{k\ell},\ k \neq \ell$	0.35	(0.09)	169	(10.38)
	$A_{k\ell},\ k \neq \ell$	0.34	(0.10)	169	(10.38)
2	$\omega_j, A_{jj}, B_{jj}\ \forall j$	0.88	(0.17)		-
	γ	0.60	(0.15)		-
	$A_{k\ell},\ k \neq \ell$	0.32	(0.10)	169	(11.86)
4	$\omega_j, A_{jj}, B_{jj}\ \forall j$	0.82	(0.18)		-
	γ	0.46	(0.16)		-
	$A_{k\ell},\ k \neq \ell$	0.31	(0.09)	169	(10.38)
11	$\omega_j, A_{jj}, B_{jj}\ \forall j$	0.80	(0.18)		-
	γ	0.43	(0.16)		-

Table 1: Median values of RAE^h and SC^h for different parameters. The MAD is given in parentheses.



Selection of λ_n^m and a^m

- Split $\{x_i\}_{i=1}^n$ in two parts: $S_1 = \{x_i\}_{i=1}^{n_1}$ and $S_2 = \{x_j\}_{j=n_1+1}^{n}$ containing 80% and 20% of the sample, respectively.
- Use S_1 to estimate $\vartheta_{1,n}(\lambda, a), \vartheta_{2,n}(\lambda, a)$, defined through *Step 1(1)* of Algorithm 2.
- Fit the tuning parameters through

$$(\lambda_n^m, a^m)^\top = \arg \max_{(\lambda,a)^\top} \mathcal{L}^m \left\{\vartheta_{1,n}(\lambda, a), \vartheta_{2,n}(\lambda, a), \vartheta_{3,n}, \dots, \vartheta_{k,n}\right\}$$

on S_2, where $\vartheta_{3,n}, \dots, \vartheta_{k,n}$ are the non-penalized estimators..

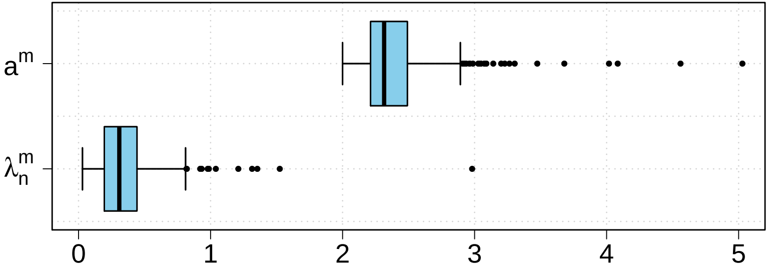
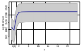


Figure 8: Boxplots for the tuning parameters of the penalization function.

▸ SCAD penalty



Pair copula construction

Example 5

Let $X = (X_1, X_2, X_3) \sim F$ with margins F_1, F_2 and F_3, and non-unique representation of the density

$$f(x_1, x_2, x_3) = f_1(x_1) f(x_2|x_1) f(x_3|x_1, x_2).$$

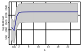
By Sklar theorem:

$$\begin{aligned} f(x_2|x_1) &= \frac{c_{1,2}\left\{F_1(x_1), F_2(x_2)\right\} f_1(x_1) f_2(x_2)}{f_1(x_1)} \\ &= c_{1,2}\left\{F_1(x_1), F_2(x_2)\right\} f_2(x_2) \end{aligned}$$

$$
\begin{aligned}
f(x_3|x_1,x_2) &= \frac{f(x_2,x_3|x_1)}{f(x_2|x_1)} \\
&= \frac{c_{2,3|1}\left\{F(x_2|x_1),F(x_3|x_1)\right\} f(x_2|x_1) f(x_3|x_1)}{f(x_2|x_1)} \\
&= c_{2,3|1}\left\{F(x_2|x_1),F(x_3|x_1)\right\} c_{1,3}\left\{F_1(x_1),F_3(x_3)\right\} f_3(x_3)
\end{aligned}
$$

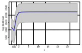
Collecting terms leads to

$$
\begin{aligned}
f(x_1,x_2,x_3) = &\prod_{i=1}^{3} f_i(x_i) \\
&\cdot c_{1,2}\left\{F_1(x_1),F_2(x_2)\right\} c_{1,3}\left\{F_1(x_1),F_3(x_3)\right\} \\
&\cdot c_{2,3|1}\left\{F(x_2|x_1),F(x_3|x_1)\right\}
\end{aligned}
$$



Clarke and Vuong test

- Tests are based related to the Kullback-Leibler divergence, see Vuong (1989, Econometrica), Clarke (2007, Political Analysis).
- H_0 : Two copula models are equivalent
- Vuong test:
  - $m_i^h \stackrel{\text{def}}{=} \ell_i^c(\vartheta_{1,n}^h, \ldots, \vartheta_{G,n}^h) - \ell_i^c(\vartheta_{1,n}^h, \ldots, \vartheta_{G-1,n}^h, \vartheta_{G,0})$
  - $\bar{m}^h = n^{-1} \sum_{i=1}^n m_i^h$
  - $V^h = \bar{m}^h / \sqrt{\sum_{i=1}^n (m_i^h - \bar{m}^h)^2} \stackrel{\mathcal{L}}{\to} \mathrm{N}(0,1)$



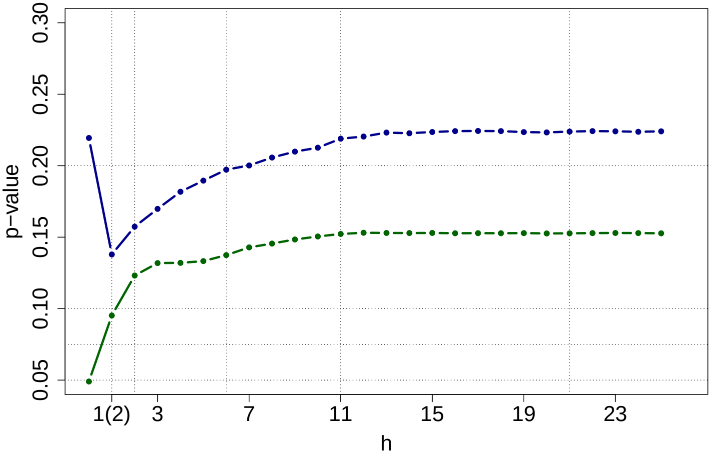


Figure 9: Average *p*-values of the Clarke and Vuong test for each step of the iteration.

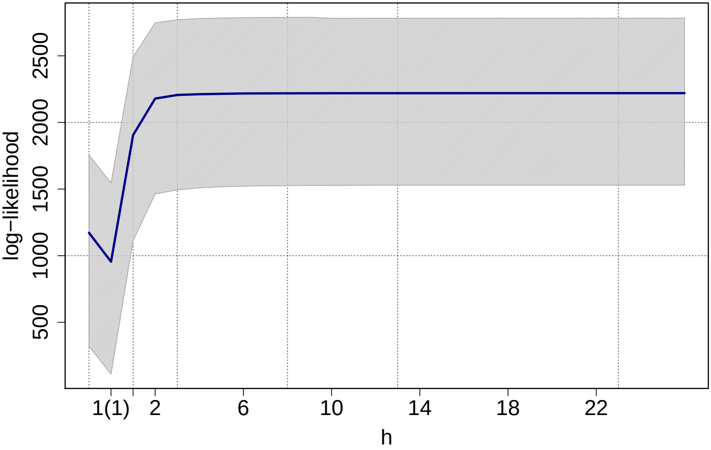


Figure 10: The median of the log likelihood for each step of the iteration. The gray area includes 0.95% of the observations.

VAR

Consider the time series model

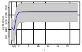
$$x_i = c + \sum_{l=1}^{q} A_l x_{i-l} + \varepsilon_i,$$

where $c = (c_1, \ldots, c_d)^\top$ and A_l is a $(d \times d)$ matrix. Given standard assumptions like

- $\mathrm{E}(\varepsilon_i \varepsilon_i^\top) = \Sigma_\varepsilon$ and $\mathrm{E}(\varepsilon_i \varepsilon_{i-l}^\top) = 0_{dd}$ for $l > 0$
- $\varepsilon = \mathrm{vec}(\varepsilon_1, \ldots, \varepsilon_d) \sim \mathrm{N}(0, I_n \otimes \Sigma_\varepsilon)$

the parameters can be efficiently estimated by OLS. But

- $r > n$ especially for a large q!



Define $\mathsf{Y} = \mathrm{vec}(x_1, \ldots, x_n)$, $Z_i = (1, x_{i-1}^\top, \ldots, x_{i-q}^\top)^\top$ and $\mathsf{Z} = (Z_1, \ldots, Z_n)$ and rewrite the model in matrix notation

$$\mathsf{Y} = (\mathsf{Z}^\top \otimes I_d)\beta + \varepsilon,$$

where $\beta = \mathrm{vec}(c, A_1, \ldots, A_q)$. We assume $\varepsilon \sim \mathrm{N}(0, \Sigma)$, with $\Sigma \neq I_n \otimes \Sigma_\varepsilon$, but the GLS estimator

$$\beta_n = \left\{(\mathsf{Z} \otimes I_d)\Sigma^{-1}(\mathsf{Z}^\top \otimes I_d)\right\}^{-1}(\mathsf{Z} \otimes I_d)\Sigma^{-1}\mathsf{Y}$$

is not feasible.

Algorithm

Step 1*:*

(1) $\beta_n^1 = \left\{(\mathrm{Z}\,\mathrm{Z}^\top)^{-1}\,\mathrm{Z} \otimes I_d\right\}\mathrm{Y}$

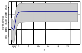
(2) $\Sigma_n^1 = \left\{\mathrm{Y} - (\mathrm{Z}^\top \otimes I_d)\beta_n^1\right\}\left\{\mathrm{Y} - (\mathrm{Z}^\top \otimes I_d)\beta_n^1\right\}^\top$

Step $h > 1$*:*

(1) $\beta_n^h = \left\{(\mathrm{Z} \otimes I_d)(\Sigma_n^{h-1})^{-1}(\mathrm{Z}^\top \otimes I_d)\right\}^{-1}(\mathrm{Z} \otimes I_d)(\Sigma_n^{h-1})^{-1}\,\mathrm{Y}$

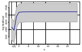
(2) $\Sigma_n^h = \left\{\mathrm{Y} - (\mathrm{Z}^\top \otimes I_d)\beta_n^h\right\}\left\{\mathrm{Y} - (\mathrm{Z}^\top \otimes I_d)\beta_n^h\right\}^\top$

Penalization of β can be embedded at *Step 1*!



Measuring volatility connectedness

- Daily realized volatilities (RVs) from January 2007 - December 2008.
- 30 U.S. blue chip companies similar to the DJIA.
- VMEM$(1,1)$ as in Simulation II ▸ VMEM.
- R-vine based on bivariate t-copulae.
- $r/n \approx 1.7$



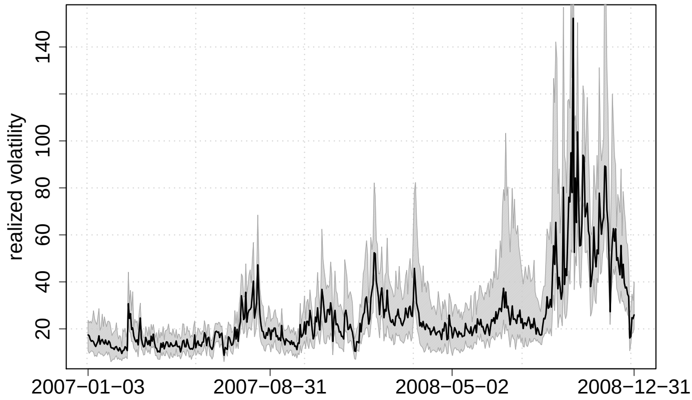


Figure 11: Median of the realized volatilities over the companies. The gray area includes 90% of the observations.

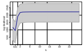
Assuming a stationary VMEM$(1,1)$ for the RVs $\{x_i\}_{i=1}^n$, whose zero-mean MA(∞) representation is

$$\begin{aligned} y_i =& \eta_i + \sum_{l=1}^{\infty} \left\{ (A+B)^l - (A+B)^{l-1} B \right\} \eta_{i-l} \\ =& \eta_i + \sum_{l=1}^{\infty} \Psi_l \eta_{i-l}, \end{aligned}$$

with $\mathsf{E}(\eta_i) = 0, \mathsf{E}(\eta_i \eta_i^\top) = \Sigma_\eta$ and $y_i = x_i - \{I_d - (A+B)\}^{-1} \omega$.

Two types of H-step prediction errors:

- $\nu_i(H) = \sum_{l=0}^{H-1} \Psi_l \eta_{i+H-l}$ and
- $\nu_{i,\ell}(H) = \sum_{l=0}^{H-1} \Psi_l \left\{ \eta_{i+H-l} - \mathsf{E}(\eta_{i+H-l} | \eta_{\ell,i+H-l} = \delta) \right\}$.



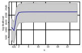
Connectedness measures

The elements of the generalized variance decomposition matrix $\widetilde{V}_H$ are

$$\tilde{v}_{k\ell,H} = \frac{e_k^\top \left[\operatorname{Var}\{\nu_i(H)\} - \operatorname{Var}\{\nu_{i,\ell}(H)\}\right] e_k}{e_k^\top \operatorname{Var}\{\nu_i(H)\}\, e_k},$$

where $e_k = (0, \ldots, 0_{k-1}, 1_k, 0_{k+1}, \ldots, 0)^\top$ is a $(d \times 1)$ vector. Standardization $v_{k\ell,H} = \tilde{v}_{k\ell,H} / \sum_{\ell=1}^{d} \tilde{v}_{k\ell,H}$ leads to, see Diebold and Yilmaz (2014, JoE):

- the total directional connectedness to others from ℓ by $C_{\bullet\leftarrow\ell,H} = \sum_{k\neq\ell} v_{k\ell,H}$,
- the total connectedness $C_H = d^{-1} \sum_{k\neq\ell} v_{k\ell,H}$.



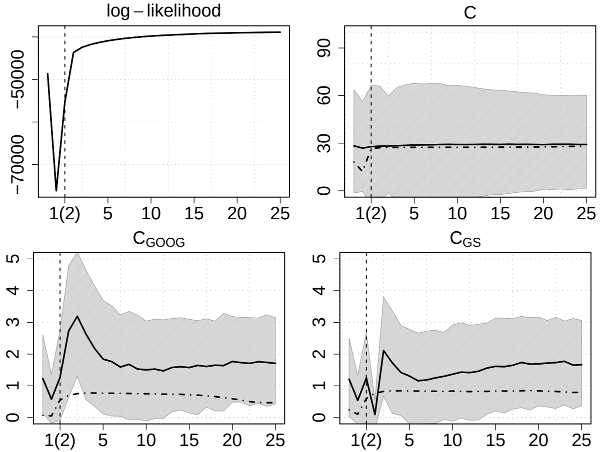
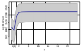


Figure 12: Upper panel: log-likelihood values and total systemic connectedness C_{12} in dependence of h. Lower panel: volatility contagion from Google $C_{\bullet\leftarrow \mathrm{GOOG},12}$ and Goldman Sachs $C_{\bullet\leftarrow \mathrm{GS},12}$ in dependence of h.

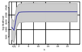


Conclusion

- Maximization strategy for complicated and high-parameterized log-likelihood functions.
- Asymptotic properties of the sparse and efficient estimator are established.
- Accuracy of the procedure is illustrated in a simulation study.
- Application emphasizes the importance of efficiency.

Future research:

- Hidden Markov models
- Risk management (DCC)
- Euro-crisis



Efficient Iterative ML Estimation of High-Parameterized Time Series Models

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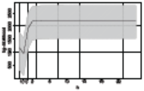

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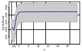

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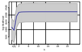


Assumptions

(1) The model is identifiable and the true value ϑ_0 is an interior point of the compact parameter space Θ. We assume that the model is correctly specified in the sense that $\mathsf{E}_\vartheta\{\partial \ell_i(\vartheta)/\partial \vartheta_g\} = 0$ and information equality holds,

$$\mathcal{I}_{i,gl}(\vartheta) \stackrel{\text{def}}{=} \mathsf{E}_\vartheta \left\{ \frac{\partial \ell_i(\vartheta)}{\partial \vartheta_g} \frac{\partial \ell_i(\vartheta)}{\partial \vartheta_l^\top} \right\} = - \mathsf{E}_\vartheta \left\{ \frac{\partial^2 \ell_i(\vartheta)}{\partial \vartheta_g \partial \vartheta_l^\top} \right\},$$

for $g, l = 1, \ldots, G$ and $i = 1, \ldots, n$.

(2) The information matrix is $\mathcal{I}(\vartheta) = \sum_{i=1}^{n} \mathcal{I}_i(\vartheta)$, with $\mathcal{I}_i(\vartheta) = \{\mathcal{I}_{i,gl}(\vartheta)\}_{g,l=1}^{G}$. Let the limit of $n^{-1}\mathcal{I}(\vartheta) \xrightarrow{\mathrm{P}} \mathcal{J}(\vartheta)$ be the asymptotic information matrix, which is finite and positive definite at ϑ_0 and $n^{-1}\ddot{\mathcal{L}}(\vartheta) \xrightarrow{\mathrm{P}} \mathcal{H}(\vartheta)$ be the asymptotic Hessian, which is finite and negative definite for $\vartheta \in \{\vartheta : ||\vartheta - \vartheta_0|| < \delta\}$, $\delta > 0$. Back

(3) The score $s(\vartheta_0) = \mathfrak{v}\{\dot{\mathcal{L}}^m(\vartheta_{1,0}, \ldots, \vartheta_{k,0}), \dot{\mathcal{L}}^c(\vartheta_0)\}$ of the decomposed log-likelihood $\mathcal{L}(\vartheta) = \mathcal{L}^m(\vartheta_1, \ldots \vartheta_k) + \mathcal{L}^c(\vartheta)$, with $\left\{n^{-1}s(\vartheta_0)s(\vartheta_0)^\top\right\} \xrightarrow{\mathrm{P}} \Sigma(\vartheta_0)$, obeys

$$n^{-1/2}s(\vartheta_0) \quad \xrightarrow{\mathcal{L}} \quad \mathrm{N}\{0, \Sigma(\vartheta_0)\}.$$

Back

(3) Define the lower block and upper block triangular matrix of $-n^{-1}\ddot{\mathcal{L}}(\vartheta_0)$ as L_n and U_n, respectively, such that $-n^{-1}\ddot{\mathcal{L}}(\vartheta_0) = L_n - U_n$ with $L_{gl,n} = 0$ for $g < l \leq G$ and $U_{gl,n} = 0$ for $l \leq g \leq G$. For the probability limits L and U of L_n and U_n, respectively, we assume $\rho(\Gamma) < 1$, where $\rho(\cdot)$ denotes the spectral radius and $\Gamma \stackrel{\text{def}}{=} \mathrm{L}^{-1}\,\mathrm{U}$. Back

(5) There exists an open subset θ of Θ that contains the true parameter ϑ_0 such that for almost all X_i, $i = 1, \ldots, n$, the density $f_i(\cdot; \vartheta)$ admits all third derivatives $\partial f_i(X_{i1}, \ldots, X_{id}; \vartheta)/\partial\vartheta_u\partial\vartheta_v\partial\vartheta_w$ for all $\vartheta \in \theta$. Furthermore, there exist functions $M_{uvw}(\cdot)$ such that

$$\left|\frac{\partial \ell_i(\vartheta)}{\partial\vartheta_u\partial\vartheta_v\partial\vartheta_w}\right| \leq M_{uvw}(X_i) \quad \text{for all} \quad \vartheta \in \theta,$$

where $\mathrm{E}\{M_{uvw}(X_i)\} < \infty$ for $u, v, w = 1, \ldots, r$. Back

Definitions

- Let the number of parameters of each subvector ϑ_g be r_g, $g = 1 \ldots, G$, s.t. $p = \sum_{g=1}^{k} r_g$ and $r = \sum_{g=1}^{G} r_g$. Define $q = r - p$ and the matrices

$$\mathcal{T}_1 = \begin{pmatrix} I_p & 0_{pp} & 0_{pq} \\ 0_{qp} & 0_{qp} & I_q \end{pmatrix} \quad \text{and} \quad \mathcal{T}_2 = \begin{pmatrix} I_p & I_p & 0_{pq} \\ 0_{qp} & 0_{qp} & I_q \end{pmatrix},$$

with identity matrix I_p, $0_{pq} = 0_p 0_q^\top$ and null vector 0_p.

- Define

$$n^{-1}\begin{Bmatrix} \ddot{\mathcal{L}}^{m}(\vartheta_{1,0},\ldots,\vartheta_{k,0}) \; 0_{pq} \\ \ddot{\mathcal{L}}^{c}_{\mathfrak{v}(\vartheta_{k+1},\ldots,\vartheta_{G}),\vartheta}(\vartheta_0) \end{Bmatrix} = \mathcal{H}^{1}(\vartheta_0) + \mathcal{O}_p(1),$$

and

$$\mathcal{B}_h = \Gamma^{h-1}\left[\mathcal{K}\mathcal{T}_1 - \{-\mathcal{H}(\vartheta_0)\}^{-1}\mathcal{T}_2\right] + \{-\mathcal{H}(\vartheta_0)\}^{-1}\mathcal{T}_2,$$

and $\quad \mathcal{K} = \left\{-\mathcal{H}^{1}(\vartheta_0)\right\}^{-1}.$

Back

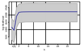
Define

$$
\begin{aligned}
\mathrm{b}_n^m &\stackrel{\text{def}}{=} \left\{ \breve{p}'_{\lambda_n^m, a^m}(\vartheta_{21,0}), \ldots, \breve{p}'_{\lambda_n^m, a^m}(\vartheta_{2r_2,0}) \right\}^\top, \\
\mathrm{b}_n^c &\stackrel{\text{def}}{=} \left\{ \breve{p}'_{\lambda_n^c, a^c}(\vartheta_{(G-1)1,0}), \ldots, \breve{p}'_{\lambda_n^c, a^c}(\vartheta_{(G-1)r_{G-1},0}) \right\}^\top.
\end{aligned}
$$

and $\mathrm{b}_n = (\mathrm{b}_n^m, 0_s, \mathrm{b}_n^c)^\top$ as well as

$$
\begin{aligned}
\Psi_n^m &= \operatorname{diag}\left\{ \breve{p}''_{\lambda_n^m, a^m}(\vartheta_{21,0}), \ldots, \breve{p}''_{\lambda_n^m, a^m}(\vartheta_{2r_2,0}) \right\}, \\
\Psi_n^c &= \operatorname{diag}\left[\breve{p}''_{\lambda_n^c, a^c}\{\vartheta_{(G-1)1,0}\}, \ldots, \breve{p}''_{\lambda_n^c, a^c}\{\vartheta_{(G-1)r_{G-1},0}\} \right].
\end{aligned}
$$

and $\Psi_n = \operatorname{diag}(\Psi_n^m, 0_{ss}, \Psi_n^c)$.



The "bread" of the covariance matrix is given by:

$$\mathcal{B}_{h,n} = \widetilde{\Gamma}^{h-1}\left[\mathcal{K}_n\mathcal{T}_1 - \{-\mathcal{H}(\tilde{\vartheta}_0)\}^{-1}\mathcal{T}_2\right] + \{-\mathcal{H}(\tilde{\vartheta}_0)\}^{-1}\mathcal{T}_2,$$

$$\text{and} \quad \mathcal{K}_n = \left\{\Psi_n - \mathcal{H}^1(\tilde{\vartheta}_0)\right\}^{-1}.$$

Back