Uniform Confidence for Pricing Kernels

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Motivation

- Arbitrage free market; riskless bond with rate r
- Underlying price process $\{S_t\}$

Price z_t at time t from a payoff $\psi(S_T)$

$$
\begin{aligned}
z_t &= \int_0^\infty exp(-r\tau)\psi(x)dQ_{S_T}(x) \\
&= \int_0^\infty exp(-r\tau)\psi(x)\frac{q_t(x)}{p_t(x)}dP_{S_T}(x)
\end{aligned}
$$

Figure 1: conditional measure at time of maturity T built opon a path of the stochastic process for underlying asset with information up to time t .

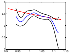
Empirical Pricing Kernel (EPK)

Pricing Kernel (PK) a stochastic discount factor, i.e.

$$\mathcal{K}_{t,\tau}(x) = exp(-r\tau)\frac{q_t(x)}{p_t(x)}$$

EPK is therefore an estimate of PK:

$$\widehat{\mathcal{K}_{t,\tau}(x)} = exp(-r\tau)\frac{\widehat{q_t(x)}}{\widehat{p_t(x)}}$$



The EPK Paradox

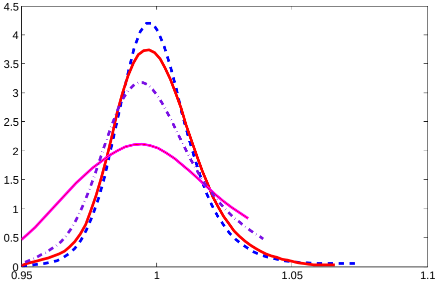
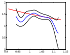


Figure 2: Examples of inter-temporal pricing kernels with maturity 0.00833($3D$) respectively on 17-Jan-2006 (blue), 18-Apr-2006 (red), 16-May-2006 (magenta), 13-June-2006 (black).



The EPK Paradox

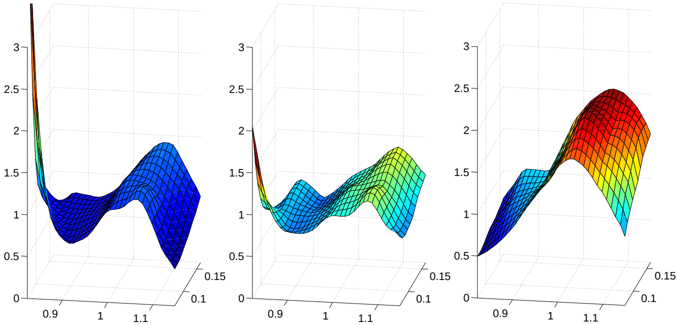


Figure 3: Estimated PK across moneyness and maturity

The EPK Paradox

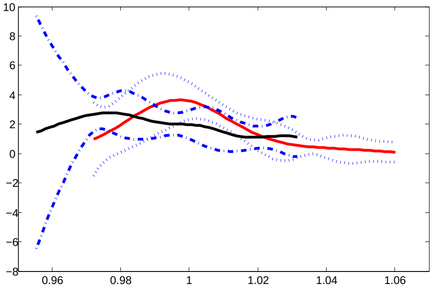


Figure 4: Examples of inter-temporal pricing kernels with various maturities in years: 0.02222 (8D) (red) 0.1(36D) (green) on 12-Jan-2006 and their confidence bands

Aims

- Nonparametric confidence band to test alternatives
- Check the statistical significance of the EPK puzzle
- Investigate shapes of EPKs: investor preference
- Understand the dynamics of risk patterns

Outline

1. Motivation ✓
2. Uniform Confidence Band
3. Monte-Carlo Study
4. Empirical Data Analysis

Risk Neutral Density (RND) Estimation

The RND may be estimated from option prices, Breeden and Litzenberger (1978):

$$q_t(S_T) = exp(r\tau)\frac{\partial^2 H_t(k,\tau)}{\partial k^2}|_{k=S_T}$$

with call price function $H_t(k,\tau)$.

Aït-Sahalia and Lo (1998) estimate $H_t(k,\tau)$ nonparametrically and differentiate it twice w.r.t. k.

Call prices (X_i, Y_i), with fixed τ, we have:

$$Y_i = H(x_i) + \varepsilon_i, i = 1, \ldots, n_q$$

(X_i, Y_i)s are i.i.d.

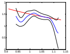
Define $L\{y; H(u)\}$ as the conditional density of Y given $K = u$

Local polynomial estimate, $(x \approx u)$:

$$H(u) \approx H(x, u) \stackrel{\text{def}}{=} \sum_{j=0}^{3} H_j(x)(u-x)^j$$

Local likelihood

$$L_{n_q}\{H(x)\} \stackrel{\text{def}}{=} \frac{1}{n_q} \sum_{i=1}^{n_q} K_{h_{n_q}}(x - x_i) \log L\{Y_i; H(x_i, x)\},$$



Data

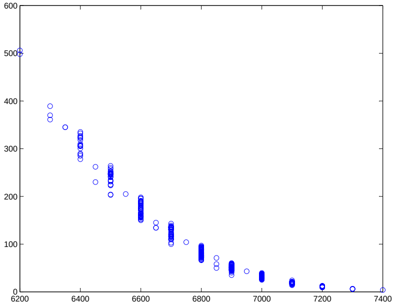


Figure 5: Plot of call prices against strikes k, $n_q = 1000$, $n_p = 500$.

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de

Solutions:

$$\widehat{\mathbf{H(x)}} \stackrel{\text{def}}{=} \text{argmax}_H L_{n_q}\{H(x)\},$$

where

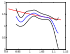
$$\widehat{\mathbf{H(x)}} = \{\widehat{H_0(x)}, \widehat{H_1(x)}, \widehat{H_2(x)}, \widehat{H_3(x)}\}^\top$$

Estimate for $q_t(x)$:

$$\widehat{q_t(x)} \propto 2!\widehat{H_2(x)}$$

The kernel density estimate for $p_t(x)$ is based on historical $\{S_t\}$:

$$\widehat{p_t(x)} = n_p^{-1} \sum_{j=1}^{n_p} K_{h_{n_p}}(x - S_j)$$

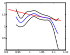


Uniform Convergence

Theorem

Under regularity conditions, for all x in an interval J, we have a.s.,

$$\begin{aligned}\sup_{x\in J}\left|\widehat{\mathcal{K}_{t,\tau}(x)}-\mathcal{K}_{t,\tau}(x)\right| &= \mathcal{O}[\max\{(n_p h_{n_p}/\log n_p)^{-0.5}, h_{n_p}^2,\\ &\quad h_{n_q}^3, h_{n_q}^{-2}\{n_q h_{n_q}/\log n_q\}^{-0.5}\}]\end{aligned}$$



Uniform Confidence Band

Theorem

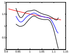
Under regularity conditions,

$$\mathcal{K}_{nt,\tau}(x) \stackrel{\text{def}}{=} n_q^{1/2} {h_{n_q}}^{5/2} \{\widehat{\mathcal{K}_{t,\tau}(x)} - \mathcal{K}_{t,\tau}(x)\} \, \mathrm{Var}\{\widehat{\mathcal{K}_{t,\tau}(x)}\}^{-1/2}.$$

We have:

$$\mathrm{P} \quad \left\{ (-2\log h_{n_q})^{1/2} \left\{ \sup_{x \in J} |\mathcal{K}_{nt,\tau}(x)| - c_{nt} \right\} < z \right\} \longrightarrow \exp\{-2\exp(-z)\},$$

where $c_{nt} = (-2\log h_{n_q})^{1/2} + (-2\log h_{n_q})^{-1/2}\{x_\alpha + \log(C/2\pi)\}$



Uniform Confidence Band

Thus, a $(1-\alpha)100\%$ confidence band for pricing kernel $\mathcal{K}_{t,\tau}$ is:

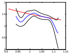
$$[f(x) : \sup_x \{ |\widehat{\mathcal{K}_{t,\tau}(x)} - f(x)| \widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau}(x)})^{-1/2} \} \leq L_\alpha]$$

where

$$L_\alpha = 2!(n_q h_{n_q}^5)^{-1/2} c_{nt}$$

and

$$x_\alpha = -\log\{-1/2 \log(1-\alpha)\}$$



Extension on τ

Let $\mathfrak{x}$ be the possible set of maturities, the extension of our results over τ :

$$[f_{t,\tau}(x) : \sup_{x \in E, \tau \in \mathfrak{x}} \{ |\widehat{\mathcal{K}_{t,\tau}(x)} - f_{t,\tau}(x)| \widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau}(x)})^{-1/2} \} \leq L_\alpha].$$

In the BS setup, the evolution of bands over time, for fixed τ_1 $(g(\tau_1 - \tau_2) = \mathcal{K}_{t,\tau_1}(x)/\mathcal{K}_{t,\tau_2}(x)))$

$$[f_{t,\tau_2} : \widehat{g}(\tau_1 - \tau_2)\{-L_\alpha \widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau_1}(x)})^{1/2} + \widehat{\mathcal{K}_{t,\tau_1}(x)}\} \leq f_{t,\tau_2}(x) \leq$$
$$\widehat{g}(\tau_1 - \tau_2)\{L_\alpha \widehat{\mathrm{Var}}(\widehat{\mathcal{K}_{t,\tau_1}(x)})^{1/2} + \widehat{\mathcal{K}_{t,\tau_1}(x)}\}],$$

Bootstrap

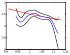
Theorem

Under regularity conditions

$$[f_{t,\tau} : \sup_{x \in E} \{ |\widehat{\mathcal{K}_{t,\tau}(x)} - f_{t,\tau}(x)| \operatorname{Var}(\widehat{\mathcal{K}_{t,\tau}})^{-1/2} \} \leq L^*_\alpha]$$

where the bound L^*_α satisfies

$$\mathrm{P}^*(-\{U_{n_q}(x)^{-1} H_{n_q}^{-1} A^*_{n_q}(x) / B_{t,\tau}(x)^{-1} N(x)^{-1} M^*(x) N(x)^{-1}\}_{3,3}$$
$$\leq L^*_\alpha) = 1 - \alpha$$



A Monte-Carlo Study

For $q_t(x)$, generate data from BS model, interest rate $r = 0.04$, $S_t = 6500$, $k \in [6200, 7400]$, $\tau = 1$M, $\varepsilon_i \in$ U$[0, 6]$, $\sigma = 0.1878$.

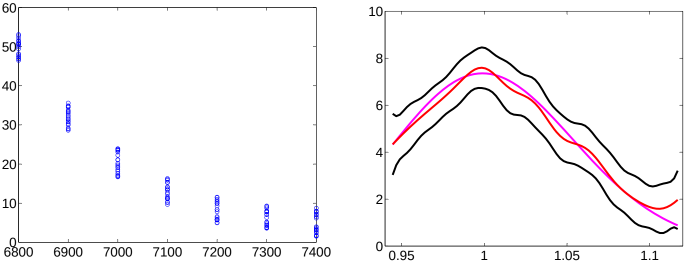
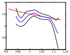


Figure 6: (Left) H against k (Right) Plot of confidence bands (black) , estimated value, the Black Scholes SPD (magenta) of the EPK, $h_{n_q} = 0.085$, $\alpha = 0.05$, $n_q = 300$.



A Monte-Carlo Study

For historical density, simulate data from Geometric Brownian Motion, with $\mu = 0.23$, $\sigma = 0.1878$.

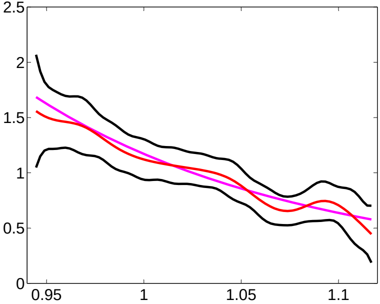
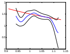


Figure 7: Plot of confidence bands, estimated value, the Black Scholes EPK (magenta), $h_{n_q} = 0.060$, $\alpha = 0.05$, $n_q = 500$, $n_p = 600$.



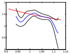
Coverage Probability

Case	$(n=)300$	450	600
$(\tau=)3$	0.9063(2.402)	0.9144(2.204)	0.9233(1.998)
6	0.8964(2.438)	0.9056(2.134)	0.9203(2.069)

Table 1: Cov. prob. (area) of the uniform confidence band for $q_t(x)$ at $\alpha = 5\%$ with $\sigma = 0.1878$, sim $= 500$

Case	$(n=)300$	450	600
3	0.7820(2.5434)	0.7980(2.4978)	0.8020(2.4131)
6	0.8602(2.4987)	0.8749(2.4307)	0.8900(2.4131)

Table 2: Same for EPK at $\alpha = 10\%$



Empirical Data Analysis

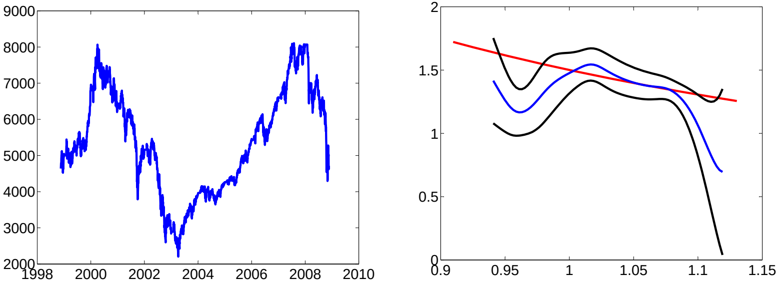
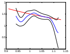


Figure 8: (Left) Plot of DAX index (Right) Plot of confidence bands (black), EPK by Black Scholes fitting, nonparametric EPK, $h_{n_q} = 0.075$, $\alpha = 0.05$, $n_p = 506$, $n_q = 715$.



Empirical Data Analysis

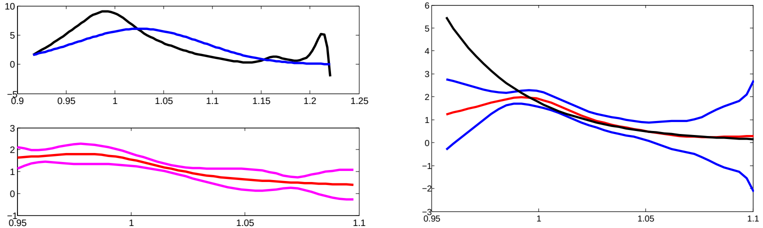
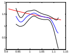


Figure 9: (Left) Plot of q and p (upper panel) Plot EPK and its bands, 2006 Feb 28th (lower panel) (Right) Plot of confidence bands (blue), EPK by Black Scholes fitting (black), EPK, 2006, April, 24th.



Empirical Data Analysis

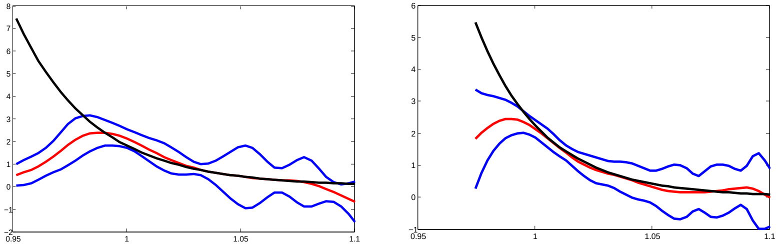


Figure 10: (Left) Plot of confidence bands (blue), EPK by Black Scholes fitting (black), EPK, 2006, July, 24th. (Right) Same for 2006, Aug, 18th.

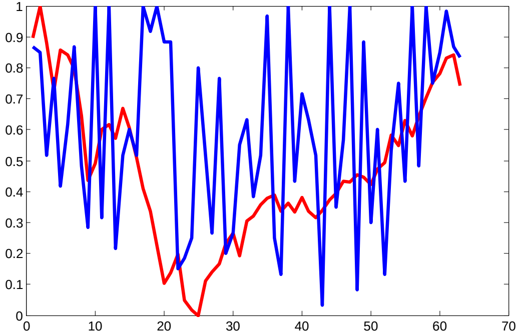


Figure 11: Plot of estimation of the BS EPK covered in band, DAX price (red) $\tau = 2$M.(200001-200006)

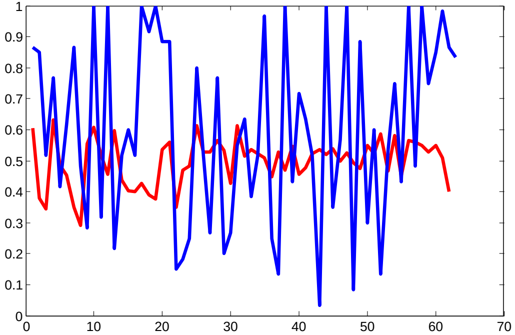


Figure 12: Plot of estimation of the BS EPK covered in band (blue), DAX price difference (red) $\tau = 2$M.(200001-200006)

Conclusions

- Uniform confidence bands tell us about risk patterns
- Smoothing of EPK is best done via IVS
- Bootstrap does not improve coverage probability significantly
- BS for $\tau = 0.5$M is mostly rejected
- Bootstrap improvement possible for robust smoothers

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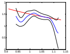

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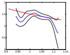

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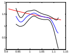

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Rookley (1997)

Let C_{it} be the price of the i^{th} option at time t and K_{it} its strike price, and define the rescaled call option $c = C/S_t$ in terms of moneyness $M = S_t/K$ s.t.

$$
\begin{aligned}
c_{it} &= c\{M_{it}; \sigma(M_{it})\} = \Phi(d_1) - \frac{e^{-r\tau}\Phi(d_2)}{M_{it}} \\
d_1 &= \frac{\log(M_{it}) + \left\{r_t + \frac{1}{2}\sigma(M_{it})^2\right\}\tau}{\sigma(M_{it})\sqrt{\tau}} \\
d_2 &= d_1 - \sigma(M_{it})\sqrt{\tau}
\end{aligned}
$$

The RND is then

$$q(\cdot) = e^{r\tau} \frac{\partial^2 C}{\partial K^2} = e^{r\tau} S \frac{\partial^2 c}{\partial K^2}$$

with

$$\frac{\partial^2 c}{\partial K^2} = \frac{\mathrm{d}^2 c}{\mathrm{d}M^2} \left(\frac{M}{K} \right)^2 + 2 \frac{\mathrm{d}c}{\mathrm{d}M} \frac{M}{K^2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2 c}{\mathrm{d}M^2} &= \Phi'(d_1) \left\{ \frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} - d_1 \left(\frac{\mathrm{d}d_1}{\mathrm{d}M} \right)^2 \right\} \\ &\quad - \frac{e^{-r\tau} \Phi'(d_2)}{M} \left\{ \frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} - \frac{2}{M} \frac{\mathrm{d}d_2}{\mathrm{d}M} - d_2 \left(\frac{\mathrm{d}d_2}{\mathrm{d}M} \right)^2 \right\} \\ &\quad - \frac{2 e^{-r\tau} \Phi(d_2)}{M^3} \end{aligned}$$

With $V = \sigma(M)$, $V' = \dfrac{\partial \sigma(M)}{\partial M}$, $V'' = \dfrac{\partial^2 \sigma(M)}{\partial M^2}$

$$
\begin{aligned}
\frac{\mathrm{d}^2 d_1}{\mathrm{d}M^2} \quad = \quad & -\frac{1}{M * V(M)\sqrt{\tau}}\left\{\frac{1}{M} + \frac{V'(M)}{V(M)}\right\} \\
& + V''(M)\left\{\frac{\sqrt{\tau}}{2} - \frac{\log(M) + r\tau}{V(M)^2\sqrt{\tau}}\right\} \\
& + V'(M)\left\{2V'(M)\frac{\log(M) + r\tau}{V(M)^3\sqrt{\tau}}\right. \\
& \left. - \frac{1}{M * V(M)^2\sqrt{\tau}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 d_2}{\mathrm{d}M^2} \quad = \quad & -\frac{1}{M * V(M)\sqrt{\tau}}\Big\{\frac{1}{M} + \frac{V'(M)}{V(M)}\Big\} \\
& - V''(M)\Big\{\frac{\sqrt{\tau}}{2} + \frac{\log(M) + r\tau}{V(M)^2\sqrt{\tau}}\Big\} \\
& + V'(M)\Big\{2V'(M)\frac{\log(M) + r\tau}{V(M)^3\sqrt{\tau}} \\
& - \frac{1}{M * V(M)^2\sqrt{\tau}}\Big\}
\end{aligned}
$$