Quantile Regression with high dimensional Single-Index Models

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Financial risk

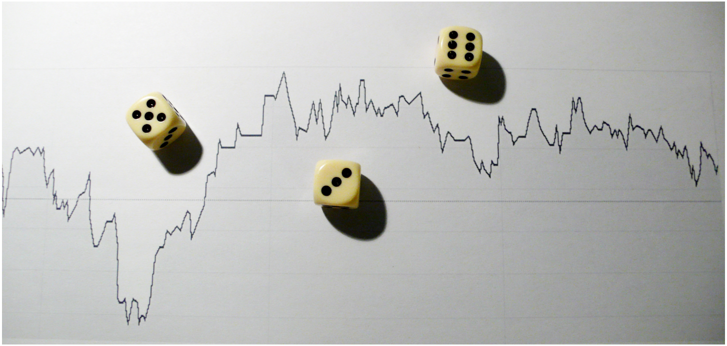
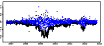
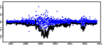


Figure 1: Financial risk



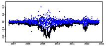
Objective

- Risk patterns depend on covariates X
- Dimensionality issues, $X \in \mathbb{R}^p$, $p \to \infty$
- Variable selection for Quantile Regression (QR)
- CoVaR, single index model (SIM)



Challenges

- Model of tails of conditional distribution
- Dimension reduction
- SIM estimation combined with variable selection
- Alternatives to MAVE (minimum average variance estimation)



Single Index Model

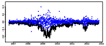
- Observations $\{X_i, Y_i\}_{i=1}^n$ with

$$Y_i = g(\beta^{*\top} X_i) + \varepsilon_i, \tag{1}$$

  where $g(\cdot)$ is the link function, and $\beta^* \in \mathbb{R}^p$. $\{\varepsilon_i\}_{i=1}^n$ are independent.
- p is possible large: $p \to \infty$.
- $\mathsf{E}_{Y|X=x}(\varepsilon) = 0$ for mean regression.
- $F^{-1}_{\varepsilon|X=x}(\tau) = 0$ for quantile regression.

What is known?

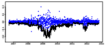
- MAVE method, Xia et al.(2002)
- Application in banking, environmental statistics
- First order "free lunch" $\sqrt{n}$ rate
- A one dimensional problem for estimating $g(\cdot)$



High dimensional SIM

- How to estimate nonzero β_l^* ?
- Which rates can we allow for p ?
- What are the consequences for estimating $g(\cdot)$?
- Sparsistency ?

Go to details



Outline

1. Motivation ✓
2. Single index model
3. Simulations
4. Applications
5. Further work

A quasi-likelihood approach

Recall (1):

$$\min_{\beta} \mathsf{E}_{(Y|X=x)} \rho_\tau\{Y - g(\beta^\top x)\}, \tag{2}$$

Quantile regression:

$$\rho_\tau(u) = \tau u \, \mathbf{I}\{u \in (0,\infty)\} - (1-\tau) u \, \mathbf{I}\{u \in (-\infty, 0)\}. \tag{3}$$

Expectile regression:

$$\rho_\tau(u) = \tau u^2 \, \mathbf{I}\{u \in (0,\infty)\} + (1-\tau) u^2 \, \mathbf{I}\{u \in (-\infty, 0)\}. \tag{4}$$

Likelihood approximations

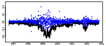
$$g(\beta^\top X_i) \approx g(\beta^\top x) + g'(\beta^\top x)\beta^\top(X_i - x) \tag{5}$$

Approximations

$$L_x(\beta) \stackrel{\text{def}}{=} \mathsf{E}\,\rho_\tau\{Y - g(\beta^\top x) - g'(\beta^\top x)\beta^\top(X - x)\} K_h\{\beta^\top(X - x)\} \tag{6}$$

$$L_{n,x}(\beta) \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^{n}\rho_\tau\{Y_i - g(\beta^\top x) - g'(\beta^\top x)\beta^\top(X_i - x)\} K_h\{\beta^\top(X_i - x)\} \tag{7}$$

where $K_h(.) = K(./h)/h$ with $K(.)$ a kernel, h bandwidth



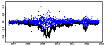
A simple trick

Minimize average contrast (w.r.t. β):

$$
\begin{aligned}
L_n(\beta) &\stackrel{\text{def}}{=} n^{-1} \sum_{j=1}^{n} L_{n,X_j}(\beta) \\
&= n^{-2} \sum_{j=1}^{n} \sum_{i=1}^{n} \rho_\tau \left\{ Y_i - g(\beta^\top X_j) - g'(\beta^\top X_j)\beta^\top (X_i - X_j) \right\} \\
&\quad K_h\{\beta^\top (X_i - X_j)\} \qquad (8)
\end{aligned}
$$

Therefore (in first approach):

$$
\widehat{\beta} \approx \arg\min_{\beta} L_n(\beta).
$$

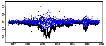


One more trick

Let $a_j = g(\beta^\top X_j)$, $b_j = g'(\beta^\top X_j)$, estimate β by:

$$\min_{(a_j,b_j)'\mathrm{s},\beta} n^{-1} \sum_{j=1}^{n} \sum_{i=1}^{n} \rho_\tau \left(Y_i - a_j - b_j X_{ij}^\top \beta \right) \omega_{ij}(\beta), \tag{9}$$

where $X_{ij} \stackrel{\mathrm{def}}{=} X_i - X_j$, $\omega_{ij}(\beta) \stackrel{\mathrm{def}}{=} K_h(X_{ij}^\top \beta) / \sum_{i=1}^{n} K_h(X_{ij}^\top \beta)$.



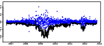
The final trick

Penalize the dimension p and estimate β by:

$$\min_{(a_j,b_j)'\mathrm{s},\beta} n^{-1}\sum_{j=1}^{n}\sum_{i=1}^{n}\rho_\tau\left(Y_i - a_j - b_j X_{ij}^\top\beta\right)\omega_{ij}(\beta) + \sum_{l=1}^{p}\gamma_\lambda(|\widehat{\beta}_l^{(0)}|)|\beta_l|, \quad (10)$$

where $\gamma_\lambda(t)$ is some non-negative function, and $\widehat{\beta}^{(0)}$ initial estimator of β^* (linear QR with variable selection).

Go to details



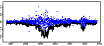
How does this work?

- $\widehat{\beta}^{(0)}$ initial estimator of β^* (linear QR with variable selection).
- For $t = 0, 1, 2, \cdots$, given $\widehat{\beta}^{(t)}$, standardize $\widehat{\beta}^{(t)}$, $\|\widehat{\beta}^{(t)}\| = 1$, $\widehat{\beta}_1^{(t)} = 1$, $\widehat{d}_l^{(t)} \stackrel{\text{def}}{=} \gamma_\lambda(|\widehat{\beta}_l^{(t)}|)$. Then compute

$$(\widehat{a}_j^{(t)}, \widehat{b}_j^{(t)}) \stackrel{\text{def}}{=} \operatorname*{arg\,min}_{(a_j, b_j)'\text{s}} \sum_{i=1}^{n} \rho_\tau\big(Y_i - a_j - b_j X_{ij}^\top \widehat{\beta}^{(t)}\big)\omega_{ij}(\widehat{\beta}^{(t)})$$

- Given $(\widehat{a}_j^{(t)}, \widehat{b}_j^{(t)})$, solve

$$\widehat{\beta}^{(t+1)} = \operatorname*{arg\,min}_{\beta}\, n^{-1} \sum_{j=1}^{n} \sum_{i=1}^{n} \rho_\tau\big(Y_i - \widehat{a}_j^{(t)} - \widehat{b}_j^{(t)} X_{ij}^\top \beta\big)\omega_{ij}(\widehat{\beta}^{(t)}) + \textstyle\sum_{l=1}^{p} \widehat{d}_l |\beta_l|.$$

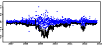


Some definitions

Let $\beta^* = (\beta^{*\top}_{(1)}, \beta^{*\top}_{(0)})^\top$ with $\beta^*_{(1)} \stackrel{\text{def}}{=} (\beta_1, \dots, \beta_q)^\top \neq 0$ and $\beta^*_{(0)} = (\beta_{q+1}, \dots, \beta_p)^\top = 0$. $X_{i(1)} \stackrel{\text{def}}{=}$ sub vector of X_i corresponding to $\beta^*_{(1)}$, $X_{i(0)}$ corresponding to $\beta^*_{(0)}$.
$\widehat{\beta}^0 \stackrel{\text{def}}{=} (\widehat{\beta}^{0\top}_{(1)}, \mathbf{0}^\top)^\top$.

$$\widehat{\beta}^0_{(1)} \stackrel{\text{def}}{=} \arg\min_{\beta_{(1)}} n^{-1} \sum_{j=1}^n \sum_{i=1}^n \rho_\tau \left\{ Y_i - a_{j(1)} - b_{j(1)} X^\top_{ij(1)} \beta_{(1)} \right\} K_h\{\beta^\top_{(1)} X_{ij(1)}\}.$$

where $a_{j(1)} = g(\beta^\top_{(1)} X_{j(1)})$, $b_{j(1)} = g'(\beta^\top_{(1)} X_{j(1)})$, $X_{ij(1)} = X_{i(1)} - X_{j(1)}$, $Z_i = X_i^\top \beta^*$.



An amuse gueule of theory

Denote $\widehat{\beta}$ as the final estimate of β^*.

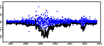
Theorem

Under A 1-5, the estimators $\widehat{\beta}^0$ *and* $\widehat{\beta}$ *exist and* $\mathsf{P}(\widehat{\beta}^0 = \widehat{\beta}) \to 1$*. Moreover,*

$$\mathsf{P}(\widehat{\beta}^0 = \widehat{\beta}) \geq 1 - (p - q)\exp(-C' n^{\alpha}), \tag{11}$$

for $p = \mathcal{O}\{\exp(n^{\alpha_2})\}$*, where* $0 < \alpha < \alpha_2/2 < 1/2$*.*

Go to details



Antipasti Theory

Theorem

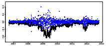
Let $\alpha_2/2 < \alpha_1 < 1$, $D_n \stackrel{\text{def}}{=} \max\{d_l : l \in \mathcal{M}_*\} = o(n^{\alpha_1 - \alpha_2/2})$, $d_l \stackrel{\text{def}}{=} \gamma_\lambda(|\beta_l^*|)$, $\mathcal{M}_* = \{l : \beta_l^* \neq 0\}$ *be the true model*, $q = \mathcal{O}(n^{\alpha_2})$. $\|\sum_i \sum_j X_{ij(1)} \omega_{ij(1)} X_{ij(0)}\|_{2,\infty} = \mathcal{O}(n^{1-\alpha_1})$, $\lambda = \mathcal{O}\big(\sqrt{q/n}\big)$. *Then*

$$\|\widehat{\beta}_{(1)} - \beta^*_{(1)}\| = \mathcal{O}_p\{(\lambda D_n + n^{-1/2})\sqrt{q}\} \tag{12}$$

For any unit vector $\boldsymbol{b}$ *in* $\mathbb{R}^q$, *we have*

$$\boldsymbol{b}^\top C_{0(1)}^{-1} \sqrt{n}(\widehat{\beta}_{(1)} - \beta^*_{(1)}) \stackrel{\mathcal{L}}{\longrightarrow} \mathrm{N}(0,\ \sigma_\tau^2) \tag{13}$$

Go to details



Antipasti Theory

where $\widehat{\beta}_{(1)} \stackrel{\text{def}}{=} (\widehat{\beta}_l)_{l \in \mathcal{M}_*}$.

$C_{0(1)} \stackrel{\text{def}}{=} \mathsf{E}\{[g'(Z_i)]^2[\mathsf{E}(X_{(1)}|Z_i) - X_{i(1)}][\mathsf{E}(X_{(1)}|Z_i) - X_{i(1)}]^\top\}$,

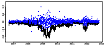
$\psi_\tau(\varepsilon)$ is a selection of the subgradient of $\rho_\tau(\varepsilon)$ and

$\sigma_\tau^2 = \mathsf{E}[\psi_\tau(\varepsilon_i)]^2/[\partial^2 \mathsf{E}\, \rho_\tau(\varepsilon_i)]^2$,

where

$$\partial^2 \mathsf{E}\, \rho_\tau(\cdot) = \left.\frac{\partial^2 \mathsf{E}\, \rho_\tau(\varepsilon_i - v)^2}{\partial v^2}\right|_{v=0} \qquad (14)$$

Go to details



Main Course

Theorem

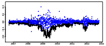
Under A 1-5, $\mathcal{B}_n \overset{\text{def}}{=} \{\widehat{\beta} = \beta^*\} : \mathrm{P}(\mathcal{B}_n) \to 1$*. Let* $\mu_j \overset{\text{def}}{=} \int u^j K(u) du$*,* $\nu_j \overset{\text{def}}{=} \int u^j K^2(u) du$*,* $j = 0, 1, \ldots$*. If* $nh^3 \to \infty$ *and* $h \to 0$*, then*

$$\sqrt{nh}\sqrt{f_Z(z)/(\nu_0 \sigma_\tau^2)} \left\{\widehat{g}(x^\top \widehat{\beta}) - g(x^\top \beta^*) - \frac{1}{2} h^2 g''(x^\top \beta^*) \mu_2 \partial \, \mathrm{E}\, \psi_\tau(\varepsilon)\right\} \xrightarrow{\mathcal{L}} \mathrm{N}(0,\ 1),$$

and

$$\sqrt{nh^3}\sqrt{\{f_Z(z)\mu_2^2\}/(\nu_2 \sigma_\tau^2)} \left\{\widehat{g}'(x^\top \widehat{\beta}) - g'(x^\top \beta^*)\right\} \xrightarrow{\mathcal{L}} \mathrm{N}(0,\ 1).$$

Go to details

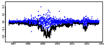


Link functions

- Model 1
$$Y_i = 5\cos(D \cdot Z_i) + \exp(-D \cdot Z_i^2) + \varepsilon_i, \tag{15}$$
$Z_i = X_i^\top \beta^*$, $D = 0.8$ is a scaling constant and ε_i is the error term.

- Model 2
$$Y_i = \sin\{\pi(a \cdot Z_i - b)\} + \varepsilon_i, \tag{16}$$
with the parameters $a = 0.1$, $b = 0.4$.

- Model 3
$$Y_i = 10\sin(D \cdot Z_i) + \sqrt{|\sin(Z_i) + \varepsilon_i|}, \tag{17}$$
with $D = 0.1$.



Criteria

1. Standardized $L2$ norm:

$$Dev \stackrel{\text{def}}{=} \sum_{l=1}^{p} \frac{\|\beta_l^* - \widehat{\beta}_l^*\|_2}{\|\beta^*\|_2}$$

2. Sign consistency:

$$Acc \stackrel{\text{def}}{=} \sum_{l=1}^{p} |sign(\beta_l^*) - sign(\widehat{\beta}_l^*)|$$

Criteria

3. Least angle:

$$\mathit{Angle} \stackrel{\mathrm{def}}{=} \frac{< \beta^*, \widehat{\beta}^* >}{\|\beta^*\|_2 \cdot \|\widehat{\beta}^*\|_2}$$

4. Relative error:

$$\mathit{Error} \stackrel{\mathrm{def}}{=} \frac{1}{n} \sum_{i=1}^{n} \left| \frac{g(Z_i) - \widehat{g}(Z_i)}{g(Z_i)} \right|$$

The estimated vs. true link functions

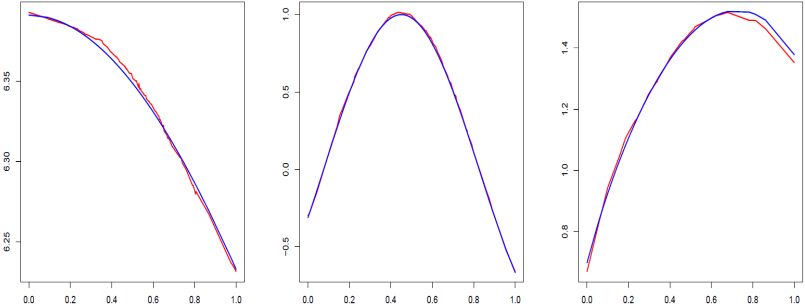
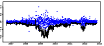


Figure 2: The true link functions (blue) and the estimated link functions (red) with $n = 100, p = 10, q = 2$, $\beta^{*\top}_{(1)} = (1, 2)$, $\tau = 90\%$ and $\varepsilon \sim \mathrm{N}(0, 0.1)$, model 1 (left), model 2 (middle) and model 3 (right).

Criteria - quantile regression (small p case)

$g(\cdot)$	τ	*Dev*	*Acc*	*Angle*	*Error*
Model 1	0.9	2.3 (0.3)	0.66 (0.40)	0.999 (0.7)	0.96 (0.1)
	0.5	3.7 (4.9)	0.04 (0.03)	0.998 (9.2)	0.15 (0.1)
Model 2	0.9	2.8 (5.6)	0.13 (0.81)	0.997 (2.9)	8.16 (1.1)
	0.5	8.2 (6.4)	0.02 (0.13)	0.995 (6.1)	7.51 (4.7)
Model 3	0.9	3.2 (5.9)	0.20 (0.92)	0.997 (4.1)	11.50 (7.9)
	0.5	1.1 (0.8)	0.07 (0.26)	0.986 (0.9)	5.34 (1.6)

Table 1: Criteria evaluated under different models and quantiles. The error ε follows a $\mathrm{N}(0, 0.1)$ distribution. In 100 simulations we set $n = 100$, $p = 10$, $q = 2$, $\beta^{*\top}_{(1)} = (1, 2)$. Standard deviations are given in brackets. *Dev*, *Error* and their standard deviations are reported in 10^{-2}. Standard deviations of *Angle* are reported in 10^{-3}.



Criteria - quantile regression (large p case)

$g(\cdot)$	*Dev*	*Acc*	*Angle*	*Error*
Model 1	6.5 (8.0)	3.5 (0.2)	0.934 (0.8)	1.6 (0.5)
Model 2	5.2 (11.1)	2.8 (0.6)	0.933 (5.2)	2.1 (5.6)
Model 3	4.1 (5.9)	0.6 (0.8)	0.992 (0.1)	2.0 (9.1)

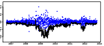
Table 2: Criteria evaluated under different models. The error ε follows a $\mathrm{N}(0, 0.1)$ distribution. In 100 simulations we set $n = 100, p = 100, q = 5, \tau = 0.9$, $\beta^{*\top}_{(1)} = (1, 5, 3, 1, 3)$. Standard deviations are given in brackets. *Dev*, *Error* and their standard deviations are reported in 10^{-2}. Standard deviations of *Angle* are reported in 10^{-2}.

Value at Risk

- Value-at-Risk (VaR) is the most known measure for quantifying and controlling the risk of a portfolio.
- The VaR of a financial institution i at $\tau \in (0,1)$:

$$\mathrm{P}(X_{i,t} \leq VaR^{\tau}_{i,t}) \stackrel{def}{=} \tau,$$

where $X_{i,t}$ represents the asset return of financial institution i at time t.

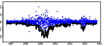


CoVaR

- Adrian and Brunnermeier (AB) (2011) proposed CoVaR.
- The CoVaR of a risk factor j given X_i at level $\tau \in (0,1)$:

$$\mathrm{P}\{X_{j,t} \le CoVaR^{\tau}_{j|i,t} | X_{i,t} = VaR^{\tau}(X_{i,t}), M_{t-1}\} \stackrel{def}{=} \tau,$$

here M_{t-1} is a vector of macroprudential variables.



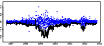
Quantile regression

- CoVaR technique (AB)
- Two linear quantile regressions

$$
\begin{aligned}
X_{i,t} &= \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t}, &(18)\\
X_{j,t} &= \alpha_{j|i} + \beta_{j|i} X_{i,t} + \gamma_{j|i}^\top M_{t-1} + \varepsilon_{j,t}. &(19)
\end{aligned}
$$

- $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau|M_{t-1}, X_{i,t}) = 0$, then

$$
\begin{aligned}
\widehat{VaR}^{\tau}_{i,t} &= \widehat{\alpha}_i + \widehat{\gamma}_i^\top M_{t-1}, &(20)\\
\widehat{CoVaR}^{\tau}_{j|i,t} &= \widehat{\alpha}_{j|i} + \widehat{\beta}_{j|i} \widehat{VaR}^{\tau}_{i,t} + \widehat{\gamma}_{j|i}^\top M_{t-1}. &(21)
\end{aligned}
$$



Quantile regression and SIM

- Generalize (19):

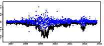
$$X_{j,t} = g(S^\top \beta_{j|S}) + \varepsilon_{j,t}, \tag{22}$$

where $S \stackrel{def}{=} [M_{t-1}, R]$, R is a vector of log returns. $\beta_{j|S}$ is a $p \times 1$ vector, p large.

- $F^{-1}_{\varepsilon_{j,t}}(\tau|S) = 0$, then:

$$\widehat{CoVaR}^{\tau}_{j|\widehat{S}} = \widehat{g}(\widehat{S}^\top \widehat{\beta}_{j|S}), \tag{23}$$

where $\widehat{S} \stackrel{def}{=} [M_{t-1}, \widehat{V}]$, where $\widehat{V}$ is the estimated VaR in (20).



Dataset

- City national corp (CYN) (as an example).
- Choose 199 financial firms and 7 macroprudential variables.
- Time period is from January 6, 2006 to September 6, 2012, $T = 1669$.

Descriptive statistics of CYN

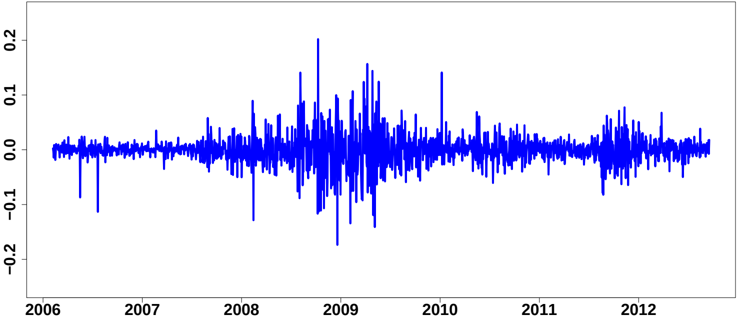
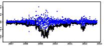


Figure 3: Log returns of CYN

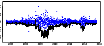


Descriptive statistics of CYN

	Mean	SD	Skewness	Kurtosis
Before crisis	-4.0×10^{-4}	0.0209	0.2408	12.1977
In crisis	-9.2×10^{-5}	0.0312	0.1326	8.9544

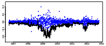
Table 3: Descriptive statistics

- Jarque Bera Test is performed: log returns of CYN are not normally distributed.
- Unit root test is conducted: log returns of CYN are stationary.



Estimation of VaR

- Window size: $n = 100$.
- 7 Macroprudential variables are applied.
- Method: quantile regression.
- $\tau = 0.05$.
- $T = 1569$ estimated VaR by moving window estimation.



Estimation of VaR

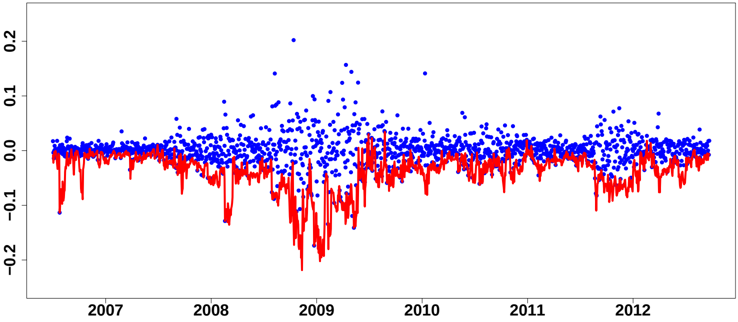
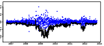


Figure 4: Log returns of CYN (blue) and VaR of log returns of CYN (red), $\tau = 0.05$, $T = 1569$, window size $n = 100$, refer to (20).



Estimation of VaR

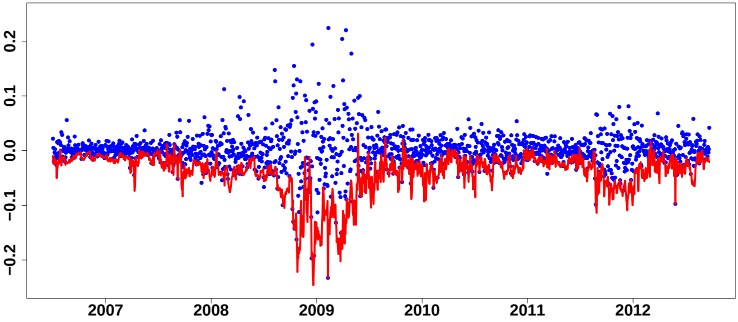
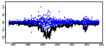


Figure 5: Log returns of JPM (blue) and VaR of log returns of JPM (red), $\tau = 0.05$, $T = 1569$, window size $n = 100$, refer to (20).

Estimation of CoVaR

- Window size: $n = 126$.
- Original variables: $p = 206$.
- Method: $L1$-norm quantile regression.
- $\tau = 0.05$.
- Bandwidth: $h_\tau = h_{mean}\left[\tau(1-\tau)\varphi\{\Phi^{-1}(\tau)\}^{-2}\right]^{0.2}$.
- Where h_{mean}: use direct plug-in methodology of a local linear regression described by Ruppert, Sheather and Wand (1995).
- Selected variables: Different $\widehat{q}$ in each window.
- $T = 1543$ estimated CoVaR by moving window estimation.



Estimation of CoVaR

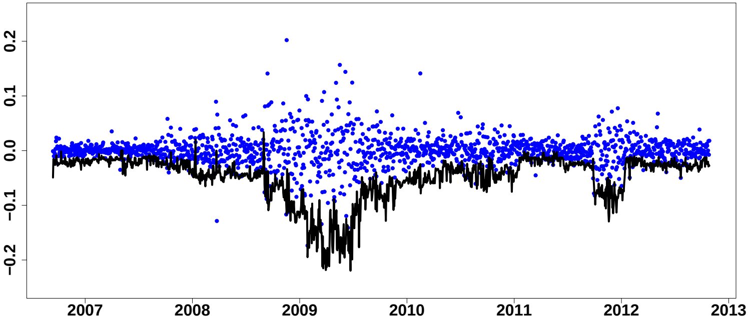


Figure 6: Log returns of CYN (blue) and the estimated CoVaR (black), $\tau = 0.05$, $T = 1543$, window size $n = 126$, refer to (23).

The link function

Figure 7: The link functions

The link function

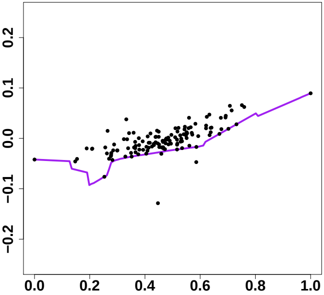
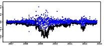


Figure 8: The estimated link function, window size $n = 126$, starting date: 20080707, $\tau = 0.05$, $h = 0.027$, $p = 206$, $\widehat{q} = 3$: FHN, MBI, RDN.



The link function

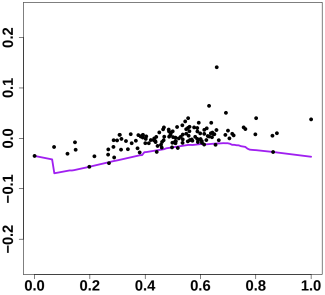
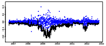


Figure 9: The estimated link function, window size $n = 126$, starting date: 20100308, $\tau = 0.05$, $h = 0.056$, $p = 206$, $\widehat{q} = 5$: ZION, EWBC, CNO, SNV, RDN.



The influential variables

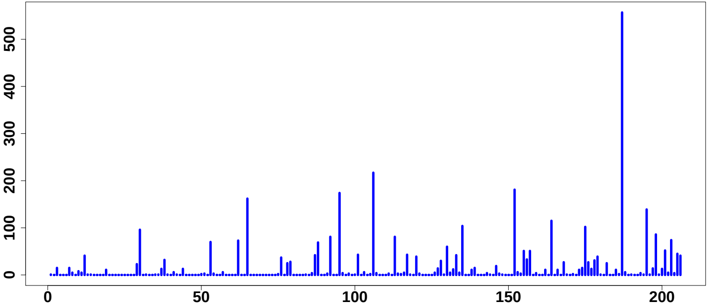
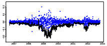


Figure 10: The frequency of the firms and macroprudential variables. The X-axis: 1 – 206 variables, and the Y-axis: the frequency of the variables selected in the moving window estimation. The variable 187, i.e. "Radian Group Inc. (RDN)" is the most frequently selected variable with frequency 557.

Go to details



Backtesting

- The violation sequence:

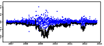
$$I_t = \begin{cases} 1, & X_{i,t} < \widehat{VaR}^{\tau}_{i,t}; \\ 0, & otherwise. \end{cases}$$

  If VaR algorithm is correct, I_t should be a martingale difference sequence.

- The CaViaR test model:

$$I_t = \alpha + \beta_1 I_{t-1} + \beta_2 VaR_t + u_t.$$

- The test procedure: estimate $\widehat{\beta}_1$ and $\widehat{\beta}_2$ by logistic regression. Then Wald's test is applied. Null hypothesis: $\widehat{\beta}_1 = \widehat{\beta}_2 = 0$, i.e. I_t is a martingale difference sequence.



Backtesting VaR

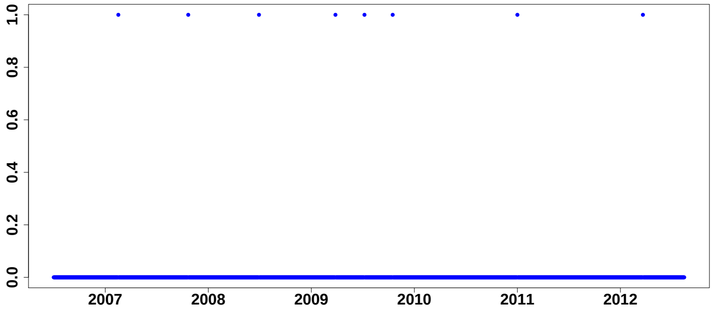
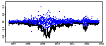


Figure 11: The top dots are the violations (i.e. $\{t : I_t = 1\}$) of $\widehat{VaR}$ of CYN, totally 8 violations, $T = 1543$.



Backtesting CoVaR

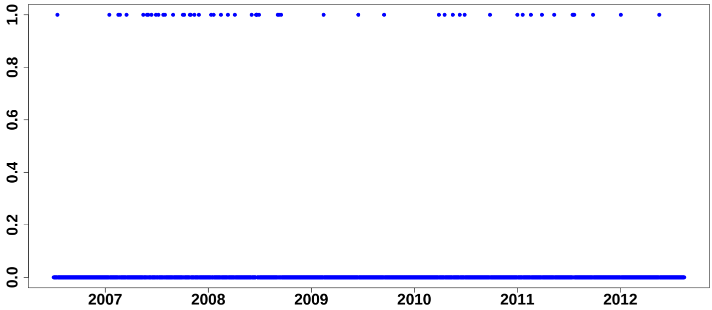


Figure 12: The top dots are the violations (i.e. $\{t : I_t = 1\}$) of $\widehat{CoVaR}$ of CYN, totally 53 violations, $T = 1543$.

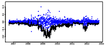
Backtesting results

- Test result

	p-value of Wald test statistics
$\widehat{VaR}$	0.87
$\widehat{CoVaR}$	0.36

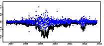
Table 4: The CaViaR test for $\widehat{VaR}$ of CYN and $\widehat{CoVaR}$ of CYN, $T = 1543$.

- Null hypothesis can not be rejected, therefore both VaR and CoVaR algorithms perform well.



Further work

- CoVaR estimation in Expectile situation.
- CoVaR estimation in Composite Quantile regression situation.
- Backtesting VaR and CoVaR.



Quantile Regression with high dimensional Single-Index Models

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Weining Wang
Lixing Zhu, Lining Yu

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http://lvb.wiwi.hu-berlin.de
http://www.case.hu-berlin.de

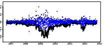
The penalty term

- Lasso, Tibshirani (1996): $\gamma_\lambda(x) = \lambda$
- SCAD, Fan and Li (2001):

$$\gamma_\lambda(x) = \lambda\{\mathbf{I}(x \leq \lambda) + \frac{(a\lambda - x)_+}{(a-1)\lambda}\mathbf{I}(x > \lambda)\},$$

- The adaptive Lasso, Zou (2006): $\gamma_\lambda(x) = \lambda|x|^{-a}$ for some $a > 0$.

Return



Assumptions

A1 K a cts symmetric pdf, $g(\cdot) \in C^2$.

A2 $\rho_\tau(x)$ convex. Suppose $\psi_\tau(x)$, subgradient of $\rho_\tau(x)$: i) Lipschitz continuous; ii) $\mathsf{E}\,\psi_\tau(\varepsilon_i) = 0$ and $\inf_{|v|\leq c} \partial\, \mathsf{E}\,\psi_\tau(\varepsilon_i - v) = C_1$.

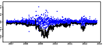
Assumptions

A3 ε_i is independent of X_i. Let $Z_i = X_i^\top \beta^*$ and $Z_{ij} = Z_i - Z_j$. Define $\mathrm{E}\{g'(Z_i)^2(\mathrm{E}(X_{i(1)}|Z_i) - X_{i(1)})(\mathrm{E}(X_{i(1)}|Z_i - X_{i(1)})\}^\top \stackrel{\mathrm{def}}{=} C_{0(1)}$, and the matrix C_0 satisfies $L_1 \le \lambda_{\min}(C_0) \le \lambda_{\max}(C_0) \le L_2$ for positive constants L_1 and L_2. There exists a constant $c_0 > 0$ such that $\sum_{i=1}^n \{\|X_{i(1)}\|/\sqrt{n}\}^{2+c_0} \to 0$.

A4 Assume $\sqrt{n}\gamma_\lambda(|\tilde{\beta}_l|) \to 0$ for $\beta_l^* \ne 0$ and $\sqrt{n}\gamma_\lambda(|\tilde{\beta}_l|) \to \infty$ for $\beta_l^* = 0$. Furthermore assume $qh \to 0$ as n goes to infinity.

A5 The error term ε_i satisfies $\mathrm{E}\,\varepsilon_i = 0$ and $\mathrm{var}(\varepsilon_i) < \infty$. Assume that $\mathrm{E}\left|\psi^m(\varepsilon_i)/m!\right| \le s_0 c^m$ where s_0 and c are constants.

Return



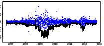
Subgradient

If $f : U \to \mathbb{R}$ is a real-valued convex function defined on a convex open set in the Euclidean space $\mathbb{R}^n$, a vector v in that space is called a subgradient at a point x_0 in U if for any x in U one has

$$f(x) - f(x_0) \geq v \cdot (x - x_0)$$

where the dot denotes the dot product.

Return

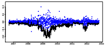


Matrix norm

Assume A is a $m \times n$ matrix

$$\|A\|_{\alpha,\beta} = \max_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}$$

Return

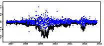


Sparsistency

The result of (11) is stronger than the oracle property defined in Fan and Li (2001) once the properties of $\widehat{\beta}^0$ are established. It was formulated by Kim et al. (2008) for the SCAD estimator with polynomial dimensionality p. It implies not only the model selection consistency and but also sign consistency (Zhao and Yu, 2006; Bickel et al., 2008, 2009):

$$\mathrm{P}\{sgn(\widehat{\beta}) = sgn(\beta^*)\} = \mathrm{P}\{sgn(\widehat{\beta}^0) = sgn(\beta^*)\} \to 1$$

▸ Return



The confidence interval

The $100(1-\alpha)\%$ confidence interval:

$$\left[\widehat{g}(z) - \frac{1}{\sqrt{nh}} \cdot \frac{\sigma_\tau \sqrt{\nu_0}}{\sqrt{\widehat{f}_Z(z)}} \cdot \mathfrak{z}_\alpha + \frac{1}{2} h^2 \widehat{g}''(z) \mu_2 \partial \widehat{\mathsf{E}} \psi_\tau(\varepsilon);\right.$$

$$\left.\widehat{g}(z) + \frac{1}{\sqrt{nh}} \cdot \frac{\sigma_\tau \sqrt{\nu_0}}{\sqrt{\widehat{f}_Z(z)}} \cdot \mathfrak{z}_\alpha + \frac{1}{2} h^2 \widehat{g}''(z) \mu_2 \partial \widehat{\mathsf{E}} \psi_\tau(\varepsilon)\right]$$

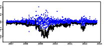
where $\mathfrak{z}_\alpha$ is the α-Quantile of the standard normal distribution, and

$\widehat{f}_Z(z) = n^{-1} \sum_{i=1}^{n} K_h(z - Z)$, where $Z = X_{i(1)}^\top \widehat{\beta}_{(1)}$.

Return

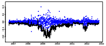
The financial firms and macroprudential variables

The financial frims:	
1. Wells Fargo & Co (WFC)	15. Franklin Resources Inc. (BEN)
2. JP Morgan Chase & Co (JPM)	16. The Travelers Companies, Inc. (TRV)
3. Bank of America Corp (BAC)	17. AFLAC Inc. (AFL)
4. Citigroup Inc (C)	18. Prudential Financial, Inc. (PRU)
5. American Express Company (AXP)	19. State Street Corporation (STT)
6. U.S. Bancorp (USB)	20. The Chubb Corporation (CB)
7. The Goldman Sachs Group, Inc. (GS)	21. BB&T Corporation (BBT)
8. American International Group, Inc. (AIG)	22. Marsh & McLennan Companies, Inc. (MMC)
9. MetLife, Inc. (MET)	23. The Allstate Corporation (ALL)
10. Capital One Financial Corp. (COF)	24. Aon plc (AON)
11. BlackRock, Inc. (BLK)	25. CME Group Inc. (CME)
12. Morgan Stanley (MS)	26. The Charles Schwab Corporation (SCHW)
13. PNC Financial Services Group Inc. (PNC)	27. T. Rowe Price Group, Inc. (TROW)
14. The Bank of New York Mellon Corporation (BK)	28. Loews Corporation (L)



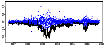
The financial firms and macroprudential variables

29. SunTrust Banks, Inc. (STI)	44. Lincoln National Corporation (LNC)
30. Fifth Third Bancorp (FITB)	45. Affiliated Managers Group Inc. (AMG)
31. Progressive Corp. (PGR)	46. Cincinnati Financial Corp. (CINF)
32. M&T Bank Corporation (MTB)	47. Equifax Inc. (EFX)
33. Ameriprise Financial Inc. (AMP)	48. Alleghany Corp. (Y)
34. Northern Trust Corporation (NTRS)	49. Unum Group (UNM)
35. Invesco Ltd. (IVZ)	50. Comerica Incorporated (CMA)
36. Moody's Corp. (MCO)	51. W.R. Berkley Corporation (WRB)
37. Regions Financial Corp. (RF)	52. Fidelity National Financial, Inc. (FNF)
38. The Hartford Financial Services Group, Inc. (HIG)	53. Huntington Bancshares Incorporated (HBAN)
39. TD Ameritrade Holding Corporation (AMTD)	54. Raymond James Financial Inc. (RJF)
40. Principal Financial Group Inc. (PFG)	55. Torchmark Corp. (TMK)
41. SLM Corporation (SLM)	56. Markel Corp. (MKL)
42. KeyCorp (KEY)	57. Ocwen Financial Corp. (OCN)
43. CNA Financial Corporation (CNA)	58. Arthur J Gallagher & Co. (AJG)



The financial firms and macroprudential variables

59. Hudson City Bancorp, Inc. (HCBK)	74. Commerce Bancshares, Inc. (CBSH)
60. People's United Financial Inc. (PBCT)	75. Signature Bank (SBNY)
61. SEI Investments Co. (SEIC)	76. Jefferies Group, Inc. (JEF)
62. Nasdaq OMX Group Inc. (NDAQ)	77. Rollins Inc. (ROL)
63. Brown & Brown Inc. (BRO)	78. Morningstar Inc. (MORN)
64. BOK Financial Corporation (BOKF)	79. East West Bancorp, Inc. (EWBC)
65. Zions Bancorp. (ZION)	80. Waddell & Reed Financial Inc. (WDR)
66. HCC Insurance Holdings Inc. (HCC)	81. Old Republic International Corporation (ORI)
67. Eaton Vance Corp. (EV)	82. ProAssurance Corporation (PRA)
68. Erie Indemnity Company (ERIE)	83. Assurant Inc. (AIZ)
69. American Financial Group Inc. (AFG)	84. Hancock Holding Company (HBHC)
70. Dun & Bradstreet Corp. (DNB)	85. First Niagara Financial Group Inc. (FNFG)
71. White Mountains Insurance Group, Ltd. (WTM)	86. SVB Financial Group (SIVB)
72. Cullen-Frost Bankers, Inc. (CFR)	87. First Horizon National Corporation (FHN)
73. Legg Mason Inc. (LM)	88. E-TRADE Financial Corporation (ETFC)

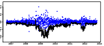


The financial firms and macroprudential variables

89. SunTrust Banks, Inc. (STI)	104. Valley National Bancorp (VLY)
90. Mercury General Corporation (MCY)	105. KKR Financial Holdings LLC (KFN)
91. Associated Banc-Corp (ASBC)	106. Synovus Financial Corporation (SNV)
92. Credit Acceptance Corp. (CACC)	107. Texas Capital BancShares Inc. (TCBI)
93. Protective Life Corporation (PL)	108. American National Insurance Co. (ANAT)
94. Federated Investors, Inc. (FII)	109. Washington Federal Inc. (WAFD)
95. CNO Financial Group, Inc. (CNO)	110. First Citizens Bancshares Inc. (FCNCA)
96. Popular, Inc. (BPOP)	111. Kemper Corporation (KMPR)
97. Bank of Hawaii Corporation (BOH)	112. UMB Financial Corporation (UMBF)
98. Fulton Financial Corporation (FULT)	113. Stifel Financial Corp. (SF)
99. AllianceBernstein Holding L.P. (AB)	114. CapitalSource Inc. (CSE)
100. TCF Financial Corporation (TCB)	115. Portfolio Recovery Associates Inc. (PRAA)
101. Susquehanna Bancshares, Inc. (SUSQ)	116. Janus Capital Group, Inc. (JNS)
102. Capitol Federal Financial, Inc. (CFFN)	117. MBIA Inc. (MBI)
103. Webster Financial Corp. (WBS)	118. Healthcare Services Group Inc. (HCSG)

The financial firms and macroprudential variables

119. The Hanover Insurance Group Inc. (THG)	134. BancorpSouth, Inc. (BXS)
120. F.N.B. Corporation (FNB)	135. Privatebancorp Inc. (PVTB)
121. FirstMerit Corporation (FMER)	136. United Bankshares Inc. (UBSI)
122. FirstMerit Corporation (FMER)	137. Old National Bancorp. (ONB)
123. RLI Corp. (RLI)	138. International Bancshares Corporation (IBOC)
124. StanCorp Financial Group Inc. (SFG)	139. First Financial Bankshares Inc. (FFIN)
125. Trustmark Corporation (TRMK)	140. Westamerica Bancorp. (WABC)
126. IberiaBank Corp. (IBKC)	141. Northwest Bancshares, Inc. (NWBI)
127. Cathay General Bancorp (CATY)	142. Bank of the Ozarks, Inc. (OZRK)
128. National Penn Bancshares Inc. (NPBC)	143. Huntington Bancshares Incorporated (HBAN)
129. Nelnet, Inc. (NNI)	144. Euronet Worldwide Inc. (EEFT)
130. Wintrust Financial Corporation (WTFC)	145. Community Bank System Inc. (CBU)
131. Umpqua Holdings Corporation (UMPQ)	146. CVB Financial Corp. (CVBF)
132. GAMCO Investors, Inc. (GBL)	147. MB Financial Inc. (MBFI)
133. Sterling Financial Corp. (STSA)	148. ABM Industries Incorporated (ABM)

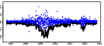


The financial firms and macroprudential variables

149. Glacier Bancorp Inc. (GBCI)	164. Citizens Republic Bancorp, Inc (CRBC)
150. Selective Insurance Group Inc. (SIGI)	165. Horace Mann Educators Corp. (HMN)
151. Park National Corp. (PRK)	166. DFC Global Corp. (DLLR)
152. Flagstar Bancorp Inc. (FBC)	167. Navigators Group Inc. (NAVG)
153. FBL Financial Group Inc. (FFG)	168. Boston Private Financial Holdings, Inc. (BPFH)
154. Astoria Financial Corporation (AF)	169. American Equity Investment Life Holding Co. (AEL)
155. World Acceptance Corp. (WRLD)	170. BlackRock Limited Duration Income Trust (BLW)
156. First Midwest Bancorp Inc. (FMBI)	171. Columbia Banking System Inc. (COLB)
157. PacWest Bancorp (PACW))	172. Safety Insurance Group Inc. (SAFT)
158. First Financial Bancorp. (FFBC)	173. National Financial Partners Corp. (NFP)
159. BBCN Bancorp, Inc. (BBCN)	174. NBT Bancorp, Inc. (NBTB)
160. Provident Financial Services, Inc. (PFS)	175. Tower Group Inc. (TWGP)
161. FBL Financial Group Inc. (FFG)	176. Encore Capital Group, Inc. (ECPG)
162. WisdomTree Investments, Inc. (WETF)	177. Pinnacle Financial Partners Inc. (PNFP)
163. Hilltop Holdings Inc. (HTH)	178. First Commonwealth Financial Corp. (FCF)

The financial firms and macroprudential variables

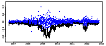
179. BancFirst Corporation (BANF)	190. Berkshire Hills Bancorp Inc. (BHLB)
180. Independent Bank Corp. (INDB)	191. Brookline Bancorp, Inc. (BRKL)
181. Infinity Property and Casualty Corp. (IPCC)	192. National Western Life Insurance Company (NWLI)
182. Central Pacific Financial Corp. (CPF)	193. Tompkins Financial Corporation (TMP)
183. Kearny Financial Corp. (KRNY)	194. BGC Partners, Inc. (BGCP)
184. Chemical Financial Corporation (CHFC)	195. Epoch Investment Partners, Inc. (EPHC)
185. Banner Corporation (BANR)	196. United Fire Group, Inc (UFCS)
186. State Auto Financial Corp. (STFC)	197. 1st Source Corporation (SRCE)
187. Radian Group Inc. (RDN)	198. Citizens Inc. (CIA)
188. SCBT Financial Corporation (SCBT)	199. S&T Bancorp Inc. (STBA)
189. WesBanco Inc. (WSBC)	



The financial firms and macroprudential variables

The macroprudential variables:
200. VIX
201. Short term liquidity spread (liquidity)
202. Daily change in the 3-month Treasury maturities (3MT)
203. Change in the slope of the yield curve (yield)
204. Change in the credit spread (credit)
205. Daily Dow Jones U.S. Real Estate index returns (D_J)
206. S&P500 returns (S&P)

Return



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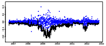

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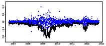

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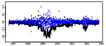

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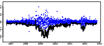

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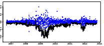

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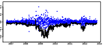

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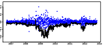

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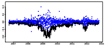

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