Quantile Regression in Risk Calibration

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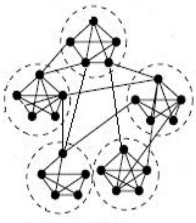
Dependence Risk



Risk Calibration and Quantile Regression

- Quantification via value-at-risk (VaR)/expected shortfall (ES)
- Quantile VaR: dependence risk?
- Parametric VaR: Chernozhukov and Umantsev (2001), Engle and Manganelli (2004)
- Nonparametric VaR: Cai and Wang (2008), Taylor (2008) and Schaumburg (2010)
- Parametric CoVaR: Adrian and Brunnermeier (2010)(AB)

Risk Calibration

- Marginal Expected Shortfall (MES): Acharya et al. (2010)
- Distressed Insurance Premium (DIP): Huang et al. (2010)
  Go to details
- AB: X_j and X_i are two asset returns,

$$\mathrm{P}\left\{ X_j \leq \mathrm{CoVaR}^{\tau}_{j|i} \,\middle|\, X_i = \mathrm{VaR}^{\tau}(X_i), M_{t-1} \right\} = \tau.$$

- Advantages:
  - Cloning property
  - Conservative property
  - Adaptiveness

Go to details

CoVaR Construction (AB)

$X_{j,t}$ and $X_{i,t}$ are two asset returns. Two linear quantile regressions:

$$X_{i,t} = \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t}, \tag{1}$$

$$X_{j,t} = \alpha_{j|i} + \beta_{j|i} X_{i,t} + \gamma_{j|i}^\top M_{t-1} + \varepsilon_{j,t}. \tag{2}$$

M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau | M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau | M_{t-1}, X_{i,t}) = 0$.

$$\widehat{VaR}_{i,t} = \hat{\alpha}_i + \hat{\gamma}_i^\top M_{t-1},$$

$$\widehat{CoVaR}_{j|i,t} = \hat{\alpha}_{j|i} + \hat{\beta}_{j|i} \widehat{VaR}_{i,t} + \hat{\gamma}_{j|i}^\top M_{t-1}.$$

CoVaR Construction Linear?

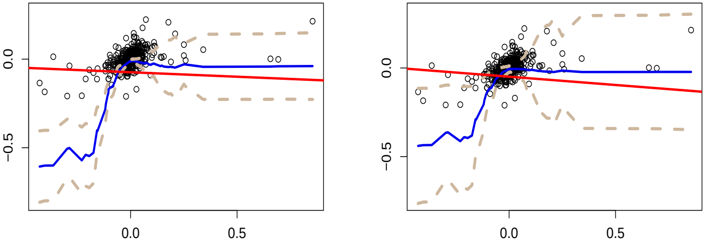


Figure 1: Goldman Sachs (GS) and Citigroup (C) weekly returns 0.05 (left) and 0.1(right) quantile functions. y-axis=GS returns; x-axis=C returns. LLQR lines. Linear quantile regression line. 95% Confidence band. $N = 546$. Data weekly returns 20050131-20100131.

Nonlinear Dependence

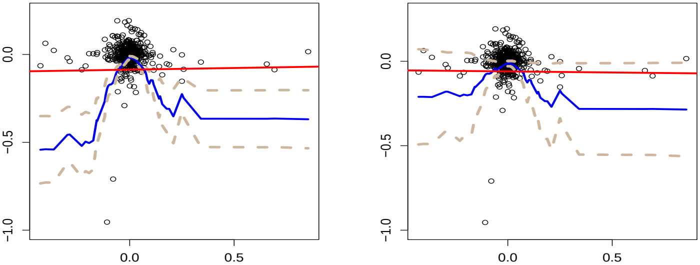


Figure 2: Lehman Brothers (LB) and C weekly returns 0.05 (left) and 0.1(right) quantile functions. y-axis=LB returns; x-axis=C returns. LLQR lines. Linear quantile regression line. 95% Confidence band. $N = 546$. Data weekly returns 20050131-20100131.

Nonlinear Dependence

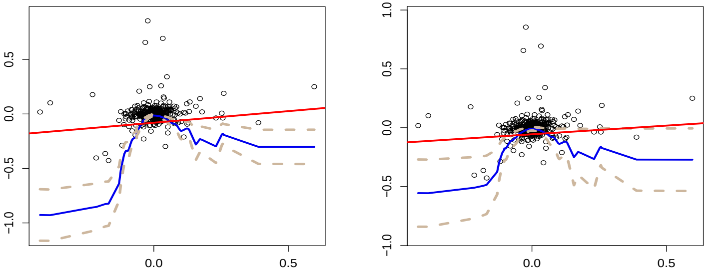


Figure 3: Bank of America (BOA) and C weekly returns 0.05 (left) and 0.1(right) quantile functions. y-axis=BOA returns; x-axis=C returns. LLQR lines. Linear quantile regression line. 95% Confidence band.. $N = 546$. Data weekly returns 20050131-20100131.

General Specification

- Nonparametric quantile regression:

$$X_{i,t} = f(M_{t-1}) + \varepsilon_{i,t}; \quad (3)$$
$$X_{j,t} = g(X_{i,t}, M_{t-1}) + \varepsilon_{j,t}. \quad (4)$$

M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau | M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau | M_{t-1}, X_{i,t}) = 0$.

- Challenges
  - The curse of dimensionality for f, g
  - Numerical Calibration of (3) and (4)

Research Questions

- Measure CoVaR in a nonparametric (semiparametric) way
- Test the performance of the CoVaR
- What can one learn from the semiparametric specification?
- Consequences for econometrical modelling?

Outline

1. Motivation ✓
2. Locally Linear Quantile Regression
3. A Semiparametric Model
4. Empirical CoVaR
5. Backtesting
6. Conclusions and Outlook

Locally Linear Quantile Estimation (LLQR)

- $\{(X_i, Y_i)\}_{i=1}^n \subset \mathbb{R}^2$ i.i.d. bivariate random variables, locally linear kernel quantile estimator estimated as $\hat{l}(x_0) = \hat{a}_{0,0}$:

$$\underset{\{a_{0,0},a_{0,1}\}}{\operatorname{argmin}} \sum_{i=1}^{N} K\left(\frac{x_i - x_0}{h}\right) \rho_\tau \{y_i - a_{0,0} - a_{0,1}(x_i - x_0)\}. \quad (5)$$

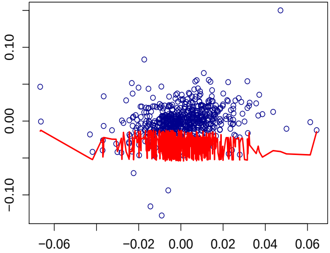
- Choice of Bandwidth: Yu and Jones (1998):

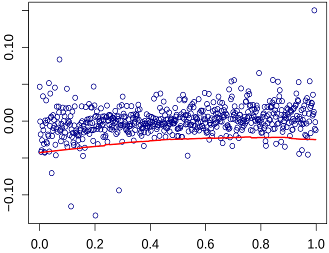
$$h_\tau = h_{mean}\left[\tau(1-\tau)\varphi\{\Phi^{-1}(\tau)\}^{-2}\right]^{1/5},$$

where h_{mean}: local mean regression bandwidth.

Stabilized Estimator

- Calculate $X_{(i:n)}$ (order statistics), then perform LLQR on $\{i/n\}_{i=1}^{n}$ and corresponding $Y_{(i:n)}$
- $\hat{l}(x)\hat{f}_X^{-1}(x)$ is a consistent estimator for the conditional quantile in the original X space





Uniform Confidence Band

Theorem (Härdle and Song (2010))

Under regularity conditions,

$$\mathrm{P}\left[(2\delta \log n)^{1/2}\left\{\sup_{x\in J} r(x)|\hat{l}(x) - l(x)|/\lambda(K)^{1/2} - d_n\right\} < z\right] \rightarrow \exp\{-2\exp(-z)\},$$

as $n \rightarrow \infty$, where $\hat{l}(\cdot)$ is the solution of (5) and d_n is a scaling constant.

Macroeconomic Drivers

Components of M_t:

1. VIX
2. Short term liquidity spread
3. Change in the 3M T-bill rate
4. Change in the slope of the yield curve
5. Change in the credit spread between 10 years BAA-rated bonds and the T-bond rate
6. S&P500 returns
7. Dow Jones U.S. Real Estate index returns

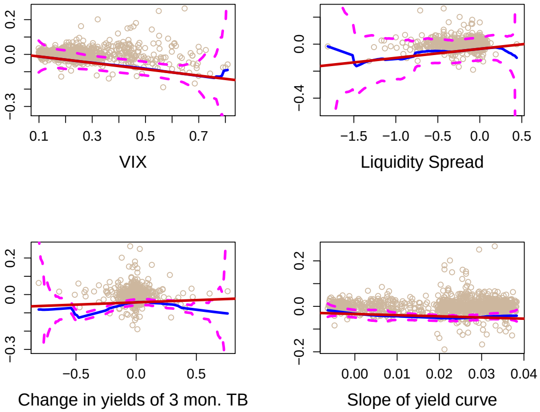
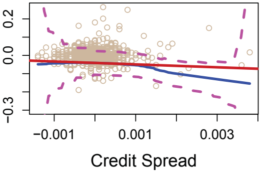
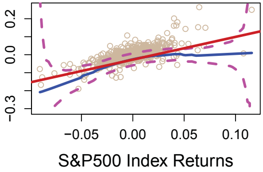


Figure 4: GS daily returns given 7 market variables and LLQR curves. Data 20060804-20110804. $n = 1260$. $\tau = 0.05$.





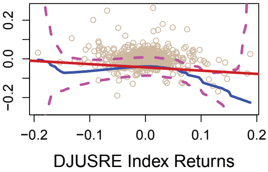


Figure 5: GS daily returns given 7 market variables and LLQR curves. Data 20060804-20110804. $n = 1260$. $\tau = 0.05$.

Partial Linear Model (PLM)

- The linearity observation (Figure 4, 5) implies:

$$
\begin{aligned}
X_{i,t} &= \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t};\\
X_{j,t} &= \tilde{\alpha}_{j|i} + \tilde{\beta}_{j|i}^\top M_{t-1} + l_{j|i}(X_{i,t}) + \varepsilon_{j,t}.
\end{aligned} \tag{6}
$$

  l: a general function. M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau|M_{t-1}, X_{i,t}) = 0$.

- Advantages
  - Capturing nonlinear asset dependence
  - Avoid curse of dimensionality

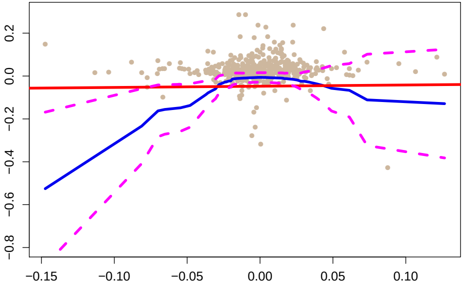


Figure 6: The nonparametric element of the PLM. y-axis=GS daily returns after filtering M_t's effect. x-axis=C daily returns. The LLQR quantile curve. Linear parametric quantile line. 95% Confidence band. Data 20080625-20081223. n=126 (window size). h =0.2003. $\tau = 0.05$.

Estimation of Partial Linear Model

- PLM model: Liang, Härdle and Carroll (1999) and Härdle, Ritov and Song (2011)

$$Y_t = \alpha + \beta^\top M_{t-1} + l(X_t) + \varepsilon_t.$$

- Consider $[0,1]$ (standard rank space). Dividing $[0,1]$ into a_n equally divided subintervals, $a_n \uparrow \infty$. On each subinterval, $l(\cdot)$ is roughly constant.

Estimation of PLM QR

Procedure:

1. Linear element β:

$$\hat{\beta} = \operatorname*{argmin}_{\beta} \min_{l_1,\ldots,l_{a_n}} \sum_{t=1}^{n} \rho_\tau \left\{ Y_t - \alpha - \beta^\top M_{t-1} - \sum_{m=1}^{a_n} l_m \mathbf{1}(X_t \in I_{nt}) \right\};$$

2. Nonlinear element $l(\cdot)$: With data $\{(X_t, Y_t - \hat{\alpha} - \hat{\beta}^\top M_{t-1})\}_{t=1}^{n}$, applying LLQR.

Empirical CoVaR

- j: GS daily returns,
  i: C daily returns
  Window Size: 126 days (half a year)
  Data 20060804-20110804
- Three types of VaR (CoVaR):
  - VaR
  - CoVaR^{AB}
  - CoVaR^{PLM}

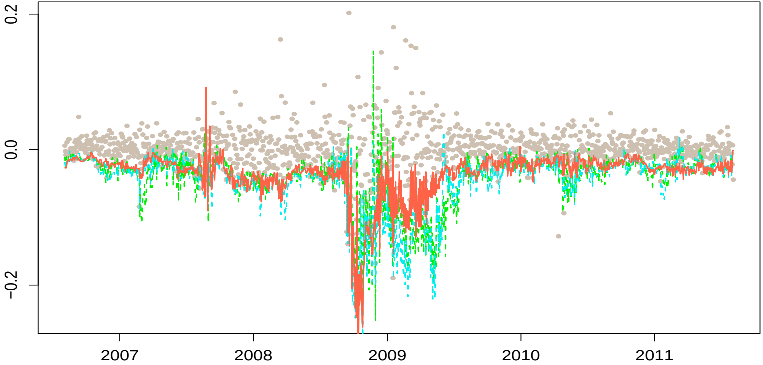


Figure 7: CoVaR of GS given the VaR of C. The x-axis is time. The y-axis is the GS daily returns. PLM CoVaR . AB (2010) CoVaR . The linear QR VaR of GS.

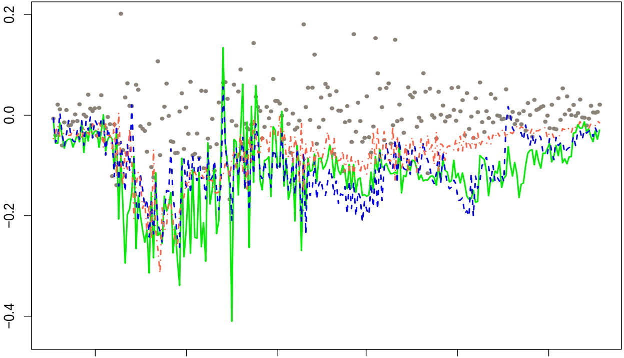


Figure 8: CoVaR of GS given the VaR of C during 20080804-20090804. The x-axis is time. The y-axis is the GS daily returns. PLM CoVaR . AB (2010) CoVaR . The VaR of GS.

Backtesting Procedure

- Berkowitz, Christoffersen and Pelletier (2011): If the VaR calibration is correct, violations

$$I_t = \begin{cases} 1, & \text{if } X_i < \widehat{(Co)VaR}^{\tau}_{t-1}(X_i) \\ 0, & \text{otherwise.} \end{cases}$$

should form a sequence of martingale difference

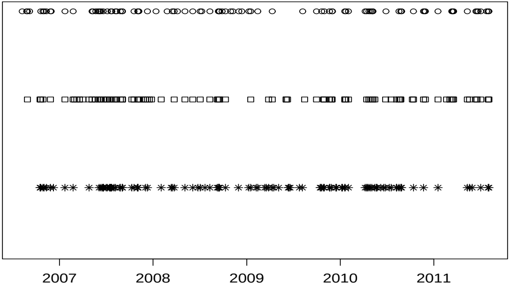


Figure 9: The timings of violations $\{t : I_t = 1\}$. The circles are the violations of the $\widehat{CoVaR}^{PLM}_{GS|C,t}$, totally 95 violations. The squares are the violations of $\widehat{CoVaR}^{AB}_{GS|C,t}$, totally 98 violations. The stars are the violations of $\widehat{VaR}_{GS,t}$, totally 109 violations. $n = 1260$.

Box Tests

- $\hat{\rho}_k$ be the estimated autocorrelation of lag k of violation $\{I_t\}$ and N be the length of the time series.
- Ljung-Box test:

$$\text{LB}(m) = N(N+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{N-k} \tag{7}$$

- Lobato test:

$$\text{L}(m) = N\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{\hat{v}_{kk}} \tag{8}$$

CaViaR Test

- Inspired by Engle and Manganelli (2004)
- Berkowitz, Christoffersen and Pelletier (2011): CaViaR performs best overall
- Test procedure:

$$I_t = \alpha + \beta_1 I_{t-1} + \beta_2 VaR_t + u_t \, ,$$

where VaR_t can be replaced by $CoVaR_t$ in the case of conditional VaR. The residual u_t follows a Logistic distribution.

- The null hypothesis is $\hat{\beta}_1 = \hat{\beta}_2 = 0$.

Summary of Backtesting Procedure

- LB(1): Ljung-Box test of lag 1
- LB(5): Ljung-Box test of lags 5
- L(1): Lobato test of lag 1
- L(5): Lobato test of lags 5
- CaViaR-O: CaViaR test, all data 20060804-20110804
- CaViaR-C: CaViaR test, data 20080804-20090804

Table 1: Goldman Sachs VaR/CoVaR backtesting *p*-values.

Measure	LB(1)	LB(5)	L(1)	L(5)	CaViaR-O	CaViaR-C
			Panel 1			
$\widehat{VaR}_{GS,t}$	0.3449	0.0253*	0.3931	0.1310	<0.0001***	0.0024**
			Panel 2			
$\widehat{CoVaR}^{AB}_{GS\|SP,t}$	0.0869	0.2059	0.2684	0.6586	<0.0001***	0.0424*
$\widehat{CoVaR}^{PLM}_{GS\|SP,t}$	0.0518	0.0006***	0.0999	0.0117*	<0.0001***	0.0019**
			Panel 3			
$\widehat{CoVaR}^{AB}_{GS\|C,t}$	0.0489*	0.2143	0.1201	0.4335	<0.0001***	0.0001***
$\widehat{CoVaR}^{PLM}_{GS\|C,t}$	0.8109	0.0251*	0.8162	0.2306	<0.0001***	0.0535

*, ** and *** denote significance at the 5, 1 and 0.1 percent levels.

Conclusions and Outlook

- Semiparametric PLM does well during financial crisis
- Nonlinear tail dependence is not negligible
- Multivariate nonlinear part in PLM

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Macroprudential Risk Measures

- Marginal Expected Shortfall (MES): Portfolio $R = \sum_i w_i X_i$ where w_i: weights, X_i: asset return, $0 < \tau < 1$,

$$\mathrm{MES}_\tau^i = \frac{\partial ES^\tau(R)}{\partial w_i} = -\mathrm{E}\left[X_i | R \leq -VaR_R^\tau\right]$$

- Distressed Insurance Premium (DIP): Huang et al. (2010) $L = \sum_{i=1}^N L_i$ total loss of a portfolio

$$DIP = \mathrm{E}^Q\left[L | L \geq L_{min}\right]$$

▸ Return

Advantages of CoVaR

- Cloning Property: if dividing X_i into several clones, then the value of CoVaR conditioning on the individual large firm does not differ from the one conditioning on one of the clones
- Conservative Property: CoVaR conditioning on some bad event, the value would be more conservative than VaR
- Adaptive to the changing market conditions

Return

Nonlinear Dependence

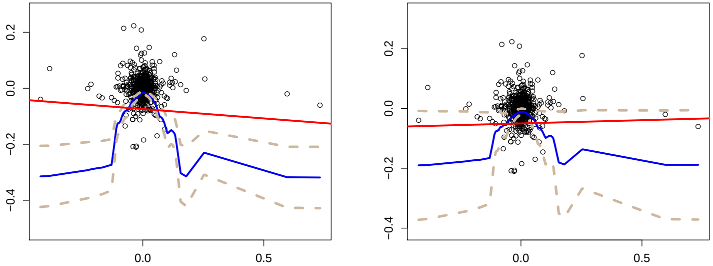

Figure 10: BOA and GS weekly returns 0.05 (left) and 0.1(right) quantile functions. y-axis=BOA returns; x-axis=GS returns. LLQR lines. Linear parametric quantile regression line. 95% Confidence band. $N = 546$.

Nonlinear Dependence

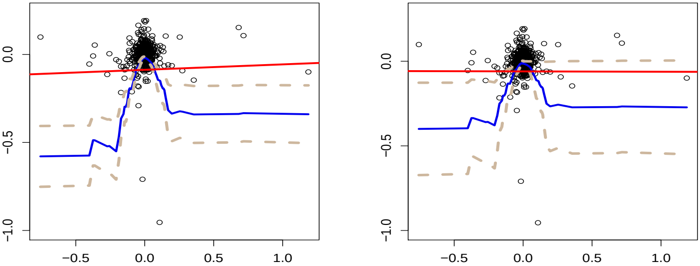

Figure 11: LB and AIG weekly returns 0.05 (left) and 0.1(right) quantile functions. y-axis=LB returns; x-axis=AIG returns. LLQR lines. Linear parametric quantile regression line. 95% Confidence band. $N = 546$.

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