Valuation of Collateralized Debt Obligations

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CDO Dynamics

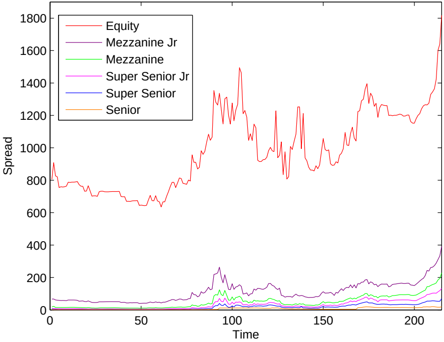

Figure 1: Time series of iTraxx spreads, Series 7, Maturity: 5 years, 21.03.2007-22.01.2008.

Dependence Matters!

The normal world is not enough.

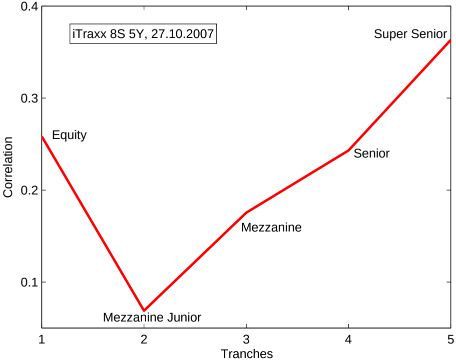

Figure 2: Gaussian one factor model with constant correlation.

Dependence Matters!

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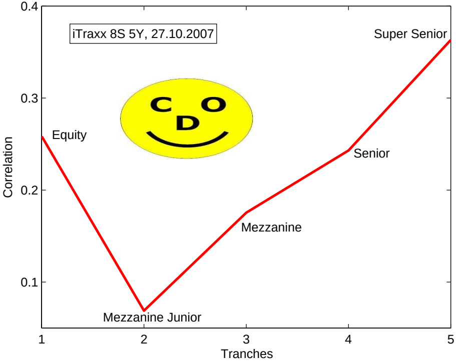

Figure 3: Gaussian one factor model with constant correlation.

Compound Correlation Over Time

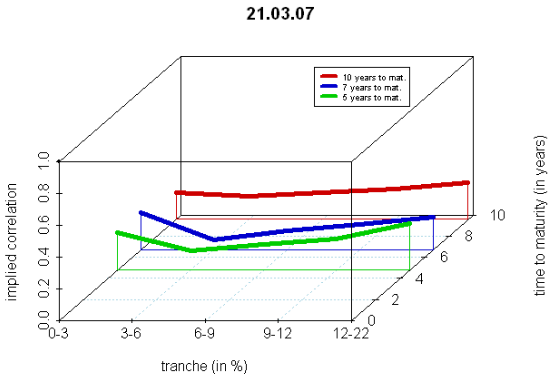

Figure 4: Film of compound correlations over time.

Base Correlation Over Time

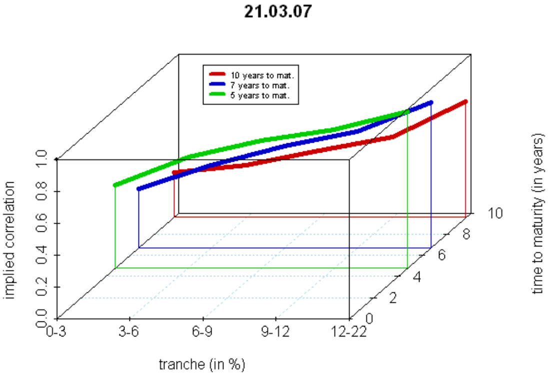

Figure 5: Film of base correlations over time.

Outline

1. Motivation ✓
2. Introduction to CDOs
3. Valuation of CDOs
4. Valuation of CDS
5. Simulation Algorithm

Cash-Flow Structure of a CDO

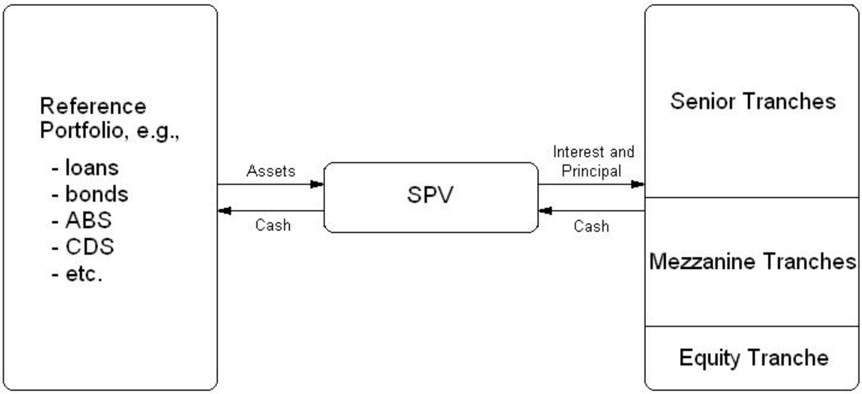

Figure 6: Illustration of a CDO transaction.

Definition

- *Collateralized debt obligation* (CDO): credit risk on a pool of d assets is tranched and sold to investors.
- *Originator* buys protection and pays a premium.
- *Tranche holder* sells credit risk protection and receives a premium.
- *Tranche* is defined by a lower and an upper attachment points.

Tranching

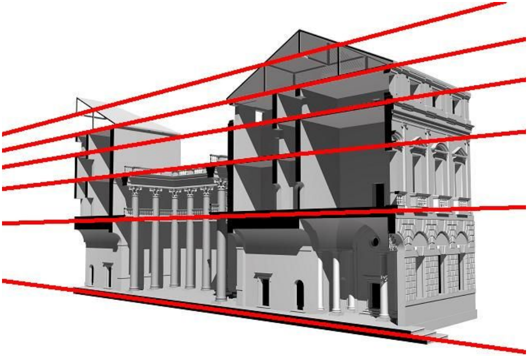

Figure 7: Palazzo Porto in Vicenza with mezzanine (entresol) under the roof.

Attachment Points

		Attachment points (%)	
Tranche number	Tranche name	Lower l	Upper u
1	Equity	0	3
2	Mezzanine Junior	3	6
3	Mezzanine	6	9
4	Senior	9	12
5	Super Senior	12	22
6	Super Super Senior	22	100

Table 1: Example of a CDO tranche structure, iTraxx.

Example

Suppose the equity tranche investor receives 500bp annually for protecting the first 3% of losses on a 10 million EUR pool.
Possible scenarios:

- no losses have occurred, then the investor is protecting the full $300,000\mathrm{EUR}$ and is paid 500bp on this amount,
- the losses of $100,000\mathrm{EUR}$ occurred, then the premium is paid on remaining $200,000\mathrm{EUR}$ that the investor is protecting.

Gaussian One Factor Model

Standardized asset log-returns:

$$X_{i,t} = \sqrt{\rho_t} Y_t + \sqrt{1-\rho_t} Z_{i,t},$$

for all, $i = 1, \ldots, d$, where Y_t (systematic risk factor), $\{Z_{i,t}\}_{i=1}^{d}$ (idiosyncratic risk factors) are i.i.d. $N(0,1)$. Hence:

$$(X_{1,t}, \ldots, X_{d,t})^\top \sim N(0, \Sigma_t),$$

with

$$\Sigma_t = \begin{pmatrix} 1 & \rho_t & \cdots & \rho_t \\ \rho_t & 1 & \cdots & \rho_t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_t & \rho_t & \cdots & 1 \end{pmatrix}$$

Default Probabilities

- Individual default probabilities of each firm defaulting in the collateral equal p_t. Default: $X_{i,t} < C_t$, C_t a threshold. Then

$$p_t = P(X_{i,t} < C_t) = P(\sqrt{\rho}Y_t + \sqrt{1-\rho}Z_{i,t} < C_t) = \Phi(C_t)$$

- Default probability given the systematic risk factor y_t:

$$\begin{aligned} Z_{i,t} &< \frac{C_t - \sqrt{\rho}y_t}{\sqrt{1-\rho_t}} \\ p(y_t) &= \Phi\left\{\frac{\Phi^{-1}(p_t) - \sqrt{\rho_t}y_t}{\sqrt{1-\rho_t}}\right\} \end{aligned}$$

Loss Distribution

- Loss variable of i-th firm until $t \in [0, T]$

$$\Gamma_{i,t} = I(\sqrt{\rho_t} Y_t + \sqrt{1-\rho_t} Z_{i,t} < C_t)$$

- Portfolio loss process

$$L_t = \frac{1-R}{d} \sum_{i=1}^{d} \Gamma_{i,t} \tag{1}$$

where R is the recovery rate equal for all credits in the portfolio.

Homogeneous Gaussian One Factor Model

In the model

$$X_i = \sqrt{\rho} Y + \sqrt{1-\rho} Z_i,$$

assume parameters equal for all credits in the portfolio and for all time horizons:

$$\rho_{i,t} = \rho \text{ and } p_{i,t} = p \text{ for all } i = 1, \ldots, d.$$

Portfolio Loss Density Function

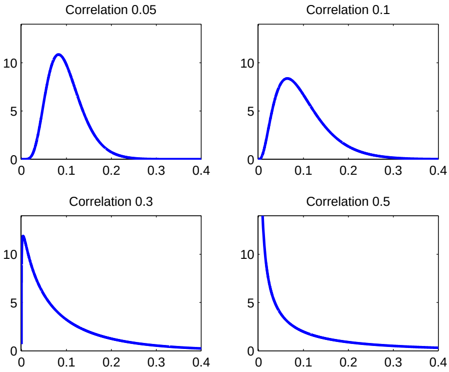

Figure 8: Portfolio Loss Density $f_L(\cdot)$ for different correlation parameters ρ and fixed probability of default $p = 10\%$.

Gaussian Multi-Factor Model

Three factor model

$$\Sigma = \begin{pmatrix} \boxed{\begin{matrix} 1 & \cdots & \rho_2 \\ & \ddots & \\ \rho_2 & \cdots & 1 \end{matrix}} & \begin{matrix} \rho_1 & \cdots & \cdots \\ \vdots & & \\ \rho_1 & & \end{matrix} & & \begin{matrix} \cdots & \cdots & \rho_1 \\ & & \vdots \\ & & \vdots \end{matrix} \\ \begin{matrix} \rho_1 & \cdots & \rho_1 \\ \vdots & & \\ \vdots & & \end{matrix} & \boxed{\begin{matrix} 1 & \cdots & \rho_2 \\ & \ddots & \\ \rho_2 & \cdots & 1 \end{matrix}} & & \\ & & \ddots & \\ & & \boxed{\begin{matrix} 1 & \cdots & \rho_3 \\ & \ddots & \\ \rho_3 & \cdots & 1 \end{matrix}} & \begin{matrix} & & \vdots \\ & & \vdots \\ \rho_1 & \cdots & \rho_1 \end{matrix} \\ \begin{matrix} \vdots & & \\ \vdots & & \\ \rho_1 & \cdots & \cdots \end{matrix} & & \begin{matrix} & & \rho_1 \\ & & \vdots \\ \cdots & \cdots & \rho_1 \end{matrix} & \boxed{\begin{matrix} 1 & \cdots & \rho_3 \\ & \ddots & \\ \rho_3 & \cdots & 1 \end{matrix}} \end{pmatrix}$$

Notation

t	the time (in years) passed since the CDO was originated
T	the maturity (in years) of the CDO
M	the initial value of the portfolio
u_j	upper attachment point of tranche j
l_j	lower attachment point of tranche j
$\{t_0, \ldots, t_K\}$	payment dates
Δt_k	frequency of payments $\Delta t_k = \frac{t_k - t_{k-1}}{360}$, $k = 1, \ldots, K$
s_j	the premium of tranche j
$\beta(t_0, t_k)$	the discount factor from t_0 to t_k

Loss of the Tranche

The loss of the tranche j at time t

$$L_{j,t} = \min\{\max(0, L_t - l_j); u_j - l_j\} \tag{2}$$

$$= \begin{cases} 0, & L_t < l_j, \\ L_t - l_j, & l_j \le L_t \le u_j, \\ u_j - l_j, & L_t > u_j. \end{cases}$$

Example Let j be the mezzanine tranche with the lower attachment point 6% and the upper attachment point 9%. Then

Loss of the portfolio	2	7	10
Loss of the tranche	0	1	3

Outstanding Notional

- The outstanding notional of the portfolio

$$F_t = 1 - \frac{L_t}{1-R},$$

- The outstanding notional of tranche j

$$F_{j,t} = u_j - l_j - L_{j,t}. \tag{3}$$

Example Let j be the mezzanine tranche with the lower attachment point 6% and the upper attachment points 9%. Then

Loss of the tranche	0	1	3
Outstanding notional of tranche	3	2	0

Cash Flows

The holders of tranche j at time t_k:

- receive an amount

$$s_j \Delta t_k F_{j,t_k} M,$$

- pay an amount

$$(L_{j,t_k} - L_{j,t_{k-1}}) M.$$

Premium

The premium is chosen in such a way that

1. fixed (premium) leg – the payments that tranche holders receive,
2. floating (protection) leg – the payments that tranche holders pay

are equal.
The premiums are constantly observed in the market!

Valuation of CDO

1. Fixed (premium) leg – the payments tranche holders receive

$$PL_j = \sum_{k=1}^{K} \beta(t_0, t_k) s_j \Delta t_k \, \mathsf{E}\{F_{j,t_k} M\}, \tag{4}$$

   where $\mathsf{E}\{F_{j,t_k} M\}$ is expected value of the outstanding notional amount of tranche j.

2. Floating (protection) leg – the payments tranche holders pay

$$DL_j = \sum_{k=1}^{K} \beta(t_0, t_k) \, \mathsf{E}\{(L_{j,t_k} - L_{j,t_{k-1}}) M\}, \tag{5}$$

   where $\mathsf{E}\{(L_{j,t_k} - L_{j,t_{k-1}}) M\}$ is expected value of the losses realized beetwen the payment day t_{k-1} and t_k.

Equity Tranche

The equity tranche is quoted in two parts:

1. an upfront fee s_1,
2. a running spread of 500 BPs.

Premium leg for the equity tranche:

$$X_{F,1} = s_1(u_1 - l_1)M + \sum_{k=1}^{K} \beta(t_0, t_k) 0.05 \Delta t_k E\{F_{j,t_k} M\}.$$

Premium

The premium s_j is chosen in such a way that (2) and (3) are equal:

$$PL_j = DL_j.$$

This leads to the solution:

$$s_j = \frac{\sum_{k=1}^{K} \beta(t_0, t_k)\{\mathrm{E}(L_{j,t_k}) - \mathrm{E}(L_{j,t_{k-1}})\}}{\sum_{k=1}^{K} \beta(t_0, t_k)\Delta t_k \, \mathrm{E}(F_{j,t_k})}. \qquad (6)$$

This is what we observe in the market!

Implied Correlation

Implied correlation is found by inverting a pricing model for CDOs and searching for a correlation parameter that match the quoted spread of a tranche.

$$s_j = \frac{\sum_{k=1}^{K} \beta(t_0, t_k)\{\mathrm{E}(L_{j,t_k}) - \mathrm{E}(L_{j,t_{k-1}})\}}{\sum_{k=1}^{K} \beta(t_0, t_k)\Delta t_k \, \mathrm{E}(F_{j,t_k})}$$

depends on ρ.
If Gaussian one factor model was correct, then the implied correlation ρ_j from s_j would be approximately constant across tranches and equal ρ.

Implied Correlation Smile!

Implied correlation for a tranche causes the value of the tranche to be zero.

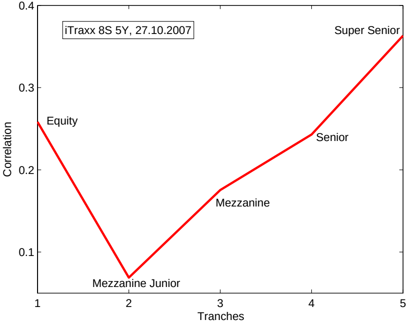

Figure 9: Implied compound correlation smile.

Bloomberg

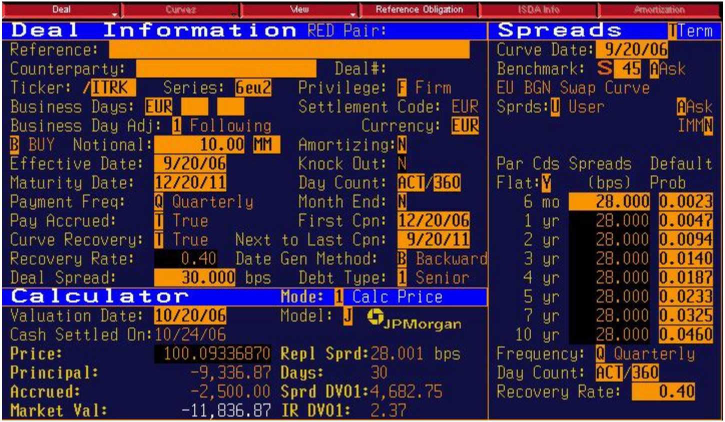

Figure 10: ITRAXX Europe, series 6EU2 with maturity 5 years.

iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10)
- Maturities: 5Y, 7Y, 10Y

Default Time

Derive the implied default probabilities from the CDS spreads and then use this risk neutral probabilities to value more complex credit structures.

Let τ be the time to default variable, with a distribution function $F(t) = P(\tau \leq t)$.

$P(t < \tau \leq t + \Delta t | \tau > t)$ is a probability that default will occur in the interval $(t, t + \Delta t)$ given that the reference entity survived up to time t.

Hazard Rate Function

Hazard rate function

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < \tau \leq t + \Delta t | \tau > t)}{\Delta t}$$

Relations with distribution of stopping time

$$\begin{aligned} F(t) &= 1 - \exp\left\{ -\int_0^t h(u)du \right\}, \\ f(t) &= \exp\left\{ -\int_0^t h(u)du \right\} \end{aligned}$$

Choice of Hazard Rate

Hazard rate $h(t)$

- Constant a ✓
- Linear $a + bt$
- Quadratic $a + bt + ct^2$
- Step function $\sum_{i=1}^{T_n} \alpha_i I(T_{i-1} < t \leq T_i)$

Survival probability

$$\bar{p}(t) = \exp\left\{ -\int_0^t h(u)du \right\}$$

Default probability

$$p(t) = 1 - \bar{p}(t). \tag{7}$$

Credit Default Swap

Credit Default Swap (CDS) is an insurance contract covering the risk that a specified credit defaults.

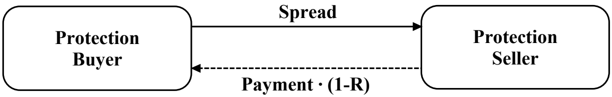

Figure 11: Illustration of a CDS transaction.

Valuation of CDS: Premium Leg

CDS premium is chosen in such a way that value of the contract is zero: expected values of payments between protection buyer and protection seller must be equal.

Stream of fixed cash flows paid at time t_k, $k = 1, \ldots, K$ either until T or until $\tau < T$ in case of default.

$$PL = \sum_{k=1}^{K} \beta(t_0, t_k) M s \Delta t_k \, \mathsf{E}\{I(\tau > t_k)\}.$$

Valuation of CDS: Protection Leg

Expected value of the present value of the contingent payment upon default (DP) minus the contingent accrued premium (AP).

$$
\begin{aligned}
DP &= \mathsf{E}\{\beta(t_0,\tau)M(1-R)I(\tau \leq T)\} \\
AP &= \sum_{k=1}^{K} \mathsf{E}\{\beta(t_0,\tau)Ms(\tau - t_{k-1})I(t_{k-1} < \tau \leq t_k)\}
\end{aligned}
$$

Fair Spread

$$DP = AP + PL \Rightarrow$$

The Fair Spread is

$$s = \left(\frac{\sum_{k=1}^{K} [\beta(t_0, t_k) \Delta t_k \, \mathrm{E}\{I(\tau > t_k)\}]}{\mathrm{E}\{\beta(t_0, \tau)(1 - R) I(\tau \leq T)\}} \right. \quad (8)$$

$$\left. + \frac{\sum_{k=1}^{K} [\mathrm{E}\{\beta(t_0, \tau)(\tau - t_{k-1}) I(t_{k-1} < \tau \leq t_k)\}]}{\mathrm{E}\{\beta(t_0, \tau)(1 - R) I(\tau \leq T)\}} \right)^{-1} \quad (9)$$

Probability of Default

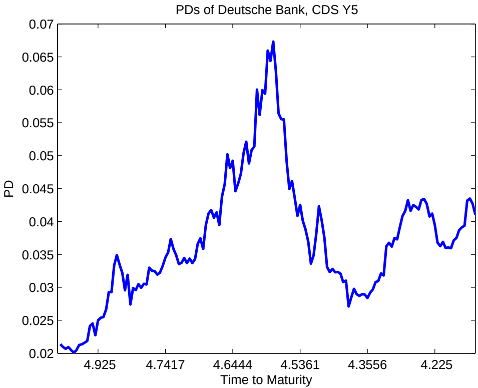

Figure 12: Probabilities of default (7) with constant hazard rate of Deutsche Bank, time period 20071022-20080812.

Index Spread

Index spread is a (risk-neutral) survival probability weighted average of the CDS spreads

$$s_{\text{Index}} = \frac{\sum_{i=1}^{125} s_i(1-p_i)}{\sum_{i=1}^{125}(1-p_i)}, \tag{10}$$

where s_i is defined in (8) and p_i has the form (7).

iTraxx Index Spread

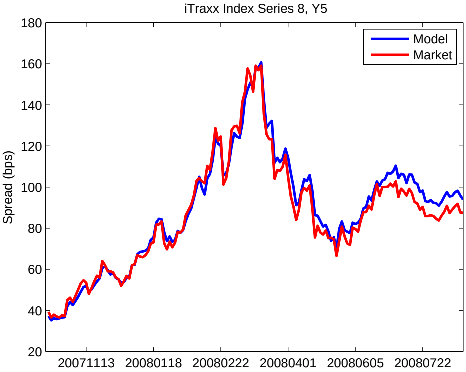

Figure 13: Comparison of market iTraxx index spread with the result of the model (10).

Intensities

- get all the CDS from the date t
- estimate individual intensities $\lambda_i, i = 1, \ldots, d$
- get the intensity of the CDO based on the individual intensities where p_i is the individual default probability and r is the risk free interest rate

Copula

Dependency between triggers can be modeled with copulae.

For a distribution function F with marginals $F_{X_1} \dots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \tag{11}$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \dots, d$. If $F_{X_1}, \dots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \dots, F_{X_d}$ are cdfs, then the function F defined in (11) is a joint cdf with marginals $F_{X_1}, \dots, F_{X_d}$.

Monte-Carlo Simulations

- Generate N samples of random numbers $(U_1, \dots, U_d) \sim C$, where C is a Gaussian copula
- compute times of default $\tau_i = p_i^{-1}(U_i)$ (7)
- compute portfolio's loss process L_t (1) and for each tranche $L_{t,j}$ (2), $F_{t,j}$ (3)
- calculate $DL_{t,j}$ (5), $PL^*_{t,j} = \sum_{k=1}^{K} \beta(t_0, t_k) s_j \Delta t_k \, \mathsf{E}\{F_{j,t_k} M\}$ for N samples and their sample average $\overline{DL}_{t,j}$ and $\overline{PL^*}_{t,j}$
- the model spread is then $s_j^c = \frac{\overline{DL}_{t,j}}{\overline{PL^*}_{t,j}}$ for $j = 2, \dots, 5$
- if $j = 1$, then $s_1^c(u_1 - l_1) + \overline{PL^*}_{t,j} = \overline{DL}_{t,j}$

Calibration

Main idea of calibration is to minimize the difference between the market spread s_j^m and the estimated spread s_j^c of the tranches $j = 2, \ldots, 5$ keeping the difference between the market upfront fee s_1^m and the estimated upfront fee s_1^c relatively small

$$D_1 \stackrel{def}{=} |s_1^c - s_1^m| \leq \epsilon$$

$$D_2 \stackrel{def}{=} \sum_{j=2}^{5} |s_j^c - s_j^m| \rightarrow \min$$

Calibration

- **One parameter model** - minimization is done over D_1. Where the optimal copula parameter $\hat{\theta} \in [\theta_l; \theta_u]$ is found by satisfying $D_1 < \epsilon$ using a bisection method
- **Two parameter model** - first assume $\rho_1 = \rho_2$, proceed as in previous method, then going along the path where $D_1 \leq \epsilon$ and changing ρ_1, ρ_2 minimize D_2

Calibration

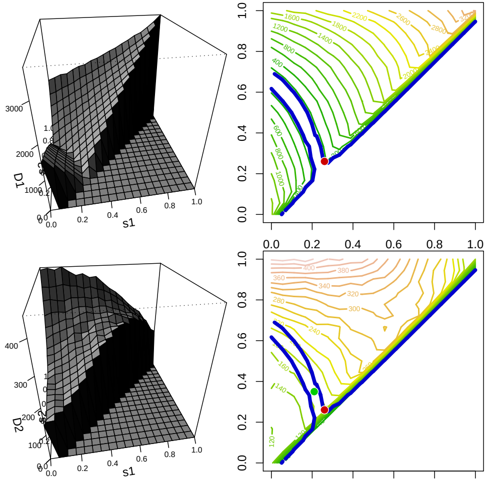

Figure 14: D_1 and D_2 as the function of parameters ρ_1 and ρ_2, 20071027.

Calibrated Spread

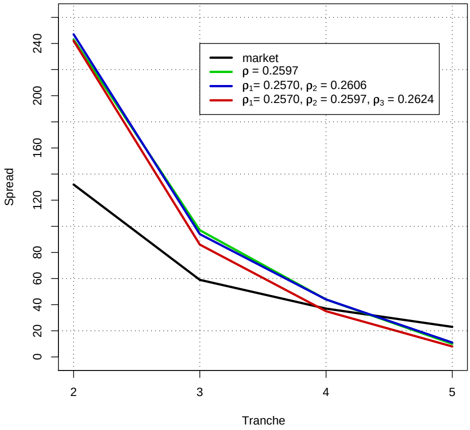

Figure 15: Calibrated spread with two and one correlation for 20071027.

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