Volatility Investing with Variance Swaps

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Why investors may wish to trade volatility?

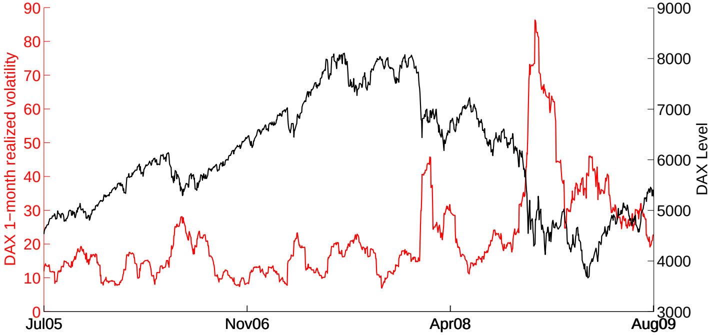


Figure 1: DAX level vs. DAX 1M realized volatility (20050103 - 20091230)

Volatility is an asset

"fear indices": VIX, VDAX, VSTOXX

3G volatility derivatives: gamma swaps, corridor variance swaps, conditional variance swaps

volatilitiy trading strategies: dispersion trading

Research questions

- How to trade volatility?
- How to hedge (replicate) volatility?
- How good can we perform?
- How does dispersion trading work?

Outline

1. Motivation ✓
2. Definition
3. Trading volatility with options
4. Replication and hedging
5. 3G volatility derivatives
6. Dispersion trading strategy
7. Conclusions

Variance swap

Figure 2: Cash flow of a variance swap at expiry

Variance swap

- forward contract
- at maturity pays the difference between realized variance σ_R^2 and strike K_{var}^2 (multiplied by notional N_{var})

$$(\sigma_R^2 - K_{var}^2) \cdot N_{var} \tag{1}$$

$$\sigma_R = \sqrt{\frac{252}{T} \sum_{t=1}^{T} \left(\log \frac{S_t}{S_{t-1}} \right)^2} \cdot 100 \tag{2}$$

Example

3-month variance swap long

Long position in 3-month variance swap. Trade size is 2500 variance notional (represents a payoff of 2500 per point difference between realized and implied variance).

If K_{var} is 20% ($K_{var}^2 = 400$) and the realized subsequent variance is $(15\%)^2$ (quoted as $\sigma_R^2 = 225$), the long position makes loss 437500 $= 2500 \cdot (400 - 225)$

Replication and hedging - intuitive approach

- European option with Black-Scholes (BS) price $V_{BS}(S, K, \sigma\sqrt{\tau})$
- variance vega:

$$\frac{\partial V_{BS}}{\partial \sigma^2} = \frac{S}{2\sigma\sqrt{\tau}}\varphi(y) \tag{3}$$

where

$$y = \frac{\log(S/K) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$$

φ - pdf of a standard normal rv.

Variance vega of options with different K

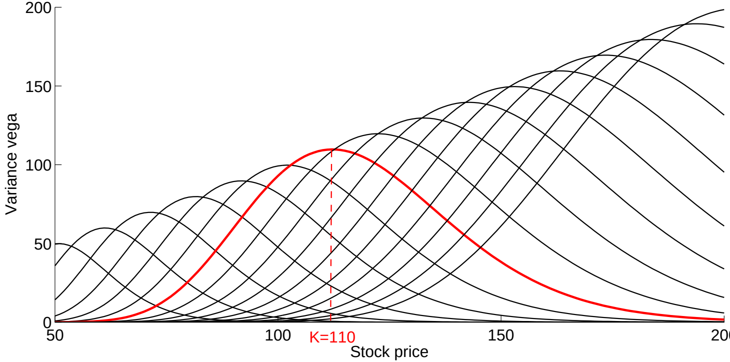


Figure 3: Dependence of variance on S for vanilla options with $K = [50, 200]$, $\sigma = 0.2$, $\tau = 1$

Equally-weighted option portfolio

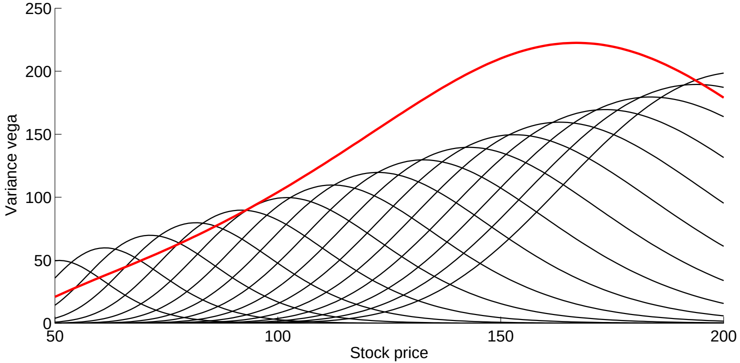


Figure 4: Variance vega of option portfolio (red line) with options weighted equally

$1/K$-weighted option portfolio

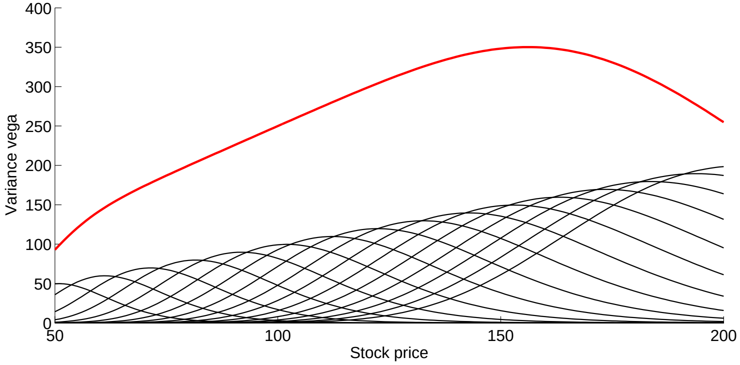


Figure 5: Variance vega of option portfolio (red line) with options weighted proportional to $1/K$

$1/K^2$-weighted option portfolio

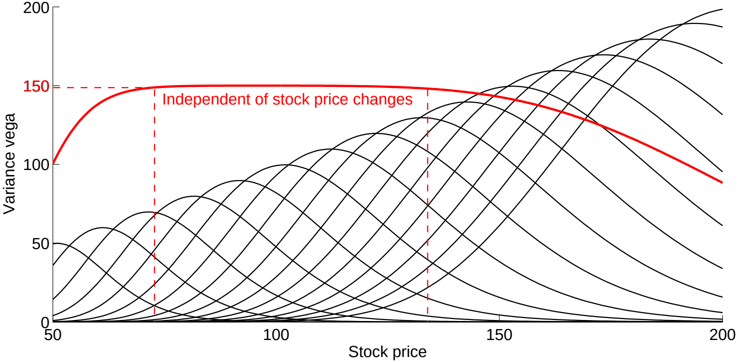


Figure 6: Variance vega of option portfolio (red line) with options weighted proportional to $1/K^2$

Replication and hedging - more rigorous approach

- existence of futures market with delivery dates $T' \geq T$
- stock price S_t (underlying) dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{4}$$

- all strikes are available (market is complete)
- continuous trading
- risk free interest rate $r = 0$, w.l.o.g.

Log contract

Define

$$f(S_t) = \frac{2}{T}\left\{\log\frac{S_0}{S_t} + \frac{S_t}{S_0} - 1\right\} \tag{5}$$

derivatives:

$$f'(S_t) = \frac{2}{T}\left(\frac{1}{S_0} - \frac{1}{S_t}\right) \tag{6}$$

and

$$f''(S_t) = \frac{2}{TF_t^2} \tag{7}$$

observe $f(S_0) = 0$

Itô's lemma

$$f(S_t) = f(S_0) + \int_0^T f'(S_t)dS_t + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma_t^2 dt \qquad (8)$$

Substituting (6), (7):

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\left(\log\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) - \qquad (9)$$

$$-\frac{2}{T}\int_0^T \left(\frac{1}{S_0} - \frac{1}{S_t}\right) dS_t$$

Equation (9) gives the value of σ_R^2 as a sum of:

$$\frac{2}{T}\int_0^T \left(\frac{1}{S_0} - \frac{1}{S_t}\right) dS_t$$

(continuously rebalanced position in underlying stock) and

$$f(S_T) = \frac{2}{T}\left(\log\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) \tag{10}$$

(**log contract**, static position).

Carr and Madan (2002) represent any twice differentiable payoff function $f(S_T)$:

$$f(S_T) = f(k) + f'(k)\left\{(S_T - k)^+ - (k - S_T)^+\right\} \tag{11}$$

$$+ \int_0^k f''(K)(K - S_T)^+ dK$$

$$+ \int_k^\infty f''(K)(S_T - K)^+ dK$$

where k is an arbitrary number.

Applying (11) to (10) with $k = S_0$ gives

$$\log\left(\frac{S_0}{S_T}\right) + \frac{S_T}{S_0} - 1 = \tag{12}$$

$$= \int_0^{S_0} K^{-2}(K - S_T)^+ dK + \int_{S_0}^{\infty} K^{-2}(S_T - K)^+ dK$$

a portfolio of OTM puts and calls weighted by K^{-2}.

What are the costs of this strategy? The strike K_{var}^2 of a variance swap is calculated via the risk-neutral expectation:

$$K_{var}^2 = \frac{2}{T} e^{rT} \int_0^{S_0} K^{-2} P_0(K) dK + \frac{2}{T} e^{rT} \int_{S_0}^{\infty} K^{-2} C_0(K) dK \quad (13)$$

where P_0 (C_0) - value of a put (call) option at $t = 0$.
Problem: vanilla options with a complete strike range (from 0 to ∞) are not traded. How to replicate a fair future realized variance in reality?

Discrete approximation

Demeterfi et al. (1998) approximate payoff (10) via piecewise linear approximation.

Example: put option with strike K_0 and 2nd closest strike K_{1p}

$$w(K_0) = \frac{f(K_{1p}) - f(K_0)}{K_0 - K_{1p}} \tag{14}$$

The second segment - combination of puts with strikes K_0 and K_{1p}:

$$w(K_{1p}) = \frac{f(K_{2p}) - f(K_{1p})}{K_{1p} - K_{2p}} - w(K_0) \tag{15}$$

where $w(K)$ amount of option with strike K in replicating portfolio (the slope of a linear segment at point K, figure 7).

Discrete approximation

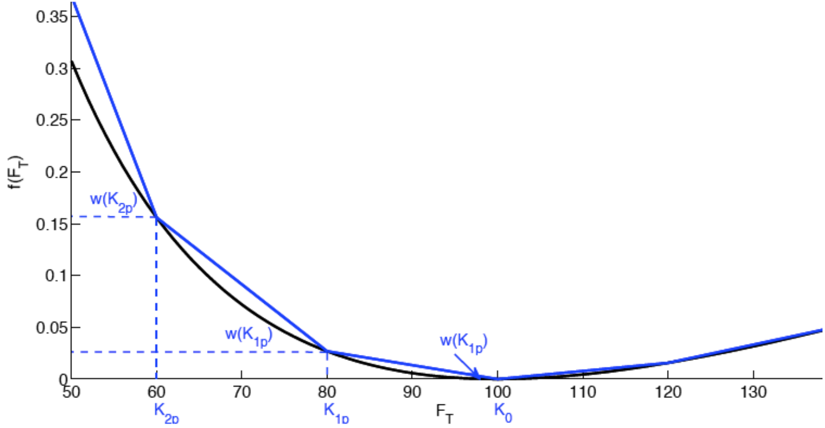


Figure 7: Discrete approximation of a log payoff (10)

Simulated payoff of 3M DAX variance swap

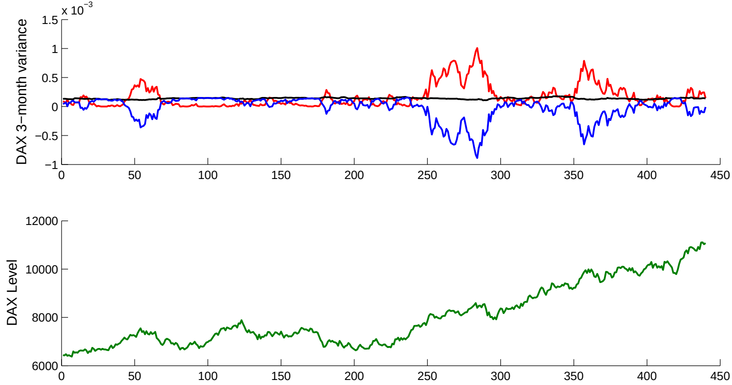


Figure 8: Strike of 3M variance swap, realized 3M variance, payoff of 3M variance swap long,price of underlying asset

3M DAX variance swap payoff statistics

Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
Min. payoff						
-0.00376	-0.00051	-0.00145	-0.00138	0.00044	-0.95685	4.33
Max. payoff						
0.00016	0.00027	0.00020	0.00020	0.00002	0.65743	3.63
Mean payoff						
-0.00018	0.00005	-0.00003	-0.00002	0.00003	-0.52441	3.30
Volatility of payoff						
0.00009	0.00044	0.00022	0.00022	0.00005	0.66285	3.37

Table 1: Summary statistics of 3M variance swap payoff simulation, duration of the strategy - 10 years (2500 days),number of paths - 1000, GBM with $\mu = 0.17$, $\sigma = 0.18$

Generalized variance swaps

Modify the floating leg of a standard variance swap (1) with a weight process w_t to obtain:

$$\sigma_R^2 = \frac{252}{T} \sum_{t=1}^{T} w_t \left(\log \frac{S_t}{S_{t-1}} \right)^2 \tag{16}$$

Corridor and conditional variance swaps

$w_t = w(S_t) = \mathbf{I}_{S_t \in C}$ defines a corridor variance swap with corridor C.

- for $C = [A, B]$ the payoff function is defined by

$$f(S_T) = \frac{2}{T}\left(\log \frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) \mathbf{I}_{S_T \in [A,B]} \tag{17}$$

  where $\mathbf{I}$ is the indicator function.

- $C = [0, B]$ gives downward variance swap
- $C = [A, \infty]$ gives upward variance swap

Simulated payoff of 3M DAX corridor swap with time-adjusting corridor

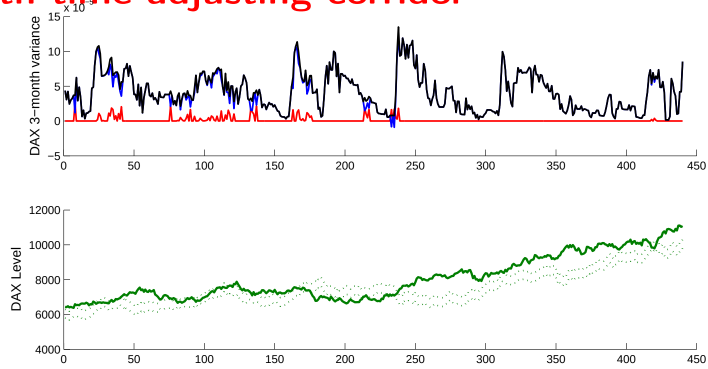


Figure 9: Strike of 3M corridor swap, realized 3M conditional variance, payoff of 3M corridor swap long, price of underlying asset

3M DAX corridor swap payoff statistics

Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
Min. payoff						
-0.00006	0.00000	-0.00001	-0.00001	0.00001	-1.70441	9.42
Max. payoff						
0.00007	0.00021	0.00014	0.00014	0.00002	0.25564	3.17
Mean payoff						
0.00001	0.00007	0.00003	0.00002	0.00001	0.75811	3.41
Volatility of payoff						
0.00001	0.00004	0.00003	0.00003	0.00001	0.16232	3.02

Table 2: Summary statistics of 3M corridor swap payoff simulation, duration of the strategy - 10 years (2500 days),number of paths - 1000, GBM with $\mu = 0.17$, $\sigma = 0.18$

Gamma swaps

$w_t = w(S_t) = S_t/S_0$ defines a price-weighted variance swap or gamma swap with realised variance paid at expiry:

$$\sigma_{gamma} = \sqrt{\frac{252}{T}\sum_{t=1}^{T}\frac{S_t}{S_0}\left(\log\frac{S_t}{S_{t-1}}\right)^2} \cdot 100 \tag{18}$$

The payoff function:

$$f(S_T) = \frac{2}{T}\left(\frac{S_T}{S_0}\log\frac{S_T}{S_0} - \frac{S_T}{S_0} + 1\right) \tag{19}$$

Simulated payoff of 3M DAX gamma swap

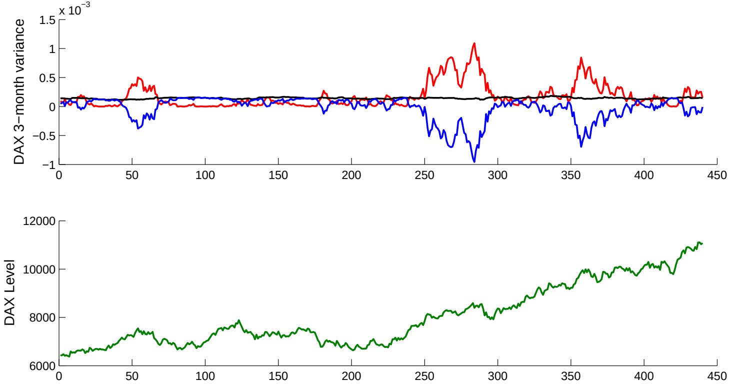


Figure 10: Strike of 3M gamma swap, realized 3M gamma-weighted variance, payoff of 3M gamma swap long, price of underlying asset

Gamma swap vs variance swap

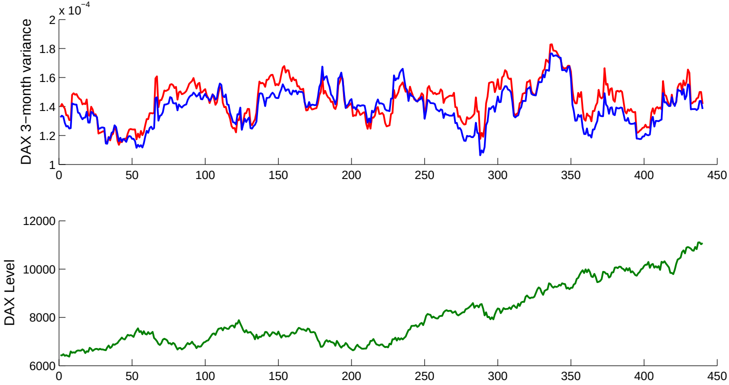


Figure 11: Strike of 3M gamma swap, Strike of 3M variance swap, price of underlying asset

3M DAX gamma swap payoff statistics

Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
Min. payoff						
-0.00435	-0.00054	-0.00158	-0.00149	0.00051	-0.99914	4.51
Max. payoff						
0.00016	0.00027	0.00020	0.00020	0.00002	0.64185	3.57
Mean payoff						
-0.00019	0.00005	-0.00003	-0.00003	0.00003	-0.55715	3.33
Volatility of payoff						
0.00009	0.00048	0.00023	0.00022	0.00005	0.72273	3.48

Table 3: Summary statistics of 3M gamma swap payoff simulation, duration of the strategy - 10 years (2500 days),number of paths - 1000, GBM with $\mu = 0.17$, $\sigma = 0.18$

Basket volatility

$$\sigma^2_{Basket} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

replace $\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$,

then $\rho = \dfrac{\sigma^2_{Basket} - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j}$ is the basket correlation ('dispersion').

Dispersion Strategy

$$\rho = \frac{\sigma^2_{Basket} - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j}$$

- Long: Variance of basket (index)
- Short: Variance of basket constituents
- Long: Dispersion

How to implement?

Dispersion Strategy

For a basket of $i = 1, \ldots, N$ stocks payoff of direct dispersion strategy is sum of:

$$(\sigma^2_{R,i} - K^2_{var,i}) \cdot N_i$$

and of short position in

$$(K^2_{var,index} - \sigma^2_{R,index}) \cdot N_{index}$$

where

$$N_i = N_{index} \cdot w_i$$

notional amount of the i-th stock.

Dispersion Strategy

Overall payoff:

$$N_{index} \cdot \left(\sum_{i=1}^{n} w_i \sigma_{R,i}^2 - \sigma_{R,Index}^2 \right) - ResidualStrike \tag{20}$$

$$ResidualStrike = N_{index} \cdot \left(\sum_{i=1}^{n} w_i K_{var,i}^2 - K_{var,Index}^2 \right)$$

Simulated payoff of 3M DAX dispersion strategy

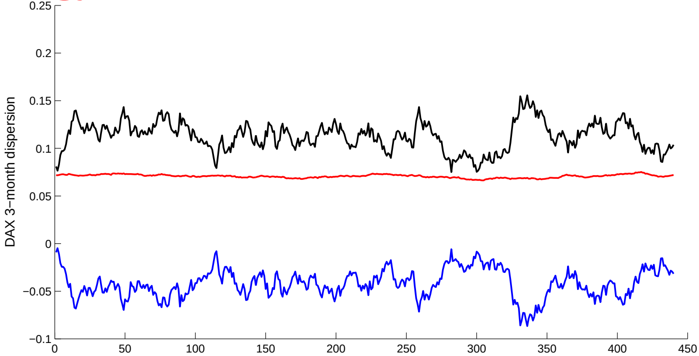


Figure 12: 3M strike dispersion, 3M realized dispersion, 3M direct dispersion strategy (dispersion long)

3M DAX dispersion strategy statistics

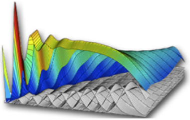
Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
Min. payoff						
-0.22189	-0.00251	-0.03967	-0.02255	0.04003	-1.58	4.79
Max. payoff						
-0.00657	0.06719	0.01258	0.00734	0.01341	1.51	4.41
Mean payoff						
-0.04815	0.00262	-0.00805	-0.00294	0.01172	-1.72	4.88
Volatility of payoff						
0.00078	0.03959	0.00868	0.00481	0.00837	1.57	4.68

Table 4: Summary statistics of 3M dispersion strategy simulation, duration of the strategy - 10 years (2500 days),number of paths - 1000

Conclusions

- Volatility can be traded as an asset
- Future realized volatility can be replicated with option portfolios
- With linear interpolation replication performs well
- The success of the volatility dispersion strategy lies in determining:
  - Direction of the strategy (GARCH volatility forecasts)
  - Constituents for the offsetting variance basket (PCA, DSFM)
  - Proper weights of the constituents (vega-flat strategy, gamma-flat strategy, theta-flat strategy)

Volatility Investing with Variance Swaps



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