Dynamics of Correlation Risk

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Market baskets

Basket correlation

$$\sigma_B^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{ij}, \tag{1}$$

where σ_i standard deviation, w_i basket weight of the i-th stock, ρ_{ij} correlation between the i-th and the j-th stock, $i, j \in \{1, \ldots, N\}$.

Replace $\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$,

then

$$\rho = \frac{\sigma_B^2 - \sum_i w_i^2 \sigma_i^2}{\sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j} \tag{2}$$

is the basket correlation.

Measure of basket diversification

Empirical evidence (N big) $0 \leq \rho \leq 1$, Bourgoin (2001). Define:

$$\sigma^2_{B,min} = \sum_i w_i^2 \sigma_i^2 \tag{3}$$

$$\sigma^2_{B,max} = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \tag{4}$$

Substituting (3) and (4) to (2) gives ρ a new interpretation:

$$\rho = \frac{\sigma^2_B - \sigma^2_{B,min}}{\sigma^2_{B,max} - \sigma^2_{B,min}} \tag{5}$$

Dynamics of DAX diversification

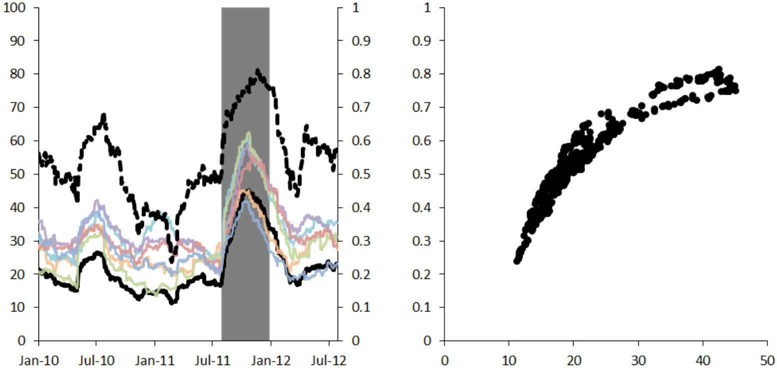

Figure 1: solid lines: $\sigma_{t+0.25}$ (18) of DAX and some constituents (Adidas, BMW, Siemens, Daimler, E.ON, Lufthansa) dashed line: $\rho_{t+0.25}$ (2), right panel: scatter plot $\sigma_{t+0.25}$ vs $\rho_{t+0.25}$, for $t + 0.25$ from 20100104 till 20120801, shaded area: Aug 2011 market fall further examples

Define RC and MFIC

- calculate the realized variance (RV) $\sigma^2_{t+\tau,B}$ of a basket and $\sigma^2_{t+\tau,i}$ (constituents) via (18)
- obtain the realized correlation (RC) $\rho_{t+\tau}$ via (2)
- calculate the model free implied variance (MFIV) $\tilde{\sigma}^2_{t,B}(\tau)$ of a basket and $\tilde{\sigma}^2_{t,i}(\tau)$ (constituents) via (19)
- obtain the model free implied correlation (MFIC) $\tilde{\rho}_t(\tau)$ via (2)

RV and MFIV

Can exposure to RC (MFIC) be profitable?

Compare:

$$\tilde{\rho}_t(\tau) = \frac{\tilde{\sigma}^2_{t,B}(\tau) - \sum_i w_i^2 \tilde{\sigma}^2_{t,i}(\tau)}{\sum_i \sum_{j \neq i} w_i w_j \tilde{\sigma}_{t,i}(\tau) \tilde{\sigma}_{t,j}(\tau)} \tag{6}$$

$$\rho_{t+\tau} = \frac{\sigma^2_{t+\tau,B} - \sum_i w_i^2 \sigma^2_{t+\tau,i}}{\sum_i \sum_{j \neq i} w_i w_j \sigma_{t+\tau,i} \sigma_{t+\tau,j}} \tag{7}$$

- $\sigma^2_{t+\tau,B} - \tilde{\sigma}^2_{t,B}(\tau) < 0$
- $\tilde{\sigma}^2_{t,i}(\tau) - \sigma^2_{t+\tau,i} \approx 0$
- expect $\tilde{\rho}_t(\tau) - \rho_{t+\tau} > 0$, how to exploit this knowledge?

▸ evidence from US market (literatire), ▸ evidence from German market (own findings)

$\tilde{\rho}_t(\tau)$ vs $\rho_{t+\tau}$ - arbitrage opportunity?

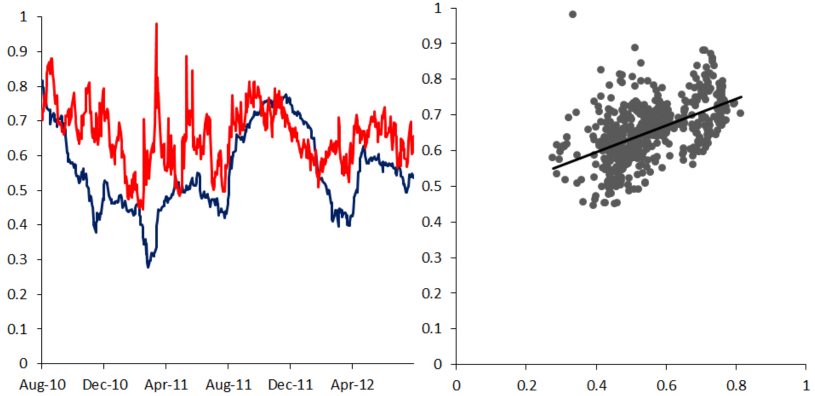

Figure 2: $\rho_{t+\tau}$ (blue) $\tilde{\rho}_t(\tau)$ (red), right panel: scatter plot $\rho_{t+\tau}$ vs $\tilde{\rho}_t(\tau)$, for $t + 0.25$ from 20100802 till 20120801

Exposure to $\tilde{\rho}_t(\tau) - \rho_{t+\tau}$

Implement dispersion strategy by trading Variance swaps:

- sell: RV of basket (index)
- buy: RVs of basket constituents

With $N_{var} = 1$ payoff at $t + \tau$:

$$D_{t+\tau} = -\left\{\sigma^2_{t+\tau,B} - \tilde{\sigma}^2_{t,B}(\tau)\right\} + \sum_{i=1}^{n} w_i \left\{\sigma^2_{t+\tau,i} - \tilde{\sigma}^2_{t,i}(\tau)\right\} = \qquad (8)$$

$$\tilde{\rho}_t(\tau) \sum_i \sum_{j \neq i} w_i w_j \tilde{\sigma}_{t,i}(\tau)\tilde{\sigma}_{t,j}(\tau) - \rho_{t+\tau} \sum_i \sum_{j \neq i} w_i w_j \sigma_{t+\tau,i}\sigma_{t+\tau,j} \approx^*$$

$$\sum_i \sum_{j \neq i} w_i w_j \tilde{\sigma}_{t,i}(\tau)\tilde{\sigma}_{t,j}(\tau) \left\{\tilde{\rho}_t(\tau) - \rho_{t+\tau}\right\} \qquad (9)$$

* justified by empirical evidence

Research questions

- $\tilde{\rho}_t(\tau) - \rho_{t+\tau} > 0$, does not always hold (Figure 1), so one needs to hedge the dispersion position (8)
- How to estimate and forecast the RC?
- Can we use these forecast to hedge the dispersion strategy?

Outline

1. Motivation ✓
2. Approximating RC with IC
3. A Dynamic Factor Model for Implied Correlation
4. Data
5. Estimation Results and Factor Modeling
6. Hedging the basket correlation
7. Conclusion

Implied volatility (IV)

We model and forecast RC with implied correlation (IC).
IC is a function of implied volatility (IV). Given the theoretical (model) price of an option V and the price observed on the market $\check{V}$, IV $\widehat{\sigma}$ can be found by solving:

$$V(\widehat{\sigma}) - \check{V} = 0.$$

IV contains incremental information beyond the historical estimate and outperforms it in forecasting future volatility, Christensen and Prabhala (1998), Fleming (1998), Blair et al. (2001)

Implied correlation (IC)

Applying (2) to IV of a basket $\widehat{\sigma}_B(\kappa,\tau)$ and its N constituents $\widehat{\sigma}_i(\kappa,\tau)$, $i \in \{1,\ldots,N\}$, we obtain the IC surface (ICS):

$$\widehat{\rho}(\kappa,\tau) = \frac{\widehat{\sigma}_B^2(\kappa,\tau) - \sum_i w_i^2 \widehat{\sigma}_i^2(\kappa,\tau)}{\sum_i \sum_{j \neq i} w_i w_j \widehat{\sigma}_i(\kappa,\tau)\widehat{\sigma}_j(\kappa,\tau)}, \tag{10}$$

where $\kappa = \dfrac{K_i}{S_i e^{r\tau}}$ is common moneyness of the options, τ - common time to maturity, r - the annualized continuously compounded risk-free interest rate, K_i - exercise price of the i-th option, S_i - current price of the i-th underlying.

S&P100 ICS: 20091210

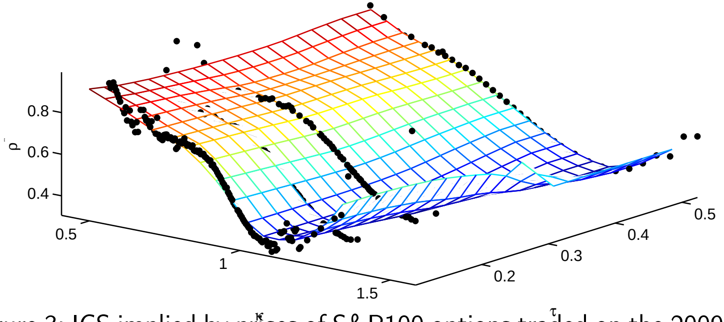

Figure 3: ICS implied by prices of S&P100 options traded on the 20091210, Nadaraya-Watson smoothing of 1-day data

Dynamic modeling of ICS

- Observe an ICS $\widehat{\rho}(\kappa_{t,j}, \tau_{t,j})$, $t = 1, \ldots, T$, $j = 1, \ldots, J_t$ (index of observations at day t)
- Apply Fisher's Z-transformation to obtain $Y_{t,j}$

  ▸ Fisher's Z-transformation
- Study the dynamics of $\{(Y_{t,j}, X_{t,j}), 1 \leq t \leq T, 1 \leq j \leq J_t\}$, where $X_{t,j} = (\kappa_{t,j}, \tau_{t,j})$

ICS with DSFM

Approximate $\mathsf{E}(Y_t|X_t)$ by the sum of $L+1$ smooth basis functions $m \stackrel{\text{def}}{=} \{m_0, \ldots, m_L\}^\top$ (factor loadings) weighted by time dependent coefficients $Z_t \stackrel{\text{def}}{=} (1, Z_{t,1}, \ldots, Z_{t,L})^\top$ (factors):

$$Y_{t,j} = m_0(X_{t,j}) + \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,j}) + \varepsilon_{t,j}, \tag{11}$$

2-step DSFM estimation procedure

1. estimate the FPCA covariance function

$$\psi(u, v) = \phi(u, v) - \mu(u)\mu(v) \tag{12}$$

   and take $\widehat{\mu}$ as $\widehat{m}_0$ and $\widehat{\gamma}_l$ as $\widehat{m}_l$, $l \in \{1, \ldots, L\}$, motivated by Hall et. al (2006).

2. estimate time series of factors $\widehat{Z}_t = (\widehat{Z}_{t1}, \ldots, \widehat{Z}_{tL})^\top$ by OLS.

1st step: estimation of space basis

1. estimate $\widehat{a}_\mu = \widehat{\mu}(u) = \widehat{\mu}(v)$:

$$\sum_{t=1}^{T}\sum_{j=1}^{J_t}\{Y_{t,j} - a_\mu - b_\mu(u - X_{t,j})\}^2 K\left(\frac{X_{t,j}-u}{h_\mu}\right)$$

2. estimate $\widehat{a}_\phi = \widehat{\phi}(u, v)$:

$$\sum_{t=1}^{T}\sum_{j,k:1\leq j\neq k\leq J_t}\{Y_{t,j}Y_{t,k} - a_\phi - b_{\phi,1}(u - X_{t,j}) - b_{\phi,2}(v - X_{t,k})\}^2$$
$$\times K\left(\frac{X_{t,j}-u}{h_\phi}\right) K\left(\frac{X_{t,k}-v}{h_\phi}\right)$$

3. use $\widehat{\mu}(u)$, $\widehat{\mu}(v)$ and $\widehat{\phi}(u, v)$ to compute (12) and take its eigenfunctions $\{\widehat{\gamma}_j\}_{j=1}^{L}$ corresponding to the L largest eigenvalues

Mean function with data

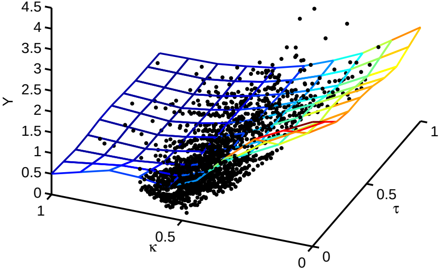

Figure 4: $\widehat{\mu}(u)$ of the DAX ICS with corresponding data points, estimated from November 2009 until October 2010 with $h_{\mu} = (h_{\mu,1}, h_{\mu,2})^{\top} = (0.12, 0.17)^{\top}$

Eigenfunctions of the covariance operator

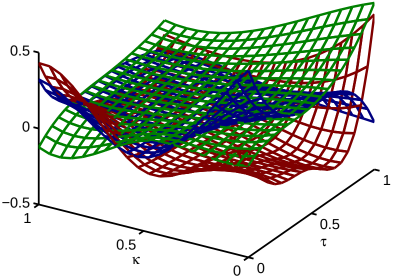

Figure 5: Three first eigenfunctions, $(\widehat{\gamma}_1,\ \widehat{\gamma}_2,\ \widehat{\gamma}_3)$, of the DAX ICS covariance operator $\widehat{\psi}$, estimated from November 2009 until October 2010 with $h_\mu = (h_{\mu,1}, h_{\mu,2})^\top = (0.12, 0.17)^\top$ and $h_\phi = (h_{\phi,1}, h_{\phi,2})^\top = (0.07, 0.085)^\top$

2nd step: estimation of factor series

Take $\widehat{m}$ from the 1st step and estimate $\widehat{Z}_t = (1, \widehat{Z}_{t,1}, \ldots, \widehat{Z}_{t,L})^\top$:

$$\widehat{Z}_t = \arg\min_{Z_t} \sum_{t=1}^{T} \sum_{j=1}^{J_t} \left\{ Y_{t,j} - Z_t^\top \widehat{m}(X_{t,j}) \right\}^2 \qquad (13)$$

IC, MFIC, RC summary statistics

Dispersion strategy from August 2010 to July 2012 on the German market represented by the DAX basket

- MFIC dataset: from daily variance swaps rates (Bloomberg) via (6), 20100802 - 20120801 (24 months), 515 trading days
- RC dataset: from daily stock returns (Bloomberg) via (7), 20100802 - 20120801 (24 months), 515 trading days
- IC dataset: from option prices (EUREX) via (10), 20100104 - 20120801 (31 months), 656 trading days, 135 obs./day

IC, MFIC, RC summary statistics

		Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt
IC	κ	0.80	1.20	0.98	0.98	0.09	0.06	2.06
	τ	0.02	0.96	0.24	0.17	0.19	1.37	4.39
	$\widehat{\rho}_t(\kappa,\tau)$	0.05	0.99	0.61	0.62	0.15	-0.27	2.61
MFIC	$\tilde{\rho}_t(0.25)$	0.44	0.97	0.65	0.65	0.08	0.06	0.16
	$\tilde{\rho}_t(0.5)$	0.49	1.47	0.70	0.69	0.08	1.81	12.13
	$\tilde{\rho}_t(1)$	0.56	1.08	0.74	0.74	0.09	0.77	0.67
RC	$\rho_{t+0.25}$	0.27	0.81	0.55	0.53	0.11	0.24	-0.80
	$\rho_{t+0.5}$	0.37	0.73	0.57	0.57	0.10	-0.02	-1.40
	ρ_{t+1}	0.43	0.65	0.59	0.60	0.05	-1.24	0.98

Table 1: IC from 20100104 to 20120801 (656 trading days, 135 obs./day), MFIC and RC from 20100802 to 20120801 (515 trading days). The figures are given after filtering and data preparation.

Market regime correction

- dependence of $\sigma_{B,t+\tau}$ and $\rho_{t+\tau}$ is stronger if the market volatility is high, empirical evidence
- not observed for $\widehat{\sigma}_{B,t}(\kappa,\tau)$ and $\widehat{\rho}_t(\kappa,\tau)$, empirical evidence
- based on regression results make a state-dependent correction of $\widehat{\rho}_t(\kappa,\tau)$:
  - if $\widehat{\sigma}_{B,t}(1,\tau) > 21$ (high volatility regime), then $\widehat{\rho}_t(\kappa,\tau) = 0.0091\widehat{\sigma}_{B,t}(\kappa,\tau)$

DSFM for DAX ICS 2010

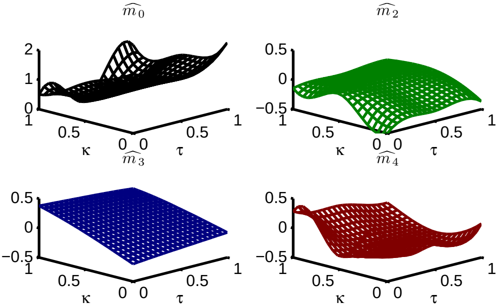

Figure 6: DAX ICS factor loadings $\widehat{m}_0, \widehat{m}_1, \widehat{m}_2, \widehat{m}_3$ from Nov. 2009 to Oct. 2010

DSFM for DAX ICS 2010

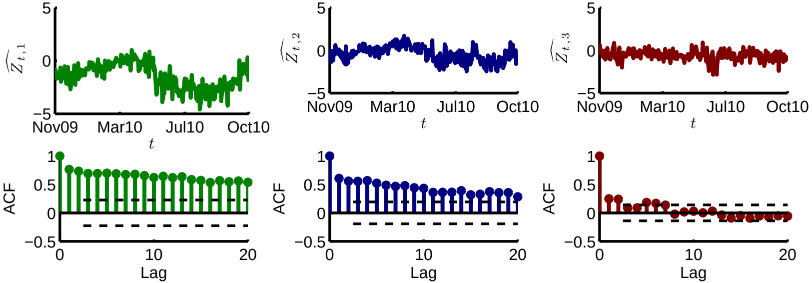

Figure 7: DAX ICS factors $\widehat{Z}_{t,1}, \widehat{Z}_{t,2}, \widehat{Z}_{t,3}$ and their ACFs from Nov. 2009 to Oct. 2010 ($\widehat{Z}_{t,0} \stackrel{\text{def}}{=} 1$)

Hedging dispersion strategy with DSFM, "naïve" hedge

At $t+\tau-\Delta t$ make a Δt-days ahead DSFM forecast $\widehat{\rho}_{t+\tau}(1, t+\tau)$ and use it as $\rho_{t+\tau}$ in (9) to obtain the value of the hedge (opposite) position to be held until $t+\tau$:

$$D^h_{t+\tau} = \sum_i \sum_{j\neq i} w_i w_j \tilde{\sigma}_{t,i}(\tau)\tilde{\sigma}_{t,j}(\tau)\left\{\tilde{\rho}_t(\tau) - \widehat{\rho}_{t+\tau}(1, t+\tau)\right\}, \quad (14)$$

then the relative hedging error

$$\varepsilon^h_{t+\tau} = \frac{D^h_{t+\tau} - D_{t+\tau}}{D_{t+\tau}} = -\frac{\widehat{\rho}_{t+\tau}(1, t+\tau) - \rho_{t+\tau}}{\tilde{\rho}_t(\tau) - \rho_{t+\tau}}, \quad (15)$$

$\varepsilon^h_{t+\tau} < 0 (> 0)$ means that (14) under-(over-)estimates (8)

Performance of "naïve" hedge, $\tau = 0.083$

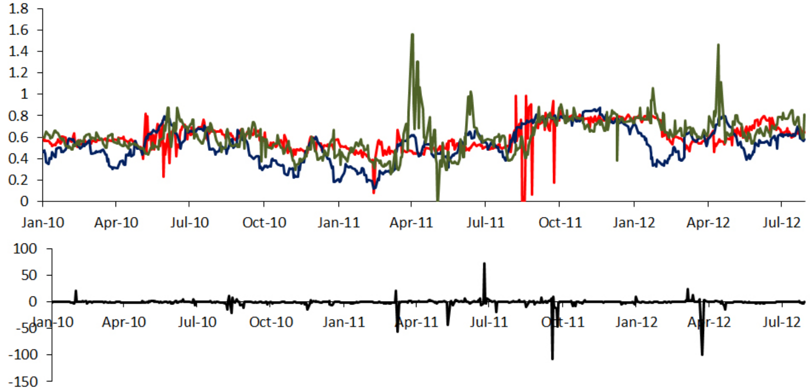

Figure 8: $\widehat{\rho}_{t+0.083}(1, t+0.083)$, $\rho_{t+0.083}$, $\tilde{\rho}_t(0.083)$ and $\varepsilon^h_{t+0.083}$, daily moving window estimates for $t+\tau$ from 20100104 until 20120801 (660 trading days), DSFM: $T = 50$, $J = 49$, $h_\mu = (0.122, 0.128)^\top$, $h_\phi = (0.153, 0.168, 0.153, 0.168)^\top$ by cross-validation

Performance of "naïve" hedge, $\tau = 0.25$

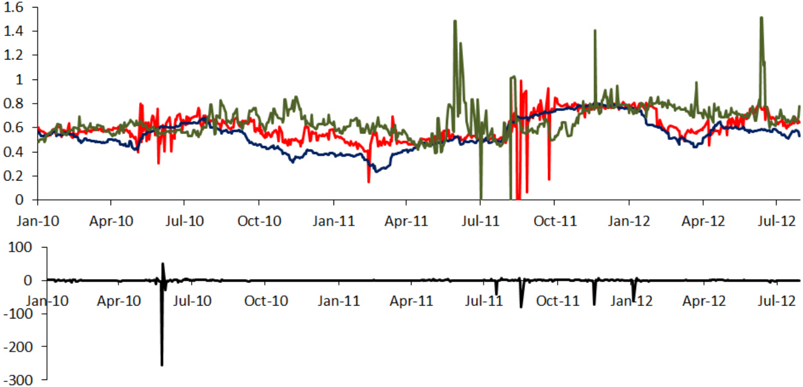

Figure 9: $\widehat{\rho}_{t+0.25}(1, t+0.25)$, $\rho_{t+0.25}$, $\tilde{\rho}_t(0.25)$ and $\varepsilon^h_{t+0.25}$, daily moving window estimates for $t+\tau$ from 20100104 until 20120801 (660 trading days), DSFM: $T = 50$, $J = 49$, $h_\mu = (0.122, 0.128)^\top$, $h_\phi = (0.153, 0.168, 0.153, 0.168)^\top$ by cross-validation

Performance of "naïve" hedge, $\tau = 0.5$

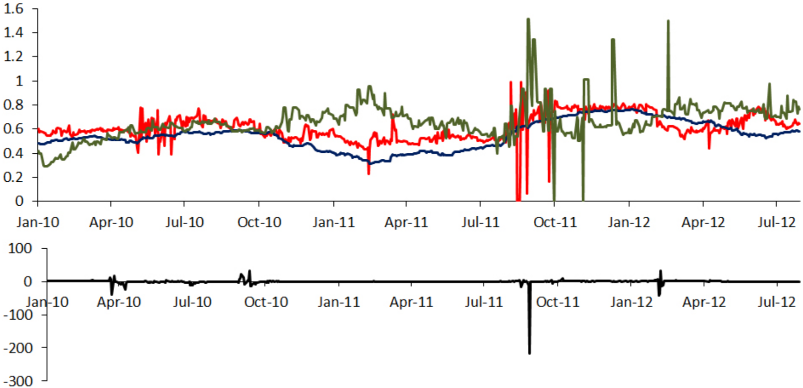

Figure 10: $\widehat{\rho}_{t+0.5}(1, t+0.5)$, $\rho_{t+0.5}$, $\tilde{\rho}_t(0.5)$ and $\varepsilon^h_{t+0.5}$, daily moving window estimates for $t+\tau$ from 20100104 until 20120801 (660 trading days), DSFM: $T = 50$, $J = 49$, $h_\mu = (0.122, 0.128)^\top$, $h_\phi = (0.153, 0.168, 0.153, 0.168)^\top$ by cross-validation

Performance of "naïve" hedge, $\tau = 1$

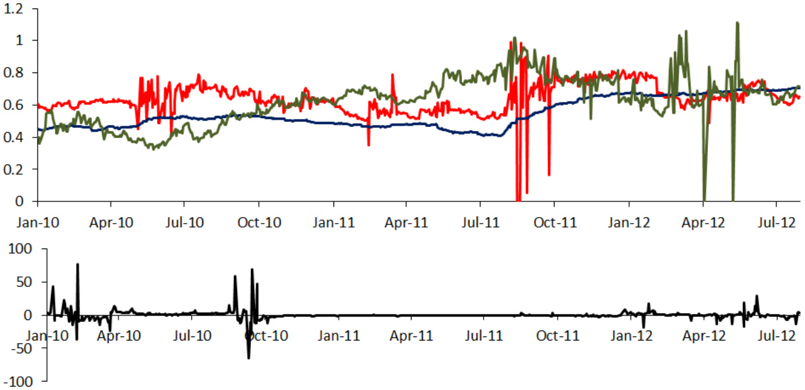

Figure 11: $\widehat{\rho}_{t+1}(1, t+1)$, ρ_{t+1}, $\tilde{\rho}_t(1)$ and ε^h_{t+1}, daily moving window estimates for $t+\tau$ from 20100104 until 20120801 (660 trading days), DSFM: $T = 50$, $J = 49$, $h_\mu = (0.122, 0.128)^\top$, $h_\phi = (0.153, 0.168, 0.153, 0.168)^\top$ by cross-validation

"Naïve" hedge summary statistics

τ	Min.	Max.	Mean.	Median	Stdd.	Skew.	Kurt.
0.083	-108.04	72.30	-1.14	-0.71	8.00	-6.61	100.49
0.25	-255.48	49.53	-1.20	-0.41	11.49	-17.58	372.33
0.5	-216.04	32.78	-0.74	-0.30	9.37	-18.66	425.86
1	-64.84	76.59	-0.01	-0.38	7.47	2.74	46.85

Table 2: Performance of "naïve" hedge, summary statistics for $\varepsilon^h_{t+\tau}$ from 20100101 until 20120801

"Advanced" hedge

- predict $\rho_{t+\tau}$ with DSFM $\widehat{\rho}_{t+\tau}(1, t+\tau)$
- if $\widehat{\rho}_{t+\tau}(1, t+\tau) \geq \tilde{\rho}_t(\tau)$ (DSFM predicts loss in dispersion strategy), take an offsetting (with negative sign) position in (14)
- if $\widehat{\rho}_{t+\tau}(1, t+\tau) < \tilde{\rho}_t(\tau)$ (DSFM predicts gain in dispersion strategy), don't hedge
- payoff of the "advanced" strategy at $t+\tau$

$$D^{adv}_{t+\tau} = \begin{cases} D_{t+\tau} - D^{h}_{t+\tau} & , \text{if} \quad \widehat{\rho}_{t+\tau}(1, t+\tau) \geq \tilde{\rho}_t(\tau) \\ D_{t+\tau} & , \text{if} \quad \widehat{\rho}_{t+\tau}(1, t+\tau) < \tilde{\rho}_t(\tau) \end{cases} \tag{16}$$

"Naïve" hedge, "advanced" hedge, no hedge $\tau = 0.083$

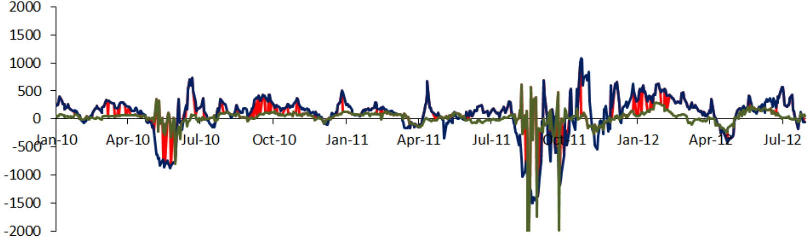

Figure 12: Payoffs of a 1-month dispersion strategy from 20100104 till 20120801 (660 trading days): $D_{t+\tau}$ (no hedge), $D_{t+\tau} - D^{h}_{t+\tau}$ ("naïve" hedge), $D^{adv}_{t+\tau}$ ("advanced" hedge)

"Naïve" hedge, "advanced" hedge, no hedge , $\tau = 0.25$

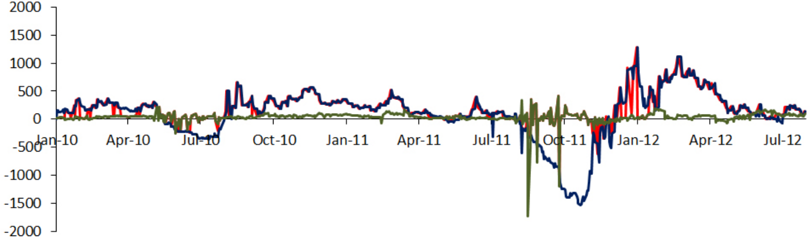

Figure 13: Payoffs of a 3-month dispersion strategy from 20100104 till 20120801 (660 trading days): $D_{t+\tau}$ (no hedge), $D_{t+\tau} - D^{h}_{t+\tau}$ ("naïve" hedge), $D^{adv}_{t+\tau}$ ("advanced" hedge)

"Naïve" hedge, "advanced" hedge, no hedge, $\tau = 0.5$

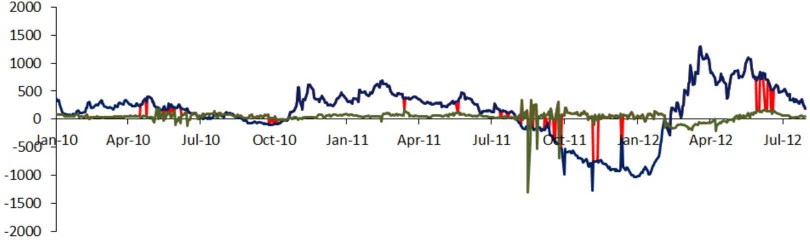

Figure 14: Payoffs of a 6-month dispersion strategy from 20100104 till 20120801 (660 trading days): $D_{t+\tau}$ (no hedge), $D_{t+\tau} - D^{h}_{t+\tau}$ ("naïve" hedge), $D^{adv}_{t+\tau}$ ("advanced" hedge)

"Naïve" hedge, "advanced" hedge, no hedge, $\tau = 1$

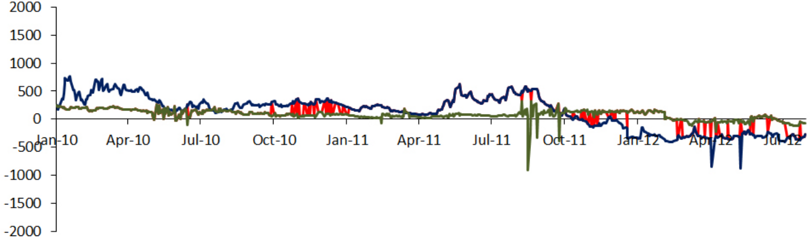

Figure 15: Payoffs of a 1-year dispersion strategy from 20100104 till 20120801 (660 trading days): $D_{t+\tau}$ (no hedge), $D_{t+\tau} - D^h_{t+\tau}$ ("naïve" hedge), $D^{adv}_{t+\tau}$ ("advanced" hedge)

Strategy	τ	Min.	Max.	Mean.	Stdd.
$D_{t+\tau}$	0.083	-1502.58	1080.23	87.09	356.94
(no hedge)	0.25	-1531.94	1282.31	101.92	440.54
	0.5	-1270.90	1301.28	136.91	456.75
	1	-872.76	760.92	134.26	299.01
$D_{t+\tau} - D^h_{t+\tau}$	0.083	-3237.72	617.40	15.35	203.09
("naïve" hedge)	0.25	-1726.53	413.28	35.90	110.14
	0.5	-1301.47	344.91	41.13	91.91
	1	-914.27	327.03	79.62	93.14
$D^{adv}_{t+\tau}$	0.083	-1375.99	1011.38	100.93	256.50
("advanced" hedge)	0.25	-1137.79	1282.31	195.09	248.41
	0.5	-760.85	1301.28	231.35	281.66
	0.083	-367.89	623.38	123.04	190.80

Table 3: Summary statistics for $D_{t+\tau}$ (no hedge), $D_{t+\tau} - D^h_{t+\tau}$ ("naïve" hedge), $D^{adv}_{t+\tau}$ ("advanced" hedge) from 20100101 until 20120801, best results (highest min, max,mean and smallest stdd.) are marked red

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Bibliography

- Blair, B.J., Poon, S. and Taylor S. J. (2001)
  Forecasting S&P 100 volatility: the incremental information content of implied volatilities and high-frequency index returns
  Journal of Econometrics, 105(1): 5 - 26

- Bourgoin, F. (2001)
  Stress-testing correlations: An application to portfolio risk management
  in Dunis, C. and Timmermann, A. and Moody, J.(eds), *Developments in forecast combination and portfolio choice*, New York: Wiley

- Carr, P. and Madan D.
  Towards a theory of volatility trading
  Volatility, 417-427, (2002)

Carr, P. and Wu L.
Variance Risk Premiums
The Review of Financial Studies, 22(3): 1311-1341, (2009)

Christensen, B.J. and Prabhala, N.R. (1998)
The relation between implied and realized volatility
Journal of Financial Economics, 50(2): 125-150

Fleming, J. (1998)
The quality of market volatility forecasts implied by S&P 100 index option prices
Journal of Empirical Finance, 5(4): 317 - 345

Hall, P., Müller, H. and Wang, J. (2006)
Properties of principal component methods for functional and longitudinal data analysis
Ann. Statist., 34(3): 1493-1517

Härdle, W. and Simar, L. (2012)
Applied Multivariate Statistical Analysis, 3rd edn,
Springer Verlag, Heidelberg

Chicago Board Options Exchange, Incorporated (2009)
CBOE S&P 500 Implied Correlation Index White Paper
http://www.cboe.com/micro/impliedcorrelation/default.aspx

Fisher's Z-transformation

$Y_{t,j} \stackrel{\text{def}}{=} T\left\{\widehat{\rho}(\kappa_{t,j}, \tau_{t,j})\right\}$, where T is Fisher's Z-transformation:

$$T(u) \stackrel{\text{def}}{=} \frac{1}{2} \log \frac{1+u}{1-u} \tag{17}$$

Härdle and Simar (2012)

Back

RV and MFIV

- RV is the variance of the asset defined at $t+\tau$ over period from t to $t+\tau$:

$$\sigma^2_{t+\tau} = \tau^{-1} \sum_{i=252t}^{252(t+\tau)} \left(\log \frac{S_i}{S_{i-1}} \right)^2 \tag{18}$$

- MFIV is the risk-neutral expectation (at t) of the integrated volatility from t to $t+\tau$, Britten-Jones and Neuberger (2000), ▸ details:

$$\tilde{\sigma}_t{}^2(\tau) = \frac{2}{\tau} e^{r\tau} \left\{ \int_0^{S_t} \frac{P_t(K,\tau)dK}{K^2} + \int_{S_t}^{\infty} \frac{C_t(K,\tau)dK}{K^2} \right\}, \tag{19}$$

where $P_t(K,\tau)$ $\{C_t(K,\tau)\}$ put (call) with strike K and maturity τ traded at t, S_t price of the asset in t, r risk free rate, t and τ are given in fractions of a year, ▸ Back

Variance swap

- forward contract opened at t that buys RV defined at $t+\tau$
- at $t+\tau$ pays the difference between RV and MFIV (multiplied by notional N_{var})

$$\left\{\sigma^2_{t+\tau} - \tilde{\sigma}^2_t(\tau)\right\} N_{var}, \tag{20}$$

where $\sigma^2_{t+\tau}$ variable leg of the variance swap defined by (18), $\tilde{\sigma}^2_t(\tau)$ fixed leg (strike) defined by (19)

Back

Variance risk premium (VRP)

Literature findings for $\sigma^2_{t+\tau} - \tilde{\sigma}^2_t(\tau)$ variance risk premium ($VRP_{t+\tau}$), Carr and Wu (2009), on US market:

- $VRP < 0$ for major US stock indexes, from January 1996 until December 2003, Carr and Wu (2009)
- $VRP < 0$ for S&P100 and constituents (less pronounced), from 1991 until 1995, Bakshi, Kapadia and Madan (2003)
- $VRP < 0$ for S&P100, but $VRP = 0$ for most constituents, from January 1996 until December 2003, Driessen and Vilkov (2009)

Back

Variance risk premium (VRP)

Empirical findings for $\sigma^2_{t+\tau} - \tilde{\sigma}^2_t(\tau)$ variance risk premium ($VRP_{t+\tau}$), Carr and Wu (2009), on German market:

- the most recent sample: German market August 2010 until August 2012
- $\sigma^2_{t+\tau} - \tilde{\sigma}^2_t(\tau) < 0$ on average for DAX and all constituents
- t-test $H_0 : VRP = 0$ ($H_1 : VRP < 0$) is strongly rejected for DAX index, but
- for 5 out of 23 stocks we cannot reject the H_0 at 5% significance level

Back

DAX constituents VRP

	$\tau = 0.25$	$\tau = 0.5$	$\tau = 1$
Allianz SE	0.0563	0.0526	0.0225
E.ON AG	0.2519	0.3176	0.0814
Metro AG	0.2931	0.1884	0.0196
RWE AG	0.6322	0.5655	0.0707
ThyssenKrupp AG	0.1964	0.0700	0.0100

Table 4: The results of t-test for H_0 that on average $RV = MFIV$ against the alternative $RV < MFIV$ of stocks for which the the H_0 is not rejected at 5% significance level. Results are presented for DAX index and 23 selected constituent stocks computed over the time period 20100802 - 20120801 for 3 different maturities/estimation windows: $\tau = 0.25, 0.5, 1$)

DAX variance risk premium

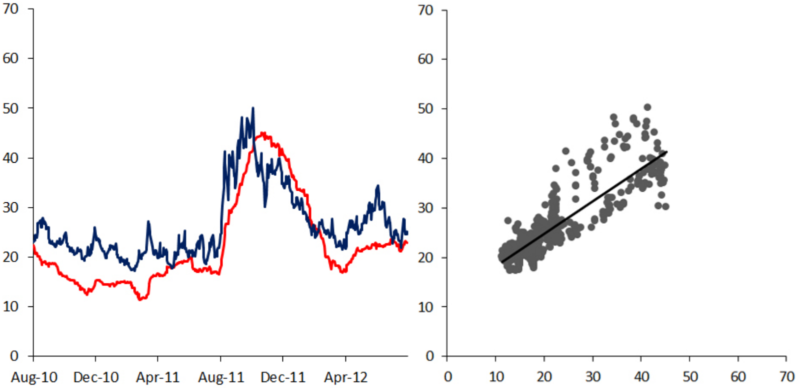


Figure 16: Left panel: $\tilde{\sigma}_{t,B}(0.083)$ vs $\sigma_{t+0.083,B}$ and $VRP_{t+0.083}$ at $t+\tau$ from 20090901 till 20120810, right panel (scatter plot): $\tilde{\sigma}_{t,B}(0.083)$ (vertical axis) vs $\sigma_{t+0.083,B}$ (horizontal axis) ▸ Back

RWE variance risk premium

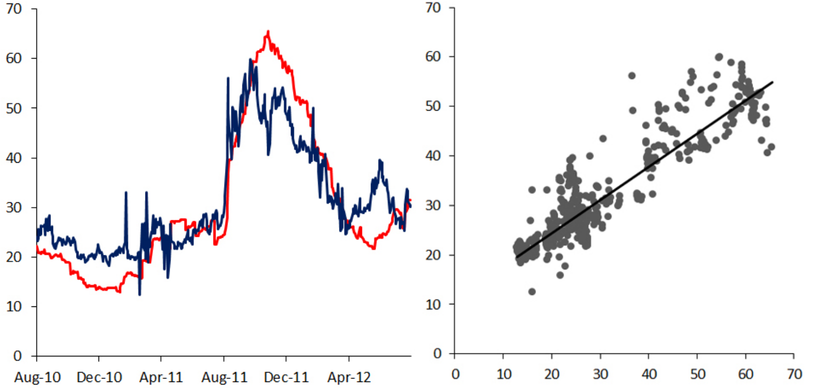


Figure 17: Left panel: $\tilde{\sigma}_{t,RWE}(0.083)$ vs $\sigma_{t+0.083,RWE}$ and $VRP_{t+0.083}$ at $t+\tau$ from 20090901 till 20120810, right panel (scatter plot): $\tilde{\sigma}_{t,RWE}(0.083)$ (vertical axis) vs $\sigma_{t+0.083,RWE}$ (horizontal axis) Ba

Switch point selection for correlation regimes

τ	$\sigma_{B,t+\tau}$	$\rho_{t+\tau}$	Slope 1	Slope 2
0.083	20.24	0.5917	0.0361	0.0085
0.25	20.34	0.5728	0.0336	0.0093
0.5	22.42	0.6008	0.0286	0.0094
Average	21.00	0.5884	0.0328	0.0091

Table 5: Segmented linear regression of $\rho_{t+\tau}$ on $\sigma_{B,t+\tau}$ with one break point, $\tau = 0083, 0.25, 0.5$ for $t+\tau$, from 20100104 till 20120801. We fit a segmented linear regression with one break point, as described in Muggeo (2003), Back to regime correction

DAX $\sigma_{B,t+0.083}$ vs $\rho_{t+0.083}$

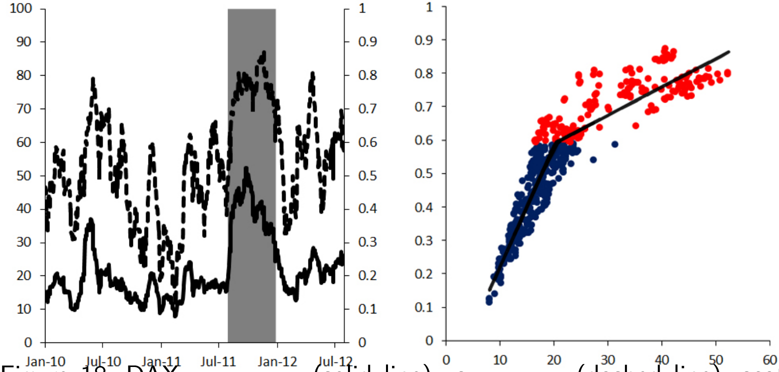


Figure 18: DAX $\sigma_{B,t+0.083}$ (solid line) vs $\rho_{t+0.083}$ (dashed line), scatter plot $\sigma_{B,t+0.083}$ vs $\rho_{t+0.083}$, for $t + 0.083$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back: ▸ Introduction ▸ Regime correction

DAX $\sigma_{B,t+0.25}$ vs $\rho_{t+0.25}$

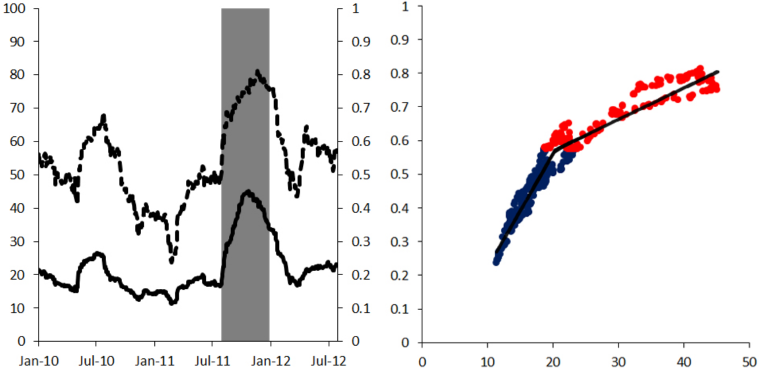
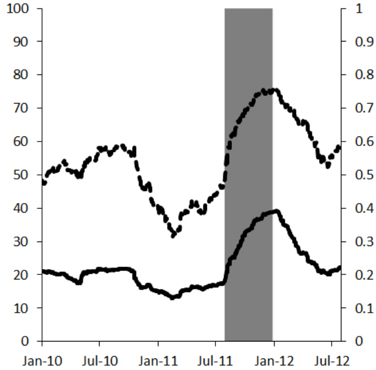


Figure 19: DAX $\sigma_{B,t+0.25}$ (solid line) vs $\rho_{t+0.25}$ (dashed line), scatter plot $\sigma_{B,t+0.25}$ vs $\rho_{t+0.25}$, for $t+0.25$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back: Introduction Regime correction

DAX $\sigma_{B,t+0.5}$ vs $\rho_{t+0.5}$



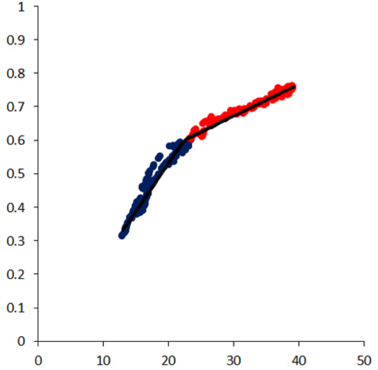


Figure 20: DAX $\sigma_{B,t+0.5}$ (solid line) vs $\rho_{t+0.5}$ (dashed line), scatter plot $\sigma_{B,t+0.5}$ vs $\rho_{t+0.5}$, for $t+0.5$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back: Introduction Regime correction

DAX $\widehat{\sigma}_{t,B}(1, 0.083)$ vs $\widehat{\rho}_t(1, 0.083)$

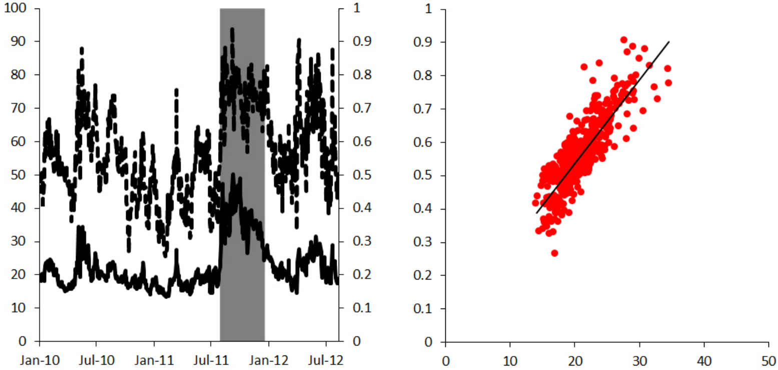


Figure 21: ATM DAX $\widehat{\sigma}_{B,t}(1, 0.083)$ (solid line) vs $\widehat{\rho}_t(1, 0.083)$ (dashed line), scatter plot $\widehat{\sigma}_{B,t}(1, 0.083)$ vs $\widehat{\rho}_t(1, 0.083)$, for $t + 0.083$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back:

Introduction Regime correction

DAX $\widehat{\sigma}_{t,B}(1, 0.25)$ vs $\widehat{\rho}_t(1, 0.25)$

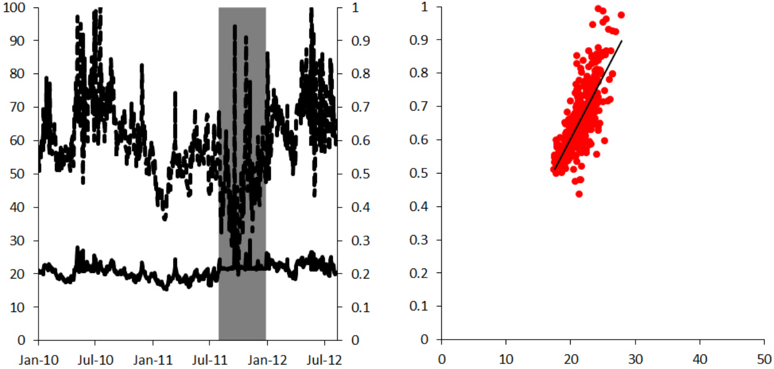


Figure 22: ATM DAX $\widehat{\sigma}_{B,t}(1, 0.25)$ (solid line) vs $\widehat{\rho}_t(1, 0.25)$ (dashed line), scatter plot $\widehat{\sigma}_{B,t}(1, 0.25)$ vs $\widehat{\rho}_t(1, 0.25)$, for $t + 0.25$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back: ▸ Introduction

▸ Regime correction

DAX $\widehat{\sigma}_{t,B}(1, 0.5)$ vs $\widehat{\rho}_t(1, 0.5)$

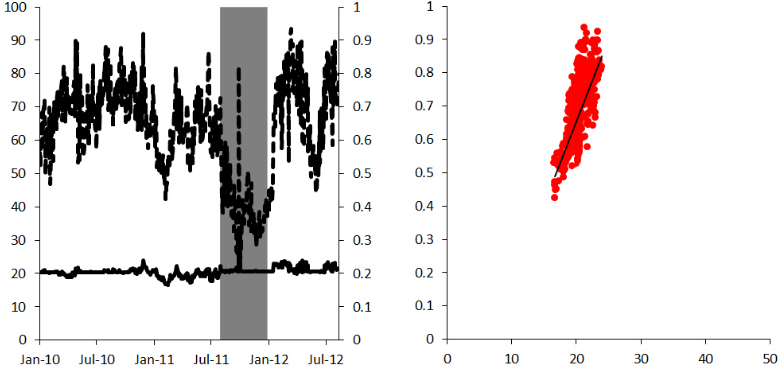


Figure 23: ATM DAX $\widehat{\sigma}_{B,t}(1, 0.5)$ (solid line) vs $\widehat{\rho}_t(1, 0.5)$ (dashed line), scatter plot $\widehat{\sigma}_{B,t}(1, 0.5)$ vs $\widehat{\rho}_t(1, 0.5)$, for $t + 0.5$ from 20100104 till 20120801, shaded area: Aug 2011 market fall, back: Introduction
Regime correction

Log contract

Define

$$f(S_t) = \frac{2}{T}\left\{\log\frac{S_0}{S_t} + \frac{S_t}{S_0} - 1\right\} \tag{21}$$

derivatives:

$$f'(S_t) = \frac{2}{T}\left(\frac{1}{S_0} - \frac{1}{S_t}\right) \tag{22}$$

and

$$f''(S_t) = \frac{2}{TF_t^2} \tag{23}$$

observe $f(S_0) = 0$ Back

Itô's lemma

$$f(S_t) = f(S_0) + \int_0^T f'(S_t)dS_t + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma_t^2 dt \tag{24}$$

Substituting (22), (23):

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\left(\log\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) - \tag{25}$$

$$-\frac{2}{T}\int_0^T \left(\frac{1}{S_0} - \frac{1}{S_t}\right) dS_t$$

Back

Equation (25) gives the value of σ_R^2 as a sum of:

$$\frac{2}{T}\int_0^T \left(\frac{1}{S_0} - \frac{1}{S_t}\right) dS_t$$

(continuously rebalanced position in underlying stock) and

$$f(S_T) = \frac{2}{T}\left(\log\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) \tag{26}$$

(**log contract**, static position).

Back

Carr and Madan (2002) represent any twice differentiable payoff function $f(S_T)$:

$$f(S_T) = f(k) + f'(k)\left\{(S_T - k)^+ - (k - S_T)^+\right\} \tag{27}$$

$$+ \int_0^k f''(K)(K - S_T)^+ dK$$

$$+ \int_k^\infty f''(K)(S_T - K)^+ dK$$

where k is an arbitrary number. Back

Applying (27) to (26) with $k = S_0$ gives

$$\log\left(\frac{S_0}{S_T}\right) + \frac{S_T}{S_0} - 1 = \tag{28}$$

$$= \int_0^{S_0} K^{-2}(K - S_T)^+ dK + \int_{S_0}^{\infty} K^{-2}(S_T - K)^+ dK$$

a portfolio of OTM puts and calls weighted by K^{-2}. Back

What are the costs of this strategy? The strike K^2_{var} of a variance swap is calculated via the risk-neutral expectation:

$$K^2_{var} = \frac{2}{T} e^{rT} \int_0^{S_0} K^{-2} P_0(K) dK + \frac{2}{T} e^{rT} \int_{S_0}^{\infty} K^{-2} C_0(K) dK \quad (29)$$

where P_0 (C_0) - value of a put (call) option at $t = 0$. Back