Adaptive Forward Intensities

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Credit Risk Modeling

- Structural Approach
  - KMV, Merton, Black-Scholes model
- Reduced-form approach
  - Discriminant Analysis, logit, probit ▸ Detail
  - SVM, ANN
  - Duration analysis, Shumway (2001)
  - Doubly stochastic Poisson intensity, Duffie et al. (2007) ▸ DSW
- Compromising approach
  - Forward intensity

CVI, VIX, S&P500

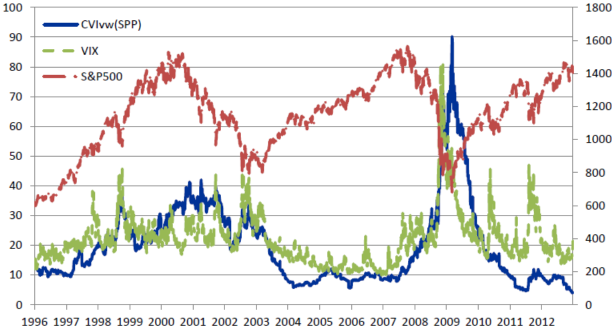


Figure 1: CVI_{vw}(SPP), S&P500 volatility index (VIX), and S&P500 index. RMI (2013)

How does it work ?

Poisson process

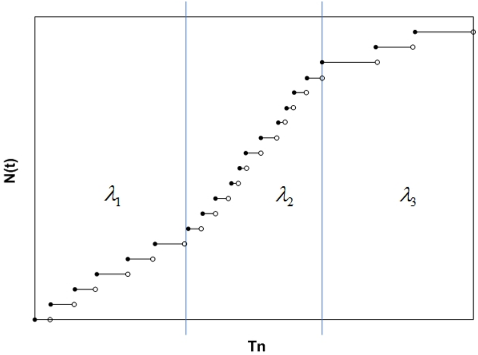


Figure 2: Sample path of Poisson process. Intensity react to $\lambda = \lambda_{it}$ by state variable X_{it}.

Forward Exit Intensity

$F_{it}(\tau)$ is cdf of default/delisted firm at $t + \tau$

Survival probability in time period $[t, t + \tau]$ ▸ intensity

$$
\begin{aligned}
1 - F_{it}(\tau) &= \exp\left\{ -\int_{t}^{t+\tau} \lambda_{is} ds \right\} \\
&\stackrel{\text{def}}{=} \exp\left\{ -\psi_{it}(\tau)\tau \right\}
\end{aligned}
$$

with

$$
\psi_{it}(\tau) = -\frac{\log\{1 - F_{it}(\tau)\}}{\tau} \tag{1}
$$

Forward Exit Intensity

Forward intensity evaluated at τ

$$g_{it}(\tau) \stackrel{\text{def}}{=} \frac{F'_{it}(\tau)}{1 - F_{it}(\tau)} = \psi_{it}(\tau) + \psi'_{it}(\tau)\tau \qquad (2)$$

▸ detail

Therefore

$$\psi_{it}(\tau)\tau = \int_0^{\tau} g_{it}(s)ds \qquad (3)$$

▸ detail

Combined exit intensities

Default and other exit are governed by two independent doubly stochastic Poisson process with different intensities λ_{it} and ϕ_{it}

▸ Double Poisson process

Now call $g_{it}(\tau)$ as combined exit intensity (default and other exit)

How to define forward default intensity, $f_{it}(\tau)$?

Forward Default Intensity

No default till s, default probability over $[t, t+\tau]$

$$\int_0^\tau \exp\{-\psi_{it}(s)s\}\, f_{it}(s)\, ds \tag{4}$$

with forward default intensity

$$f_{it}(\tau) \stackrel{\text{def}}{=} e^{-\psi_{it}(\tau)\tau} \lim_{\Delta t \to 0} \frac{\mathsf{P}_t\left(t+\tau < \tau_{Di} = \tau_{Ci} \le t+\tau+\Delta t\right)}{\Delta t}$$

$$= e^{-\psi_{it}(\tau)\tau} \lim_{\Delta t \to 0} \frac{\mathsf{E}_t\left[\int_{t+\tau}^{t+\tau+\Delta t} \exp\left\{-\int_t^s (\lambda_{iu} + \phi_{iu}) du\right\} \lambda_{is} ds\right]}{\Delta t}$$

with

$\mathsf{P}_t(\cdot)$ denote the time-t conditional probability

τ_{Di} is default time of the i-th firm, and

τ_{Ci} is combined exit time, $\tau_{Ci} < \tau_{Di}$

Forward Intensities approach

Define new $f_{it}(\tau)$ and $g_{it}(\tau)$ as functions of state variables X_{it} and evaluated at horizon τ

Let $X_{it} = (x_{it,1}, x_{it,2}, \dots, x_{it,p}) = (W_t, U_{it})$ is state variable that affects the forward intensities for the i-th firm

It may include macroeconomic factors (W_t) as share common elements and firm-specific attributes (U_{it})

No need to specify the dynamic of X_{it} to avoid estimating the model of X_{it}

Forward Intensities approach

Function $f_{it}(\tau)$ can be all kinds of function of X_{it} as long as $f_{it}(\tau) > 0$ and $g_{it}(\tau) \geq f_{it}(\tau)$,

$$
\begin{aligned}
f_{it}(\tau) &= \exp\left\{\alpha_0(\tau) + \alpha_1(\tau) x_{it,1} + \ldots + \alpha_p(\tau) x_{it,p}\right\} \\
g_{it}(\tau) &= f_{it}(\tau) + \exp\left\{\beta_0(\tau) + \beta_1(\tau) x_{it,1} + \ldots + \beta_p(\tau) x_{it,p}\right\}
\end{aligned}
$$

where $f_{it}(\tau)$ and $g_{it}(\tau)$ do not need to share the same set of X_{it} by setting some coefficients to zero

What can go wrong ?

- Given fixed τ,
  - forward intensity model are time homogeneous, i.e. $f_{it}(\tau)$ and $g_{it}(\tau)$ follow the same structural equation at each t, $\alpha_j(\tau)$ and $\beta_j(\tau)$ constant over the time
  - If true parameters are constant, in which period it hold ?
- For prediction horizon $\tau > 1$,
  - Overlapping pseudo-likelihood makes the inference is not clear
  - Default correlation through intensities are conspicuously absent

What can go wrong ?

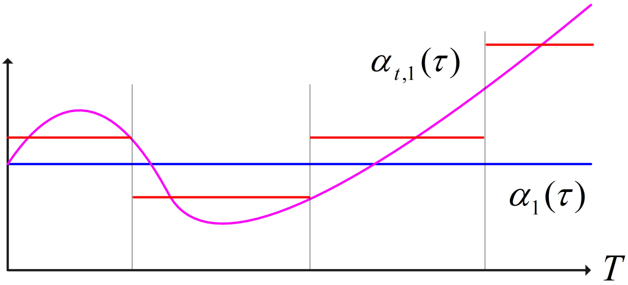


Figure 3: Adaptive forward intensity with parameters vary over the time approximated by piecewise constant in homogeneous interval

Outline

1. Motivation ✓
2. Local Parametric Approach
   - Local Change Point detection
3. Output

Local Parametric Approach (LPA)

- Involving processes that are stationary only locally
  - Consider just most recent data
  - Imply subsetting of data using some localization scheme that can itself either global or local and adaptive

- LPA methods
  - Mercurio and Spokoiny (2004): Local Change Point (LCP)
  - Katkovnik and Spokoiny (2008): Local Model Selection (LMS), also known as Intersection of Confidence Intervals (ICI)
  - Belomestny and Spokoiny (2007): Stagewise Aggregation (SA)

Objective: Localising adaptive forward intensity

For each t exist interval $I_k = [t - m_k + 1, t]$ in which forward intensity model described the process adequately (parameters are constant)

Approximate adaptive $\alpha_{t,j}(\tau)$ and $\beta_{t,j}(\tau)$ by constant in I_k,
$\alpha_{t \in I_k, j}(\tau) = \alpha_{I_k, j}(\tau)$ and $\beta_{t \in I_k, j}(\tau) = \beta_{I_k, j}(\tau)$

Estimation windows with potentially varying length. Find the longest stable (homogeneity) interval

Allow the structural breaks and jumps in parameters value

Interval Selection

Given time t, go back and split time series into K intervals,

$$
\begin{array}{ccccccc}
I_K & \supset \cdots \supset & I_k & \supset \cdots \supset & I_1 & \supset & I_0 \\
\widetilde{\boldsymbol{\theta}}_K(\tau) & \cdots & \widetilde{\boldsymbol{\theta}}_k(\tau) & \cdots & \widetilde{\boldsymbol{\theta}}_1(\tau) & & \widetilde{\boldsymbol{\theta}}_0(\tau)
\end{array}
$$

where $I_k = [t - m_{k+1} + 1, t]$, with length of interval $|I_k| = m_k$

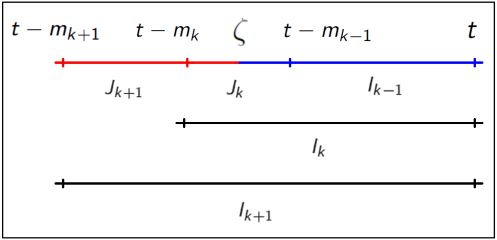
Example: Fix t and τ,

$I_k = [t - m_k + 1, t], \quad m_k = [m_0 c^k], \quad c > 1$

$\{m_k\}_{k=1}^K = \{24, 30, 38, \ldots, 264\}$ months, $m_0 = 24,\ c = 1.25,\ K = 11$

Sequential Test ($k = 1, \ldots, K$), fix t & τ

H_0 : parameter homogeneity within I_k
H_1 : change point within I_k



$$T_{k,\tau} = \sup_{\zeta \in J_k} \left\{ L_{A_{k,\zeta,\tau}} \left(\widetilde{\theta}_{A_{k,\zeta}}(\tau) \right) + L_{B_{k,\zeta,\tau}} \left(\widetilde{\theta}_{B_{k,\zeta}}(\tau) \right) - L_{I_{k+1},\tau} \left(\widetilde{\theta}_{I_{k+1}}(\tau) \right) \right\},$$

Detail

with $J_k = I_k \setminus I_{k-1}$, $A_{k,\zeta} = [t - m_{k+1}, \zeta]$ and $B_{k,\zeta} = (\zeta, t]$

Algorithm: Find $\zeta \in I_k$

1. Fix t and τ, determine estimate $\widehat{\boldsymbol{\theta}}(\tau) = \widetilde{\boldsymbol{\theta}}_0(\tau)$
   $\widetilde{\boldsymbol{\theta}}_0(\tau)$ is estimates obtained from pseudo-likelihood (PMLE)
2. Increase interval to I_k, $k > 1$. Obtain $\widetilde{\boldsymbol{\theta}}_k(\tau)$
3. Compare test statistic $T_{k,\tau}$ with critical value $\mathfrak{z}_{k,\tau}$
   If $T_{k,\tau}$ is accepted go to step 4, otherwise go to step 5
4. Let $\widehat{\boldsymbol{\theta}}_k(\tau) = \widetilde{\boldsymbol{\theta}}_k(\tau)$, set $k = k + 1$, repeat step 2
5. Detect change point ζ in J_k

Note:

Rejecting H_0 at $k = 1$, $\widehat{\theta}(\tau)$ equals PMLE at I_0

If the algorithm goes until K, $\widehat{\theta}(\tau)$ equals PMLE at I_K

Pseudo-MLE

Pseudo-maximum likelihood estimates (PMLEs) of $\boldsymbol{\theta}_k$

$$\widetilde{\boldsymbol{\theta}}_k(\tau) = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L_{k,\tau}\left(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k; \tau_C, \tau_D, X\right) \tag{5}$$

where $L_{k,\tau}(\cdot)$ is log likelihood for interval I_k evaluated at τ

$$L_{k,\tau}(\cdot) = \prod_{i=1}^{N} \prod_{k=0}^{K} \prod_{t\in I_k} L_{I_k,\tau,i,t}\left(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k; \tau_C, \tau_D, X\right) \mathbf{I}\left\{t \in I_k\right\} \tag{6}$$

where N is number of companies in t. ▸ Likelihood

Consistency check, critical value $\mathfrak{z}_\tau$

- Lepski et al. (1997):
  - Test statistic, $\widetilde{\boldsymbol{\theta}}_k(\tau) - \widetilde{\boldsymbol{\theta}}_\ell(\tau)$, interval $\ell < k$

- Polzehl and Spokoiny (2006):
  - Apply localized likelihood ratio type test
  - Check whether $\widetilde{\boldsymbol{\theta}}_k(\tau)$ in confidence sets $\mathcal{E}_\ell(\mathfrak{z}_\tau)$ of $\widetilde{\boldsymbol{\theta}}_\ell(\tau)$, $\ell < k$

$$\mathcal{E}_\ell(\mathfrak{z}_\tau) \stackrel{\text{def}}{=} \left\{ \boldsymbol{\theta}^* : L(\widetilde{\boldsymbol{\theta}}_\ell) - L(\boldsymbol{\theta}^*) \leq \mathfrak{z}_\tau \right\}$$

Critical value, $\mathfrak{z}$

- Wilks phenomenon
  - LRT is nearly χ^2 and its asymptotic distribution depends only on the dimension of the parameter space ▸ Simulation
  - Do not apply in finite samples under possible model misspecification
- Fixing critical value based on propagation condition
  - Allow width of $\mathcal{E}_\ell(\mathfrak{z})$ depends on ℓ, i.e. $\mathfrak{z} = \mathfrak{z}_\ell$

Critical value, $\mathfrak{z}_k$

'Propagation' condition (under H_0)

$$\mathsf{E}_{\theta^*} \left| L_{k,\tau}(\widetilde{\theta}_k(\tau)) - L_{k,\tau}(\widehat{\theta}_k(\tau)) \right|^r \leq \frac{k\ \rho}{K} \mathcal{R}_r\left(\theta^*\right), \quad \forall k \leq K$$

ρ and r are two hyper-parameters ▸ Hyper-par.

'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)
'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

constant risk bound $\mathcal{R}_r\left(\theta^*\right)$ w.r.t. true parameter θ^* ▸ Risk Bound

Steps to Compute $\mathfrak{z}_k$

(i) Generate survival time T based on true parameter θ^* ▸ Cox

(ii) Generate time of default and other exit, τ_D and τ_C

(iii) For each interval I_k and horizon τ, apply sequential choice of $\mathfrak{z}_k$

Sequential Choice of $\mathfrak{z}_k$

- Consider first only $\mathfrak{z}_{1,\tau}$, set $\mathfrak{z}_{2,\tau}, \ldots, \mathfrak{z}_{K-1,\tau} = \infty$. Leads to $\widehat{\theta}_k(\mathfrak{z}_1, \tau)$ for $k = 2, 3, \ldots, K$
- The value $\mathfrak{z}_{1,\tau}$ is selected as the minimal one for which

$$\mathsf{E}_{\theta^*} \left| L_{k,\tau}(\widetilde{\theta}_k(\tau), \widehat{\theta}_k(\mathfrak{z}_1, \tau)) \right|^r \leq \frac{\rho \mathcal{R}_r(\theta^*)}{K}, \quad k = 1, \ldots, K$$

- Set $\mathfrak{z}_{k+1} = \ldots, \mathfrak{z}_K = \infty$ and adjust $\mathfrak{z}_k$, leads to set of $\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \infty, \ldots, \infty$ and estimates $\widehat{\theta}_l(\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \tau)$ for $l = k+1, \ldots, K$. Select $\mathfrak{z}_{k,\tau}$ as minimal value which fulfills

$$\mathsf{E}_{\theta^*} \left| L_{l,\tau}(\widetilde{\theta}_l(\tau), \widehat{\theta}_l(\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \tau)) \right|^r \leq \frac{k \rho \mathcal{R}_r(\theta^*(\tau))}{K}, \quad l = k, \ldots, K$$

Sequential Choice of $\mathfrak{z}_k$

- Generate Q samples using data generating process with parameter θ^*
- Compute and estimates $\widetilde{\theta}_{k,\tau}^{(q)}$, $T_{k,\tau}^{(q)}$, and $L_{k,\tau}\left(\widetilde{\theta}_{k,\tau}^{(q)}\right)$ for every $q = 1, \ldots, Q$ and $k \leq K$
- Provided by $\{\mathfrak{z}_{k,\tau}\}$, running the procedure and computing $\widetilde{\theta}_{k,\tau}$ and $L_{k,\tau}\left(\widetilde{\theta}_{k,\tau}, \widehat{\theta}_{k,\tau}\right)$ requires only a fixed number of operation proportional to K

Output

Provided by local estimate $\widehat{\boldsymbol{\theta}}_k, \quad k = 1, \ldots, K,$

In $[t+\tau, t+\tau+1]$ with discretized time interval $\Delta t = 1/12$

(i) Forward default probability

$$\mathsf{P}_t(t+\tau < \tau_{Di} = \tau_{Ci} \leq t+\tau+1) = e^{-\psi_{it}(\tau)\tau\Delta t}\left\{1 - e^{-f_{it}(\tau)\Delta t}\right\}$$

(ii) Forward combined exit probability

$$\mathsf{P}_t(t+\tau < \tau_{Ci} \leq t+\tau+1) = e^{-\psi_{it}(\tau)\tau\Delta t}\left\{1 - e^{-g_{it}(\tau)\Delta t}\right\}$$

Output

In $[t, t+\tau]$

(iii) Cumulative default probability

$$\mathsf{P}_t(t < \tau_{Di} = \tau_{Ci} \le t + \tau) = \sum_{s=0}^{\tau-1} e^{-\psi_{it}(s) s \Delta t} \left\{ 1 - e^{-f_{it}(s)\Delta t} \right\}$$

(iv) Spot combined exit intensity

$$\psi_{it}(\tau) = \frac{1}{\tau} \left\{ \psi_{it}(\tau - 1)(\tau - 1) + g_{it}(\tau - 1) \right\}$$

No need to specify $\psi_{it}(0)$ since it is irrelevant

References

Bender, R., Augustin, T., and Blettner, M.
Generating survival times to simulate Cox proportional hazards models
Statistics in Medicine, 2005, 24, 1713 - 1723

Duan, J-C., Sun, J., and Wang, T.
Multiperiod Corporate Default Prediction – A Forward Intensity Approach
Journal of Econometrics **170**(1): 191–209, 2012

Spokoiny, V.
Estimation of a function with discontinuities via local polynomial fit with an adaptive window choice
The Annals of Statistics **26**(4): 1356–1378, 1998

References

Spokoiny, V.
Multiscale Local Change Point Detection with Applications to Value-at-Risk
The Annals of Statistics **37**(3): 1405–1436, 2009

Risk Management Institute
Construction and Application of the Corporate Vulnerability Index
CVI White Paper: Jan, 2013

A Crisis Barometer

▸ Back

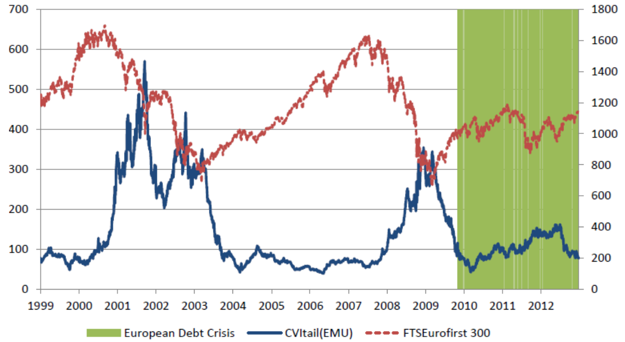


Figure 4: $\mathrm{CVI}_{\mathrm{tail}}$(EMU) and the FTSEEurofirst300 during downturns. RMI (2013)

Indicator of Corporate Default

Back

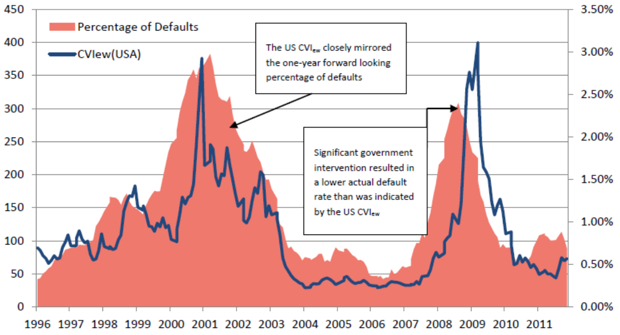


Figure 5: CVI_{ew}(USA) and realized defaults in US. RMI (2013).

Indicator of Recession

▸ Back

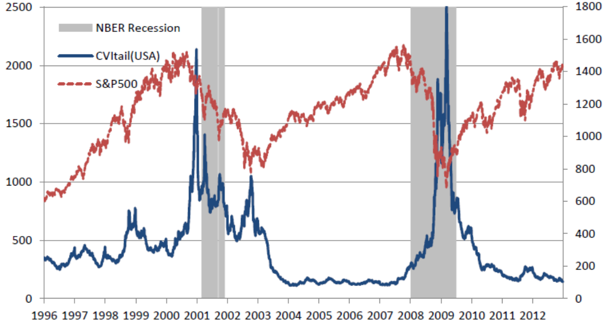


Figure 6: $\mathrm{CVI}_{\mathrm{tail}}$(USA) and S&P500 index during NBER recessions. RMI (2013).

Hedging Tool

Back

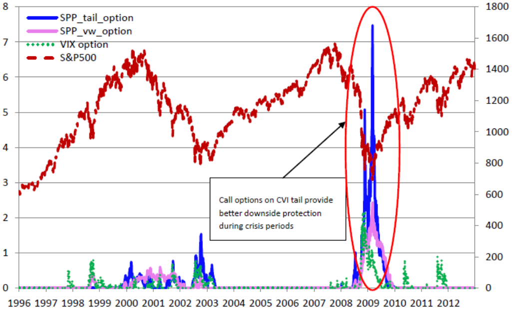


Figure 7: Daily scaled payoffs of synthetic one-year CVI_{tail}(SPP) call option, one-year VIX call option and the S&P500 index. RMI (2013).

Logit and Probit

Back

Binary logit and probit model assess a firm's likelihood of default in the next period but remain silent for default prediction beyond one period.

Campbell et al. (2008) employed a multiple logit model to predict bankruptcy for different time horizons.

DSW

Back

State variables governing the Poisson intensities are assumed to follow a specific high-dimensional VAR process

The dynamics of the state variables is related to multiperiod default prediction, i.e. generating term structure of default probabilities

Poisson process

Back

Let D_i are times between default from different firms, therefore $D_1, D_2, \cdots$ be i.i.d. exponentially distributed with intensity λ

Let $T_n = D_1 + D_2 + \cdots + D_n, \quad T_0 = 0, \; n = 1, 2, \ldots$

A Poisson process with intensity λ is continuous time stochastic process $\{N(t), t \geq 0\}$, where

$$N(t) = \sup \{n \geq 0 : T_n \leq t\} \text{ for } t \geq 0$$

where $N(t)$ counts the number of default in $[0, t]$

Poisson process

Back

Intensity λ is expected number of default per unit time

Poisson process is characterized by intensity λ such that number of default in $[t, t+\tau]$ follow Poisson distribution with intensity $\lambda\tau$

$$\mathrm{P}\left[N(t+\tau) - N(t) = d\ \right] = \frac{e^{-\lambda\tau}(\lambda\tau)^d}{d!}$$

where $d = 0, 1, \ldots$ and $N(t)$ is number of event at time t

Forward Intensity

Poisson distribution

Back

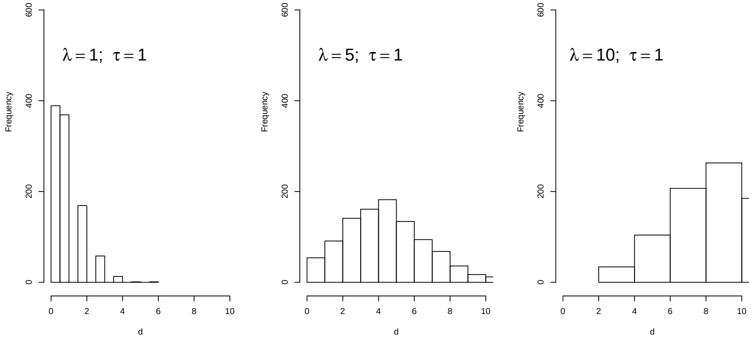


Figure 8: Distribution of number of default in $[t, t+\tau]$ follow Poisson distribution. Sample size $n = 1000$.

Non-homogeneous Poisson process

▸ Back

Intensity $\lambda(t)$ may change over the time

$$\mathrm{E}\left[N(\tau) | \lambda(t), 0 \leq t \leq \tau\right] = \int_0^{\tau} \lambda(t) dt$$

In $[t, t+\tau]$, number of default follow Poisson distribution with

intensity $\int_t^{t+\tau} \lambda(s) ds$

$$\mathrm{P}\left[N(t+\tau) - N(t) = d\right] = \frac{e^{-\int_t^{t+\tau} \lambda(s) ds} \left(\int_t^{t+\tau} \lambda(s) ds\right)^d}{d!}$$

Forward intensity at τ

Back

$$
\begin{aligned}
F_{it}(\tau) &= 1 - \exp\left\{-\psi_{it}(\tau)\tau\right\} \\
F'_{it}(\tau) &= -\exp\left\{-\psi_{it}(\tau)\tau\right\}\left\{-\psi'_{it}(\tau)\tau - \psi_{it}(\tau)\right\} \\
&= \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi'_{it}(\tau)\tau + \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi_{it}(\tau)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{F'_{it}(\tau)}{1 - F_{it}(\tau)} &= \frac{\exp\left\{-\psi_{it}(\tau)\tau\right\}\psi_{it}(\tau) + \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi'_{it}(\tau)\tau}{\exp\left\{-\psi_{it}(\tau)\tau\right\}} \\
&= \psi_{it}(\tau) + \psi'_{it}(\tau)\tau
\end{aligned}
$$

Forward intensity at τ

Back

$$g_{it}(\tau) = \psi_{it}(\tau) + \psi'_{it}(\tau)\tau$$

Therefore

$$\begin{aligned}
\int_0^\tau g_{it}(s)ds &= \int_0^\tau \psi_{it}(s)ds + \int_0^\tau \psi'_{it}(s)s\,ds \\
&= \int_0^\tau \psi_{it}(s)ds + \psi_{it}(\tau)\tau - \int_0^\tau \psi_{it}(s)ds \\
&= \psi_{it}(\tau)\tau
\end{aligned}$$

Doubly stochastic Poisson process

▸ Back

For the i-th firm, default (D) and other exit (E) with stochastic intensity λ_{it} and ϕ_{it}. In $[t, t+\tau]$, the probability of survival

$$\mathrm{E}_t \left[\exp\left\{ -\int_t^{t+\tau} (\lambda_{is} + \phi_{is}) ds \right\} \right],$$

If a firm survive till time s, with $t < s < t+\tau$, then probability of default in $[t, t+\tau]$

$$\mathrm{E}_t \left[\int_t^{t+\tau} \exp\left\{ -\int_t^{s} (\lambda_{iu} + \phi_{iu}) du \right\} \lambda_{is} \, ds \right]$$

Problem: How do we know λ_{it} and ϕ_{it} ?

Pseudo-Likelihood

Back

In interval I_k

$$
\begin{aligned}
L_{I_k,\tau,i,t}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) &= \mathrm{P}\,(\text{Survive}) + \mathrm{P}\,(\text{default}) + \mathrm{P}\,(\text{other exit}) \\
&= 1_{\{t_{0i} \leq t,\ \tau_{Ci} > t+\tau\}}\, \mathrm{P}_t(\tau_{Ci} > t + \tau) \\
&\quad + 1_{\{t_{0i} \leq t,\ \tau_{Di} = \tau_{Ci} \leq t+\tau\}}\, \mathrm{P}_t(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \leq t + \tau) \\
&\quad + 1_{\{t_{0i} \leq t, \tau_{Di} \neq \tau_{Ci}, \tau_{Ci} \leq t+\tau\}}\, \mathrm{P}_t(\tau_{Ci}; \tau_{Di} \neq \tau_{Ci} \& \tau_{Ci} \leq t + \\
&\quad + 1_{\{t_{0i} > t\}} + 1_{\{\tau_{Ci} \leq t\}}
\end{aligned}
$$

If the firm does not appear in sample in month t, then we set the pseudo likelihood to 1 and is transformed to 0 in log-$L_\tau(\cdot)$

Pseudo-Likelihood

▸ Back

$$\mathsf{P}_t\left(\tau_{Ci} > t+\tau\right) = \exp\left\{-\sum_{s=0}^{\tau-1} g_{it}(s)\Delta t\right\}$$

$$\mathsf{P}_t\left(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \leq t+\tau\right)$$

$$= \begin{cases} 1-\exp\left\{-f_{it}(0)\Delta t\right\} & \text{if} \quad \tau_{Ci} = t+1, \\ & \\ \left[1-\exp\left\{-f_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\}\right] & \\ \times \exp\left\{-\sum_{s=0}^{\tau_{Ci}-t-2} g_{it}(s)\Delta t\right\} & \text{if} \quad t+1 < \tau_{Ci} \leq t+\tau \end{cases}$$

Pseudo-Likelihood

▸ Back

$$
\begin{aligned}
&\mathsf{P}_t\left(\tau_{Ci};\tau_{Di}\neq\tau_{Ci}\&\tau_{Ci}\leq t+\tau\right)\\
&=\begin{cases}
\exp\left\{-f_{it}(0)\Delta t\right\}-\left\{-g_{it}(0)\Delta t\right\} & \text{if}\quad \tau_{Ci}=t+1,\\
\\
\exp\left\{-f_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\} & \\
-\exp\left\{-g_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\} & \\
\times\exp\left\{-\sum_{s=0}^{\tau_{Ci}-t-2} g_{it}(s)\Delta t\right\} & \text{if}\quad t+1<\tau_{Ci}\leq t+\tau
\end{cases}
\end{aligned}
$$

Decomposition

Back

The pseudo-likelihood is decomposible into separate $\alpha_k(\tau)$ and $\beta_k(\tau)$ corresponding to different τ's represented by s,

$$
\begin{aligned}
L\left\{\alpha_k(s)\right\} &= \prod_{i=1}^{N} \prod_{t=0}^{T-s-1} L_{i,t}\left\{\alpha_k(s)\right\} \\
L\left\{\beta_k(s)\right\} &= \prod_{i=1}^{N} \prod_{t=0}^{T-s-1} L_{i,t}\left\{\beta_k(s)\right\}
\end{aligned}
$$

where $s = 0, 1, \ldots, \tau - 1$

Decomposable Pseudo-Likelihood

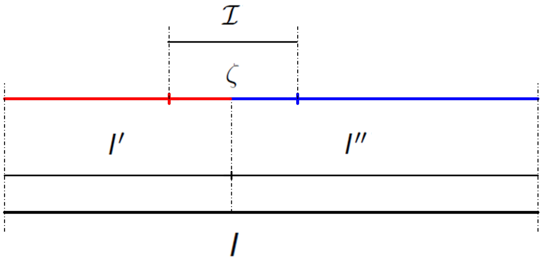
Back

$$
\begin{aligned}
L\{\alpha_k(s)\} \\
= \quad & 1_{\{t_{0i} \le t,\ \tau_{Ci} > t+s+1\}} \exp\{-f_{it}(s)\Delta t\} \\
& + 1_{\{t_{0i} \le t,\ \tau_{Di} = \tau_{Ci} \le t+s+1\}} [1 - \exp\{-f_{it}(s)\Delta t\}] \\
& + 1_{\{t_{0i} \le t, \tau_{Di} \ne \tau_{Ci}, \tau_{Ci} \le t+s+1\}} \exp\{-f_{it}(s)\Delta t\} \\
& + 1_{\{t_{0i} > t\}} + 1_{\{\tau_{Ci} \le t+s+1\}} \\
L\{\beta_k(s)\} \\
= \quad & 1_{\{t_{0i} \le t,\ \tau_{Ci} > t+s+1\}} \exp\{-[g_{it}(s) - f_{it}(s)]\Delta t\} \\
& + 1_{\{t_{0i} \le t,\ \tau_{Di} = \tau_{Ci} \le t+s+1\}} \\
& + 1_{\{t_{0i} \le t, \tau_{Di} \ne \tau_{Ci}, \tau_{Ci} \le t+s+1\}} [1 - \exp\{-[g_{it}(s) - f_{it}(s)]\Delta t\}] \\
& + 1_{\{t_{0i} > t\}} + 1_{\{\tau_{Ci} \le t+s+1\}}
\end{aligned}
$$

where $g_{it}(s) - f_{it}(s) = \exp\{\beta_0(s) + \beta_1(s)x_{it,1} + \ldots + \beta_p(s)x_{it,p}\}$

Test Statistics

Back



$\mathcal{I}$: tested interval possibly contain change point

$I = [I', I'')$: larger testing interval

Test Statistics

Back

H_0 : homogeneity within $\mathcal{I}$ vs. H_1 : change point within $\mathcal{I}$
LRT Statistics, $L(\cdot)$ is log likelihood function

$$\begin{aligned} T_{\mathcal{I},\zeta} &= \max_{\theta',\theta''} \left\{ L_{I''}(\theta'') + L_{I'}(\theta') \right\} - \max_{\theta} L_I(\theta) \\ &= L_{I'}(\widetilde{\theta}_{I'}) + L_{I''}(\widetilde{\theta}_{I''}) - L_I(\widetilde{\theta}_I) \end{aligned}$$

Reject H_0 if $T_{\mathcal{I},\zeta} \geq \mathfrak{z}$
Thus,

$$T_{\mathcal{I}} = \max_{\zeta \in \mathcal{I}} T_{\mathcal{I},\zeta}$$

Let $\mathcal{I} = I_k \setminus I_{k-1}$,

$$\widehat{\boldsymbol{\theta}} = \widetilde{\boldsymbol{\theta}}_{\widehat{k}}, \quad \widehat{k} = \max_{k \leq K} \{ k : T_\ell \leq \mathfrak{z}_\ell, \ell \leq k \}$$

LRT: Poisson distribution

▸ LRT

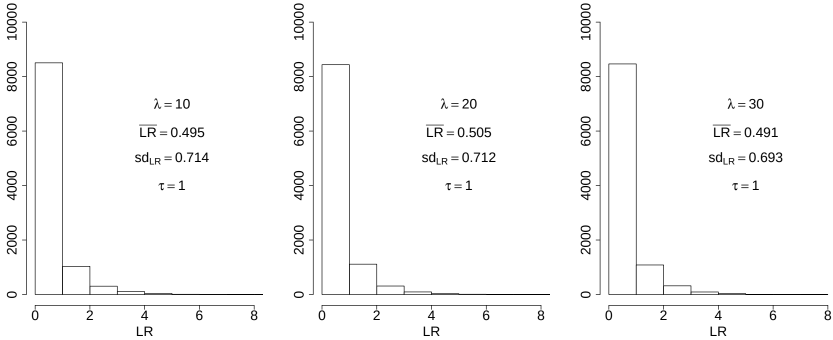


Figure 9: Monte Carlo simulation, similar result for $\lambda = 1, 2, \ldots, 9$

LRT: Exponential distribution

▸ LRT

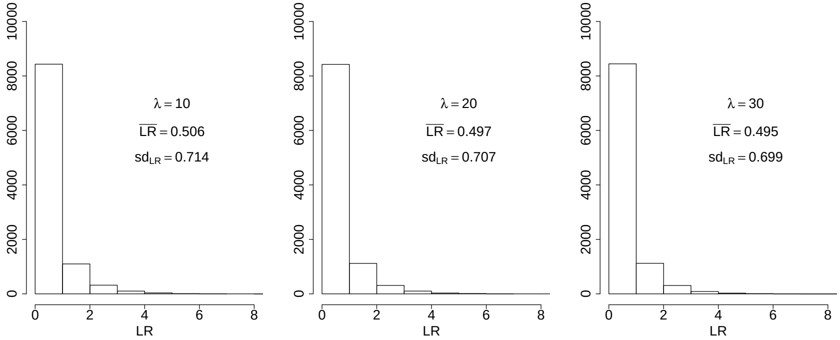


Figure 10: Monte Carlo simulation, similar result for $\lambda = 1, 2, \ldots, 9$

Hyper Parameters

Back

- The role of ρ is similar to the significance level of a test
- The r denotes the power of loss function

$$\mathrm{E}_{\theta^*} L^r_{k,\tau} \left\{ \widetilde{\theta}_k(\tau), \widehat{\theta}_k(\tau) \right\} \to \mathrm{P}_{\theta^*} \left\{ \widetilde{\theta}_k(\tau) \neq \widehat{\theta}_k(\tau) \right\}, \quad r \to 0.$$

- The $\mathfrak{z}_{1,\tau}, \ldots, \mathfrak{z}_{K-1,\tau}$ enter implicitely in the propagation condition: if false alarm event $\left\{ \widetilde{\theta}_k(\tau) \neq \widehat{\theta}_k(\tau) \right\}$ happen too often, it is indication that some $\mathfrak{z}_{1,\tau}, \ldots, \mathfrak{z}_{k-1,\tau}$ are too small
- Note: propagation condition relies on artificial parametric model P_{θ^*} instead of the true model P

Parametric Risk Bound

Propagation

$$
\begin{aligned}
\mathsf{E}_{\theta^*} \left| L(\widetilde{\theta}, \theta^*) \right|^r &= r \int_0^\infty \mathfrak{z}^{r-1} \, \mathsf{P} \left\{ \left| L(\widetilde{\theta}, \theta^*) \right| > \mathfrak{z} \right\} d\mathfrak{z} \\
&= r \int_0^\infty \mathfrak{z}^{r-1} \, \mathsf{P} \left\{ \left| L(\widetilde{\theta}, \theta^*) \right| > \mathfrak{z}, \widetilde{\theta} \in \mathcal{E}(\mathfrak{z}) \right\} d\mathfrak{z} \\
&\quad + r \int_0^\infty \mathfrak{z}^{r-1} \, \mathsf{P} \left\{ \left| L(\widetilde{\theta}, \theta^*) \right| > \mathfrak{z}, \widetilde{\theta} \notin \mathcal{E}(\mathfrak{z}) \right\} d\mathfrak{z} \\
&\le \mathcal{R}_r \left(\theta^* \right) \\
&< \infty
\end{aligned}
$$

Note: $\mathcal{E}(\mathfrak{z}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{\theta}^* : L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*) \le \mathfrak{z} \right\}$

Standard Cox model

▸ Back

- Observe $Y = \min\{T, C\}$ on time interval $[0, T_{\text{max}}]$ with survival time T and censoring time C
- Set for $t \in [0, T_{\text{max}}]$

$$\delta = \mathbf{I}\{T \leq C\}, \; N(t) = \delta \, \mathbf{I}\{Y \leq t\}$$

- Covariates: X
- Observations: $\{Y_i, \delta_i, X_i\}$

Standard Cox model

Back

- $X \in \mathbb{R}^p$: p-dimensional state variable
- Hazard function of failure time T

$$\lambda(t) = \lambda_0(t) \exp\{\alpha^\top X\}$$

  - Baseline hazard function $\lambda_0(t)$
  - Coefficient function $\alpha = (\alpha_0, \ldots, \alpha_p)^\top$

Survival time, T

Back

- Survival function, with $\Lambda_0(t) = \int_0^t \lambda_0(s)\, ds$

$$S(t) = \exp\left[-\Lambda_0(t) \exp\{\alpha^\top X\}\right]$$

- Let $F(t) = 1 - S(t)$, $F \sim \mathrm{U}[0,1]$, therefore $S \sim \mathrm{U}[0,1]$
- Replace $S(t) = U \sim \mathrm{U}[0,1]$

$$\begin{aligned} U &= \exp\left[-\Lambda_0(t) \exp\{\alpha^\top X\}\right] \\ \Lambda_0(t) &= -\log(U)\ \exp\{-\alpha^\top X\} \end{aligned}$$

- Survival time, $T = \Lambda_0^{-1}(t)\left[-\log(U)\ \exp\{-\alpha^\top X\}\right]$

Generate T

Back

- Let $T \sim \text{Exp}(\lambda^*)$, scale parameter $\lambda^* > 0$. Set $\lambda^* = 5$

$$T_i = -\frac{\log(U_i)}{\lambda^* \exp\{\alpha^\top X_i\}}$$

- Let $T \sim \text{Weibull}(\lambda^*, \nu)$, scale and shape parameters $\lambda^* > 0$, $\nu > 0$. Set $\lambda^* = 5$ and $\nu = 2$

$$T_i = \left[-\frac{\log(U_i)}{\lambda^* \exp\{\alpha^\top X_i\}}\right]^{1/\nu}$$

Generate T

Back

- Let $T \sim \text{Gompertz}(\lambda^*, \gamma)$, scale and shape parameters $\lambda^* > 0$, $\gamma \in (-\infty, \infty)$. Set $\lambda^* = 5$ and $\gamma = 2$

$$T_i = \frac{1}{\gamma} \log \left[1 - \frac{\gamma \log(U_i)}{\lambda^* \exp\left\{\alpha^\top X_i\right\}} \right]$$