Copula Dynamics in Collateralized Debt Obligations

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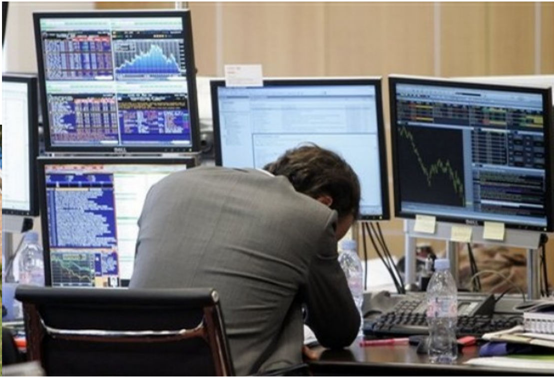
Collateralized Debt Obligation

Triggered the financial crisis.



Risk Transfer

iTraxx Europe Tranches

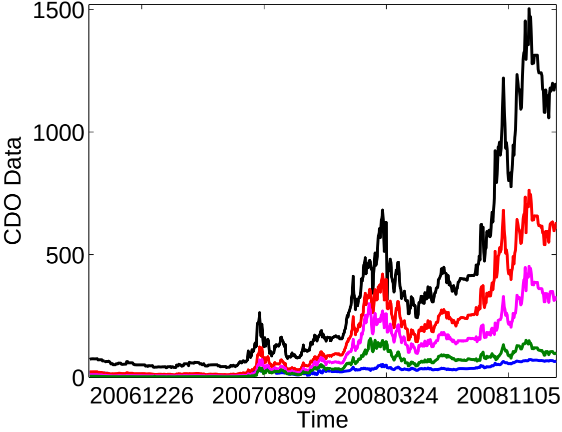


Figure 1: Generic iTraxx Europe, 20060920-20090202. Tranche: 1, **2**, 3, 4, 5.

Research Goals

- How to model the varying dependency?
- How to model the dynamics of implied copula model parameters?
- How well do models asses the risk of CDO tranches?
- How to calculate the Value-at-Risk for CDO tranches?

Outline

1. Motivation ✓
2. CDO valuation models
3. Modelling parameters dynamics
4. Value-at-Risk backtesting
5. Empirical study
6. Conclusions

iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities;
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10);
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted;
- Maturities: 3Y, 5Y, 7Y, 10Y.

Tranching

Tranche name	Attachment points (%) Lower l	Upper u
Super Super Senior	22	100
Super Senior	12	22
Senior	9	12
Mezzanine	6	9
Mezzanine Junior	3	6
Equity	0	3

Table 1: iTraxx tranches.

Collateralized Debt Obligations

Consider a CDO of d underlying entities, J tranches, and with a maturity of T years. The spreads $s_j(t)$, $j = 2, \ldots, J$, and the UFF $s_1(t)$ are observed at $t = 1, \ldots, \widetilde{T}$.

Consider a large portfolio framework.

A homogeneous default probability within an interval $[t_0, t]$ is

$$p(t) = 1 - \exp\{-\lambda(t - t_0)\},$$

where λ is an intensity parameter equal to all obligors. ▸ Valuation

CDO Valuation One-Factor Models

Linear factor models

- Gaussian copula model ▸ Gaussian
- Normal Inverse Gaussian (NIG) copula model ▸ NIG
- Double-t copula model ▸ DoubleT

Archimedean copula models

- Gumbel copula model ▸ Gumbel

Correlation's Types

Compound correlation $\rho(l_j, u_j)$, $j = 1, \ldots, J$.

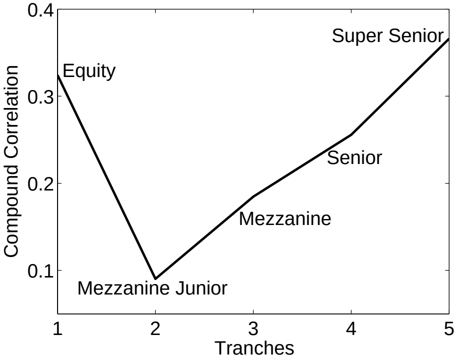


Figure 2: Implied correlation smile in the Gaussian one factor model, 20071022.

Correlation's Types

Base correlation (BC) $\rho(0, u_j)$, $j = 1, \ldots, J$.

Represent the expected loss $\mathsf{E}\{L_{(l_j,u_j)}\}$ as a difference:

$$\mathsf{E}\{L_{(l_j,u_j)}\} = \mathsf{E}_{\rho(0,u_j)}\{L_{(0,u_j)}\} - \mathsf{E}_{\rho(0,l_j)}\{L_{(0,l_j)}\}, \; j = 2, \ldots, J.$$

of the expected losses of two fictive tranches $(0, u_j)$ and $(0, l_j)$.
Bootstrapping process: $\mathsf{E}\{L_{(0,3\%)}\}$ is traded on the market,

$$\begin{aligned} \mathsf{E}\{L_{(3\%,6\%)}\} &= \mathsf{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\} - \mathsf{E}_{\rho(0,3\%)}\{L_{(0,3\%)}\}, \\ \mathsf{E}\{L_{(6\%,9\%)}\} &= \mathsf{E}_{\rho(0,9\%)}\{L_{(0,9\%)}\} - \mathsf{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\}, \ldots \end{aligned}$$

Base Correlations

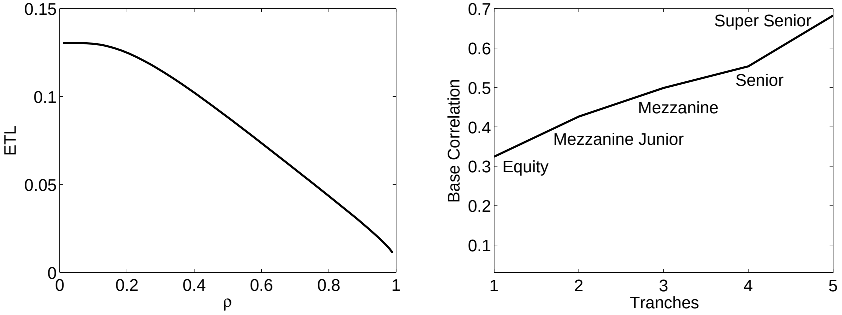


Figure 3: Expected loss of the equity tranche calculated using the Gaussian copula model with a one-year default probability computed from the iTraxx index Series 8 with 5 years maturity (left) and the base correlation smile (right) on 20071022.

Value-at-Risk for CDO Tranches

For a given level α, the VaR is an α-quantile of the profit-loss distribution

$$\mathsf{P}\{\Delta s_j(t) < \mathrm{VaR}^{\alpha}_{s_j}(t)\} = \alpha,$$

where $\Delta s_j(t) = s_j(t) - s_j(t-1)$ is a profit-loss process of the jth tranche.

The calculation of the one-day $\mathrm{VaR}^{\alpha}_{s_j}(t)$ requires forecasts of $s_j(t)$.

Time Dynamics

Recall that the tranche spread is a function of:

- intensity parameter,
- dependence parameters.

By calibrating the CDO copula models over $t = 1, \ldots, \widetilde{T}$ construct time series of base parameters.

Model 1: Endogenous Base Parameter Model

Let $\theta_j(t)$ denote a base parameter implied from the tranche j at t. Further, let $X_j(t) = \theta_j(t) - \theta_j(t-1)$ or $X_j(t) = \log \theta_j(t) - \log \theta_j(t-1)$.

Consider a standard ARMA(R, M)-GARCH(P, Q) model

$$X_j(t) = C_j + \sum_{i=1}^{R} \phi_{j,i} X_j(t-i) + \sum_{l=1}^{M} \psi_{j,l} \varepsilon_j(t-l) + \varepsilon_j(t), \qquad (1)$$

where $\varepsilon_j(t) = \sigma_j(t) Z_j(t)$ are innovations, $Z_j(t)$ are standardised innovations that follow the standard normal or the t distribution, and $\sigma_j^2(t) = K_j + \sum_{i=1}^{P} G_{j,i} \sigma_j^2(t-i) + \sum_{l=1}^{Q} A_{j,l} \varepsilon_j^2(t-l)$ with the following constraints $\sum_{i=1}^{P} G_{j,i} + \sum_{l=1}^{Q} A_{j,l} < 1$, $K_j > 0$, $G_{j,i} \geq 0$, and $A_{j,i} \geq 0$.

Estimation and Forecasting

- Estimation in moving windows of $h = 250$ elements, i.e. for any $t_0 \in [h, \widetilde{T}]$ analyse $\{X_j(t)\}_{t=t_0-h}^{t_0-1}$
- In the first window $\{X_j(t)\}_{t=1}^{h}$ determine the lag orders and the distribution of the standardised residuals.
- In each following window re-estimate and forecast the model.

Next day spread $\hat{s}_j\{t; \hat{\theta}_j(t), \hat{\theta}_{j-1}(t), \hat{\lambda}(t)\}$, $j = 2, \ldots, J$.

- The dependence between $\theta_j(t)$ and $\lambda(t)$ is not considered.
- The base parameters of neighbor tranches are strongly dependent.

Innovation Copula

Define an innovation copula as

$$C_{G_j}(u_1, u_2) = G_j\{F_{\varepsilon_j}^{-1}(u_1), F_{\varepsilon_{j-1}}^{-1}(u_2)\}, \quad j = 2, \dots, J, \tag{2}$$

where $\varepsilon_j(t)$ are the innovations of $X_j(t)$ with the marginals $\varepsilon_j \sim F_{\varepsilon_j}$.

The copula chosen here are a bivariate Gaussian, Gumbel, and Clayton.

VaR Calculation

1. Generate N random bivariate vectors from C_{G_j} (2).
2. Calculate $\left[\widehat{X}_j(t), \widehat{X}_{j-1}(t)\right]$ as in (1).
3. Transform them into $\left[\hat{\theta}_j(t), \hat{\theta}_{j-1}(t)\right]$.
4. Forecast $\lambda(t)$ with an ARMA$(1,1)$ model.
5. Calculate $\hat{s}_j\{t; \hat{\theta}_j(t), \hat{\theta}_{j-1}(t), \hat{\lambda}(t)\}$.
6. Calculate $\widehat{\text{VaR}}_{s_j}^{\alpha}(t)$ and $\text{VaR}_{s_j}^{1-\alpha}(t)$ as sample quantiles.

Model 2: Exogeneous Index Spread Model

Include the iTraxx index as an exogenous variable. Consider an ARMAX($R, M, 1$)-GARCH(P, Q) model where X_j is represented as

$$X_j(t) = C_j + \beta_j Y(t) + \sum_{i=1}^{R} \phi_{j,i} X_j(t-i) + \sum_{l=1}^{M} \psi_{j,l} \varepsilon_j(t-l) + \varepsilon_j(t), \quad (3)$$

where $Y(t)$ is the first difference of the iTraxx index and β_j is an exogenous parameter.

Forecast $Y(t)$ using an independent ARMA model.

Model 3: Intensity Model with Fixed Base Parameters

Assume that the tranche risk is determined by the dynamics of $\lambda(t)$ and that the correlation is constant, $\hat{\theta}_j(t+1) = \theta_j(t)$.

Consider $\Lambda(t) = \log\lambda(t) - \log\lambda(t-1)$ that follows ARMA$(1,1)$

$$\Lambda(t) = D + \phi\Lambda(t-1) + \psi\varepsilon(t-1) + \varepsilon(t) \tag{4}$$

with the normally distributed innovations.

Backtesting

Testing VaR

1. Kupiec (1995) likelihood ratio test ▸ KT
2. Dynamic Quantile (DQ) test of Engle & Manganelli (2004) ▸ DQ

Testing density forecasts

1. Diebold, Gunther & Tay (1998) test based on Kolmogorov–Smirnov test ▸ KS
2. Berkowitz (2001) test ▸ BT

Data Set

iTraxx Europe with the maturity of 5 years, series:

1. S6 20060920-20070322
2. S7 20070323-20070919
3. S8 20070920-20080320
4. S9 20080321-20090202

Together 619 days (132, 129, 131, 227).
Discount curves calculated from rates of Euribor and Euro Swaps.

Implied Base Parameters

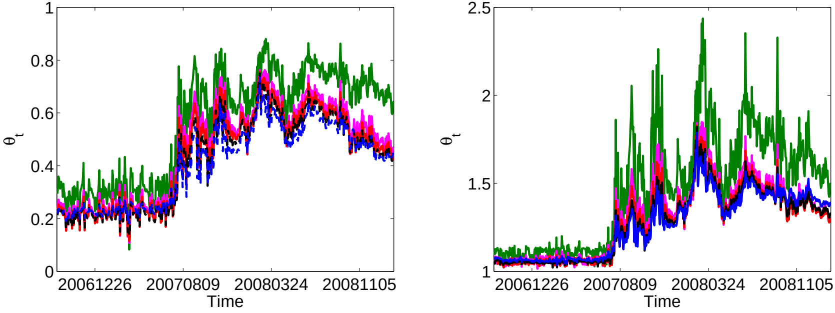


Figure 4: Double-t (left) and Gumbel (right) model. Tranche: 1, **2**, 3, 4, 5.

implBC implNIG

VaR for Spreads

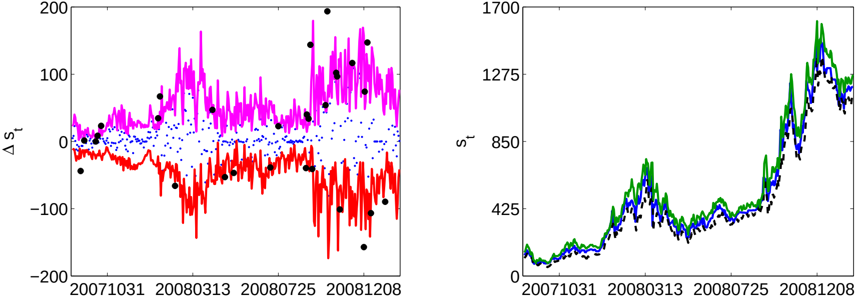


Figure 5: Model 1 results. Tranche 2 in double-t model with the Gumbel innovation copula. Left: Market spread difference, VaR(5%), VaR(95%), **exceedances**. Right: Quantiles of predicted spreads at the level 5% and **95%** compared with the market values.

DQ Test for Model 1

Model	Tr.	VaR(5%) Gauss	NIG	double-t	Gumbel	VaR(95%) Gauss	NIG	double-t	Gumbel
	1	0.755	0.137	0.653	0.605	0.011	0.068	0.020	0.085
Gauss	2	0.735	0.119	0.574	0.482	0.000	0.009	0.223	0.001
	3	0.513	0.126	0.740	0.137	0.000	0.000	0.000	0.009
	4	0.188	0.014	0.382	0.055	0.063	0.001	0.001	0.026
	5	0.244	0.484	0.036	0.050	0.000	0.335	0.269	0.018
Gumbel	2	0.723	0.000	0.741	0.596	0.008	0.000	0.001	0.132
	3	0.425	0.000	0.738	0.118	0.003	0.000	0.000	0.035
	4	0.037	0.165	0.221	0.018	0.037	0.000	0.089	0.034
	5	0.474	0.000	0.019	0.251	0.000	0.052	0.031	0.033
Clayton	2	0.111	0.068	0.182	0.110	0.005	0.000	0.240	0.003
	3	0.184	0.780	0.054	0.009	0.000	0.067	0.005	0.033
	4	0.009	0.026	0.033	0.009	0.009	0.122	0.510	0.018
	5	0.019	0.037	0.000	0.009	0.000	0.165	0.114	0.017

Table 1: DQ test's p-values. Model 1. Ex1 KT1 Ex2 KT2 DQ2

Results

DQ Test for lower & upper VaR

- Model 1: Gaussian model with the Gaussian innovation copula & NIG valuation model with the Clayton innovation copula
- Model 2: double-t model with the Gumbel innovation copula & double-t model with the Clayton innovation copula

DQ Test for Model 1 and 3

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.755	0.137	0.653	0.605	0.011	0.068	0.020	0.085
2	0.735	0.119	0.741	0.596	0.008	0.009	0.240	0.132
3	0.513	0.780	0.740	0.137	0.003	0.067	0.005	0.035
4	0.188	0.165	0.382	0.055	0.063	0.122	0.510	0.034
5	0.474	0.484	0.036	0.251	0.000	0.335	0.269	0.033

Table 2: DQ test. Model 1 with the best innovation copulae.

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.015	0.000	0.006	0.003	0.000	0.000	0.000	0.000
2	0.775	0.000	0.829	0.657	0.072	0.000	0.111	0.001
3	0.670	0.000	0.532	0.031	0.471	0.000	0.001	0.008
4	0.648	0.000	0.000	0.011	0.062	0.000	0.000	0.000
5	0.430	0.168	0.101	0.000	0.038	0.000	0.000	0.000

Table 3: DQ test's p-values. Model 3. FKT1 FKT2 FKT3 FDQ2

Results

DQ Test for lower & upper VaR

- Model 1: NIG valuation model
- Model 2: double-t & Gaussian valuation model
- Model 3: Gaussian valuation model

Distribution Tests for Model 1

Model	Tr.	KS Gauss	KS NIG	KS double-t	KS Gumbel	Berkowitz Gauss	Berkowitz NIG	Berkowitz double-t	Berkowitz Gumbel
	1	0.061	0.000	0.095	0.011	0.399	0.000	0.102	0.370
Gauss	2	0.004	0.025	0.139	0.000	0.036	0.338	0.092	0.000
	3	0.046	0.003	0.245	0.000	0.006	0.000	0.004	0.000
	4	0.000	0.000	0.261	0.000	0.000	0.000	0.173	0.000
	5	0.403	0.015	0.003	0.000	0.000	0.003	0.000	0.000
Gumbel	2	0.000	0.003	0.040	0.002	0.023	0.000	0.116	0.008
	3	0.014	0.000	0.059	0.000	0.007	0.000	0.001	0.000
	4	0.000	0.002	0.079	0.000	0.000	0.005	0.183	0.000
	5	0.453	0.023	0.002	0.000	0.000	0.000	0.000	0.000
Clayton	2	0.009	0.352	0.008	0.001	0.002	0.280	0.021	0.000
	3	0.001	0.000	0.043	0.000	0.000	0.001	0.002	0.000
	4	0.000	0.010	0.000	0.000	0.000	0.000	0.000	0.000
	5	0.000	0.030	0.004	0.000	0.000	0.000	0.000	0.000

Table 4: Kolmogorov-Smirnov & Berkowitz tests. Model 1. KSB2 KSB3

Results

Kolmogorov-Smirnov & Berkowitz tests

- Model 1: double-t valuation model
- Model 2: Gaussian & double-t valuation model
- Model 3: double-t & Gaussian valuation model

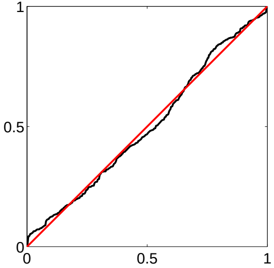


Figure 6: PP-plot of the NIG model with the Clayton innovations copula for tranche 2, Model 1.

MSE for Model 1

Tranche	Gauss	NIG	double-t	Gumbel
1	0.145	0.145	0.137	0.142
2	3.024	2.996	2.924	3.040
3	1.804	1.993	1.799	2.286
4	1.288	1.613	1.238	2.910
5	0.449	0.523	0.442	0.616
Mean	1.342	1.454	1.308	1.799

Table 5: Mean squared error of the spread predictions relative to the average tranche spread. Model 1. MSE2 MSE3 MSEf

Conclusions

- Investigated the dynamics of CDO copulae over time.
- Proposed a method for calculating VaR for spreads based on modelling the evolution of the dependence parameters (Model 1: Endogenous Base Parameter Model and Model 2: Exogeneous Index Spread Model) and of the intensity parameter (Model 3: Intensity Model with Fixed Base Parameters).
- The performance of the Gaussian and the more advanced valuation models are comparable.
- Model 1 with the NIG valuation model achieves the best VaR forecasts and with the double-t valuation model the best density forecast and the best next day spread predictions.

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Default

Consider a CDO with a maturity of T years, J tranches, and a pool of d entities. Define a loss variable of i-th obligor until $t \in [t_0, T]$ as

$$I_i(t) = \mathbf{1}(\tau_i < t), \; i = 1, \ldots, d,$$

where τ_i is a time to default variable

$$\begin{aligned} F_i(t) &= \mathrm{P}(\tau_i \leq t) \\ &= 1 - \exp\left\{ - \int_{t_0}^{t} \lambda_i(u) du \right\} \end{aligned}$$

and λ_i is a deterministic intensity function. Talk

Large Pool Approach for Linear Factor Models

Default times are calculated from a vector $(X_1, \ldots, X_d)^\top$

$$X_i = \sqrt{\rho}Y + \sqrt{1-\rho}Z_i,$$

where Y (systematic risk factor), $\{Z_i\}_{i=1}^d$ (idiosyncratic risk factors) are i.i.d. Assume that

- obligors have the same default probability p and LGD,
- one dependence parameter ρ,
- d is large.

Talk

Large Pool Approximation

Computations are simplified significantly when the portfolio loss distribution is approximated:

$$P(L \leq x) = 1 - F_Y \left\{ \frac{F_X^{-1}(p) - \sqrt{\rho} F_Z^{-1}(x)}{\sqrt{1-\rho}} \right\},$$

where $X_i \sim F_X$, $Z_i \sim F_Z$, $Y \sim F_Y$.

Talk

Gaussian Copula Model

The factors Y and $\{Z_i\}_{i=1}^{d}$ are i.i.d. $N(0,1)$. Thus, $X_i \sim N(0,1)$
The cdf of the portfolio loss equals

$$\mathrm{P}(\tilde{L} \leq x) = \Phi \left\{ \frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}} \right\}.$$

Default times are given by $\tau_i = F_i^{-1}\{\Phi(X_i)\}$.

▸ Talk

NIG Model

Factors:

$$
\begin{aligned}
Y &\sim \mathrm{NIG}\left(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right), \gamma = \sqrt{\alpha^2 - \beta^2}, \\
Z_i &\sim \mathrm{NIG}\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right).
\end{aligned}
$$

Because of the stability under convolution

$$
X_i \sim \mathrm{NIG}\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right) = \mathrm{NIG}_{(1/\sqrt{\rho})}.
$$

Default times are given by $\tau_i = F_i^{-1}\{\mathrm{NIG}_{(1/\sqrt{\rho})}(X_i)\}$.

▸ Talk

Double-t Model

Define

$$X_i = \sqrt{\rho}\sqrt{\frac{\nu_Y - 2}{\nu_Y}}Y + \sqrt{1-\rho}\sqrt{\frac{\nu_Z - 2}{\nu_Z}}Z_i, \; i = 1, \ldots, d,$$

where Y and Z_i are t distributed with ν_Y and ν_Z DoF respectively.

The t distribution is not stable under convolution: X_i are not t distributed and the copula is not a t copula, $X_i \sim F_X$ has to be computed numerically. Default times are computed as

$$\tau_i = F_i^{-1}\{F_X(X_i)\}.$$

▸ Talk

Large Pool Approach for Archimedean Copulae

d-dimensional Archimedean copula $C : [0,1]^d \to [0,1]$ is

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0,1],$$

where $\phi \in \{\phi : [0; \infty) \to [0,1] \,|\, \phi(0) = 1,\ \phi(\infty) = 0;\ (-1)^j \phi^{(j)} \geq 0;\ j = 1, \dots, \infty\}$ is a copula generator.

Each ϕ is a Laplace transform of a cdf of a positive random variable $Y \sim F_Y$

$$\phi(t) = \int_0^\infty e^{-tw} dF_Y(w),\ \ t \geq 0.$$

▸ Talk

Large Pool Approach for Archimedean Copulae

If X_i, $i = 1, \ldots, d$, i.i.d. U$[0,1]$ and Y's Laplace transform is ϕ, then the Archimedean Copula C is a joint cdf of $U_i = \phi\left(-\frac{\log X_i}{Y}\right)$.
Conditional on the realisation of Y, U_i are independent.
The large pool approximation of the loss distribution is

$$\mathrm{P}(\tilde{L} \leq x) = F_Y \left\{ -\frac{\log(1-x)}{\phi^{-1}(\bar{p})} \right\}.$$

For the Gumbel copula

$$C(u_1, \ldots, u_d; \theta) = \exp\left[-\left\{ \sum_{i=1}^{d} (-\log u_i)^{\theta} \right\}^{\theta^{-1}} \right],$$

F_Y is an α-stable distribution with $\alpha = 1/\theta$. ▸ Talk

Simple Backtesting

Exceedance ratios

$$\widehat{\alpha_j^u} = \frac{1}{\widetilde{T} - h} \sum_{t=h+1}^{\widetilde{T}} \mathbf{1}\{\Delta s_j(t) > \widehat{\mathrm{VaR}}_{s_j}^{1-\alpha}(t)\},$$

$$\widehat{\alpha_j^l} = \frac{1}{\widetilde{T} - h} \sum_{t=h+1}^{\widetilde{T}} \mathbf{1}\{\Delta s_j(t) < \widehat{\mathrm{VaR}}_{s_j}^{\alpha}(t)\}.$$

Kupiec (1995) likelihood ratio test

$$LR_j = -2\log\{(1-\alpha_j)^{\widetilde{T}-h-n}\alpha_j^n\} + 2\log\{(1-\widehat{\alpha}_j)^{\widetilde{T}-h-n}\widehat{\alpha}_j{}^n\} \sim \chi^2(1),$$

where α_j is either α_j^u or α_j^l and n is the number of exceedences.

▸ Backtesting

Dynamic Quantile of Engle & Manganelli (2004)

Define a hit H_j as $\{H_j^u(t)\}_{t=h+1}^{\widetilde{T}}$ for the upper VaR or $\{H_j^l(t)\}_{t=h+1}^{\widetilde{T}}$ for the lower VaR such that

$$
\begin{aligned}
H_j^u(t) &= \mathbf{1}\{\Delta s_j(t) > \widehat{\text{VaR}}_{s_j}^{1-\alpha}(t)\} - \alpha, \\
H_j^l(t) &= \mathbf{1}\{\Delta s_j(t) < \widehat{\text{VaR}}_{s_j}^{\alpha}(t)\} - \alpha.
\end{aligned}
$$

Regress the hits against possible explanatory variables V_j: lags of H_j, $\{\widehat{\text{VaR}}_{s_j}^{1-\alpha}(t)\}_{t=t_0-h}^{t_0-1}$ or $\{\widehat{\text{VaR}}_{s_j}^{\alpha}(t)\}_{t=t_0-h}^{t_0-1}$.

Test statistics:

$$
DQ_j = H_j V_j^\top (V_j^\top V_j)^{-1} V_j^\top H_j / \{\alpha(1-\alpha)(\widetilde{T} - h - n)\}, \qquad (5)
$$

H_0: H_j and V_j are orthogonal. $DQ_j \sim \chi^2(q_j)$, where $q_j = \text{rank}(V_j)$.

▸ Backtesting

Diebold, Gunther & Tay (1998) Test

Assume $s_j(t) \sim F_j$, $j = 1, \ldots, J$. Then a probability integral transform (PIT) of the spread is

$$y_j(t) = \hat{F}_j(s_j(t)),$$

where $\hat{F}_j$ is the forecasted cdf.

Test if $y_j(t)$ are iid U$(0, 1)$ using Kolmogorov–Smirnov test.
Large sample size required.

▸ Backtesting

Berkowitz (2001) Test

Let $z_j(t) = \Phi^{-1}\{y_j(t)\}$ so that $z_j(t) \sim \mathrm{N}(0,1)$ under the null. The null is tested against the first order alternative with the mean μ_j, variance σ_j^2 and autocorrelation ρ_j possibly different than (0,1,0)

$$z_j(t) - \mu_j = \rho_j\{z_j(t-1) - \mu_j\} + \varepsilon_j(t) \tag{6}$$

and the test statistic is given by

$$\mathrm{LR}_j^B = -2\left\{\mathcal{L}(0,1,0) - \mathcal{L}(\hat{\mu}_j, \hat{\sigma}_j^2, \hat{\rho}_j)\right\} \sim \chi^2(3), \tag{7}$$

where $\mathcal{L}$ denotes the Gaussian log-likelihood function.

Backtesting

Implied Base Parameters

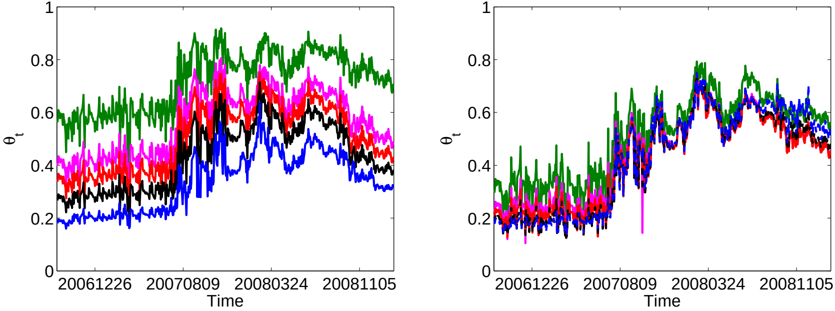

Figure 7: Gaussian (left) and NIG (right) model. Tranche: 1, **2**, 3, 4, 5.

Talk

Calibration of the NIG Copula Model

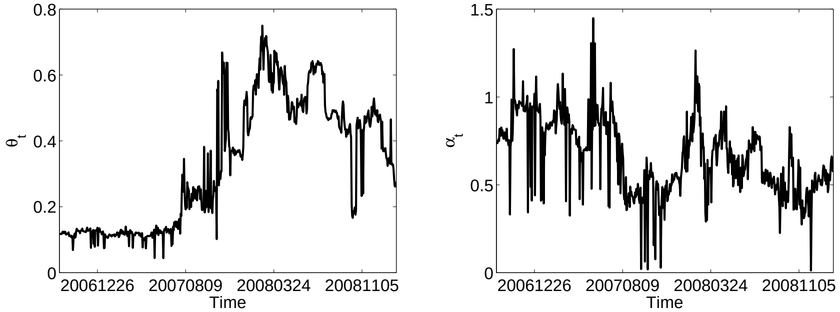

Figure 8: Base correlation (left) and α parameter (right) calibrated for all tranches using the NIG model.

Exceedance Ratios for Model 1

Model	Tr.	VaR(5%)				VaR(95%)			
		Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
	1	0.035	0.019	0.033	0.043	0.054	0.030	0.057	0.046
Gauss	2	0.038	0.054	0.030	0.024	0.065	0.052	0.038	0.027
	3	0.033	0.068	0.033	0.019	0.068	0.082	0.060	0.011
	4	0.016	0.030	0.022	0.011	0.016	0.043	0.054	0.014
	5	0.019	0.038	0.008	0.011	0.060	0.038	0.035	0.005
Gumbel	2	0.038	0.087	0.033	0.030	0.054	0.087	0.049	0.030
	3	0.038	0.120	0.033	0.014	0.052	0.120	0.065	0.008
	4	0.008	0.038	0.024	0.005	0.008	0.057	0.046	0.008
	5	0.027	0.087	0.022	0.024	0.071	0.057	0.038	0.008
Clayton	2	0.014	0.011	0.016	0.014	0.046	0.046	0.041	0.033
	3	0.016	0.038	0.011	0.003	0.052	0.035	0.043	0.008
	4	0.003	0.016	0.008	0.003	0.003	0.016	0.030	0.005
	5	0.005	0.008	0.000	0.003	0.052	0.016	0.030	0.005

Table 6: Exceedance ratios. Model 1. DQ1

Kupiec Test for Model 1

Model	Tr.	VaR(5%)				VaR(95%)			
		Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
	1	0.174	0.002	0.103	0.557	0.706	0.056	0.543	0.735
Gauss	2	0.273	0.706	0.056	0.013	0.200	0.886	0.273	0.028
	3	0.103	0.133	0.103	0.002	0.133	0.011	0.403	0.000
	4	0.001	0.056	0.005	0.000	0.001	0.557	0.706	0.000
	5	0.002	0.273	0.000	0.000	0.403	0.273	0.174	0.000
Gumbel	2	0.273	0.003	0.103	0.056	0.706	0.003	0.924	0.056
	3	0.273	0.000	0.103	0.000	0.886	0.000	0.200	0.000
	4	0.000	0.273	0.013	0.000	0.000	0.543	0.735	0.000
	5	0.028	0.003	0.005	0.013	0.086	0.543	0.273	0.000
Clayton	2	0.000	0.000	0.001	0.000	0.735	0.735	0.401	0.103
	3	0.001	0.273	0.000	0.000	0.886	0.174	0.557	0.000
	4	0.000	0.001	0.000	0.000	0.000	0.001	0.056	0.000
	5	0.000	0.000	0.000	0.000	0.886	0.001	0.056	0.000

Table 7: Kupiec test's p-values. Model 1. DQ1

Kupiec Test for Model 1

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.174	0.002	0.103	0.557	0.706	0.056	0.543	0.735
2	0.273	0.706	0.103	0.056	0.735	0.886	0.924	0.103
3	0.273	0.273	0.103	0.002	0.886	0.174	0.557	0.000
4	0.001	0.273	0.013	0.000	0.001	0.557	0.735	0.000
5	0.028	0.273	0.005	0.013	0.886	0.543	0.273	0.000

Table 8: Kupiec test. Model 1 with the best innovation copulae. FDQ1

Exceedance Ratios for Model 2

Model	Tr.	VaR(5%)				VaR(95%)			
		Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
	1	0.035	0.022	0.035	0.054	0.065	0.046	0.071	0.084
Gauss	2	0.054	0.068	0.057	0.041	0.082	0.079	0.071	0.041
	3	0.046	0.095	0.043	0.008	0.079	0.122	0.043	0.019
	4	0.038	0.060	0.038	0.038	0.082	0.073	0.038	0.019
	5	0.041	0.079	0.022	0.027	0.041	0.060	0.043	0.046
Gumbel	2	0.052	0.117	0.052	0.038	0.054	0.120	0.057	0.030
	3	0.060	0.158	0.043	0.008	0.063	0.168	0.041	0.011
	4	0.054	0.095	0.035	0.027	0.057	0.125	0.024	0.014
	5	0.016	0.109	0.038	0.052	0.016	0.071	0.052	0.057
Clayton	2	0.016	0.043	0.016	0.016	0.057	0.073	0.049	0.030
	3	0.014	0.049	0.008	0.005	0.054	0.082	0.035	0.011
	4	0.011	0.033	0.005	0.005	0.030	0.038	0.014	0.003
	5	0.008	0.035	0.011	0.008	0.000	0.035	0.030	0.035

Table 9: Exceedance ratios. Model 2. DQ1

Kupiec Test for Model 2

Model	Tr.	VaR(5%) Gauss	VaR(5%) NIG	VaR(5%) double-t	VaR(5%) Gumbel	VaR(95%) Gauss	VaR(95%) NIG	VaR(95%) double-t	VaR(95%) Gumbel
	1	0.174	0.005	0.174	0.706	0.200	0.735	0.086	0.006
Gauss	2	0.706	0.133	0.543	0.401	0.011	0.019	0.086	0.401
	3	0.735	0.000	0.557	0.000	0.019	0.000	0.557	0.002
	4	0.273	0.403	0.273	0.273	0.011	0.054	0.273	0.002
	5	0.401	0.019	0.005	0.028	0.401	0.403	0.557	0.735
Gumbel	2	0.886	0.000	0.886	0.273	0.706	0.000	0.543	0.056
	3	0.403	0.000	0.557	0.000	0.289	0.000	0.401	0.000
	4	0.706	0.000	0.174	0.028	0.543	0.000	0.013	0.000
	5	0.001	0.000	0.273	0.886	0.001	0.086	0.886	0.543
Clayton	2	0.001	0.557	0.001	0.001	0.543	0.054	0.924	0.056
	3	0.000	0.924	0.000	0.000	0.706	0.011	0.174	0.000
	4	0.000	0.103	0.000	0.000	0.056	0.273	0.000	0.000
	5	0.000	0.174	0.000	0.000	0.000	0.174	0.056	0.174

Table 10: Kupiec test's p-values. Model 2. DQ1

DQ Test for Model 2

Model	Tr.	VaR(5%) Gauss	VaR(5%) NIG	VaR(5%) double-t	VaR(5%) Gumbel	VaR(95%) Gauss	VaR(95%) NIG	VaR(95%) double-t	VaR(95%) Gumbel
	1	0.294	0.228	0.782	0.010	0.378	0.399	0.009	0.000
Gauss	2	0.005	0.001	0.050	0.691	0.000	0.000	0.000	0.001
	3	0.466	0.000	0.448	0.033	0.000	0.000	0.000	0.002
	4	0.036	0.005	0.253	0.000	0.000	0.000	0.006	0.000
	5	0.005	0.000	0.077	0.000	0.108	0.000	0.008	0.000
Gumbel	2	0.104	0.000	0.075	0.695	0.019	0.000	0.409	0.022
	3	0.153	0.000	0.516	0.033	0.000	0.000	0.000	0.010
	4	0.002	0.000	0.500	0.010	0.006	0.000	0.006	0.008
	5	0.002	0.000	0.077	0.000	0.168	0.000	0.001	0.000
Clayton	2	0.188	0.396	0.172	0.185	0.000	0.000	0.008	0.007
	3	0.118	0.806	0.036	0.019	0.000	0.000	0.014	0.010
	4	0.068	0.189	0.019	0.016	0.378	0.000	0.021	0.009
	5	0.000	0.000	0.007	0.000	0.000	0.000	0.020	0.000

Table 11: DQ test's p-values. Model 2. DQ1

Kupiec Test for Model 2

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.174	0.005	0.174	0.706	0.200	0.735	0.086	0.006
2	0.886	0.557	0.886	0.401	0.706	0.054	0.924	0.401
3	0.735	0.924	0.557	0.000	0.706	0.011	0.557	0.002
4	0.706	0.403	0.273	0.273	0.543	0.273	0.273	0.002
5	0.401	0.174	0.273	0.886	0.401	0.403	0.886	0.735

Table 12: Kupiec test. Model 2 with the best innovation copulae. FDQ1

DQ Test for Model 2

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.294	0.228	0.782	0.010	0.378	0.399	0.009	0.000
2	0.188	0.396	0.172	0.695	0.019	0.000	0.409	0.022
3	0.466	0.806	0.516	0.033	0.000	0.000	0.014	0.010
4	0.068	0.189	0.500	0.016	0.378	0.000	0.021	0.009
5	0.005	0.000	0.077	0.000	0.168	0.000	0.020	0.000

Table 13: DQ test. Model 2 with the best innovation copulae. FDQ1

Distribution Tests for Model 2

Model	Tr.	KS				Berkowitz			
		Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
	1	0.045	0.089	0.098	0.145	0.020	0.000	0.044	0.038
Gauss	2	0.639	0.005	0.470	0.083	0.013	0.016	0.267	0.001
	3	0.582	0.000	0.009	0.000	0.000	0.000	0.276	0.000
	4	0.700	0.191	0.000	0.000	0.000	0.002	0.001	0.000
	5	0.006	0.010	0.445	0.003	0.002	0.004	0.000	0.000
Gumbel	2	0.145	0.000	0.325	0.302	0.036	0.000	0.407	0.000
	3	0.277	0.000	0.002	0.000	0.000	0.000	0.169	0.000
	4	0.457	0.003	0.001	0.000	0.000	0.000	0.000	0.000
	5	0.000	0.000	0.080	0.396	0.000	0.000	0.000	0.000
Clayton	2	0.013	0.137	0.046	0.002	0.002	0.108	0.084	0.000
	3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4	0.000	0.109	0.000	0.000	0.000	0.000	0.000	0.000
	5	0.000	0.001	0.011	0.000	0.000	0.018	0.000	0.000

Table 14: Kolmogorov-Smirnov & Berkowitz tests' p-values. Model 2. KSB1

Exceedance ratios for Model 3

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.073	0.065	0.063	0.046	0.082	0.073	0.076	0.043
2	0.057	0.054	0.038	0.049	0.054	0.073	0.035	0.052
3	0.063	0.065	0.057	0.084	0.057	0.090	0.049	0.073
4	0.060	0.063	0.092	0.092	0.063	0.109	0.071	0.084
5	0.063	0.071	0.065	0.106	0.052	0.084	0.068	0.101

Table 15: Exceedance ratios. Model 3.

Kupiec Test for Model 3

Tranche	VaR(5%)				VaR(95%)			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.054	0.200	0.289	0.735	0.011	0.054	0.032	0.557
2	0.543	0.706	0.273	0.924	0.706	0.054	0.174	0.886
3	0.289	0.200	0.543	0.006	0.543	0.002	0.924	0.054
4	0.403	0.289	0.001	0.001	0.289	0.000	0.086	0.006
5	0.289	0.086	0.200	0.000	0.886	0.006	0.133	0.000

Table 16: Kupiec test's p-values. Model 3. FDQ1

Distribution Tests for Model 3

Tranche	KS				Berkowitz			
	Gauss	NIG	double-t	Gumbel	Gauss	NIG	double-t	Gumbel
1	0.256	0.110	0.099	0.000	0.000	0.000	0.000	0.000
2	0.002	0.002	0.001	0.003	0.130	0.000	0.016	0.172
3	0.003	0.022	0.001	0.022	0.314	0.000	0.468	0.001
4	0.044	0.040	0.082	0.109	0.055	0.000	0.000	0.000
5	0.031	0.158	0.050	0.076	0.847	0.000	0.020	0.000

Table 17: Kolmogorov-Smirnov & Berkowitz tests' p-values. Model 3. KSB1

MSE for Model 2

Tranche	Gauss	NIG	double-t	Gumbel
1	0.176	0.152	0.177	0.163
2	3.342	3.786	3.123	3.803
3	1.988	2.052	2.168	2.247
4	1.572	1.620	1.716	8.404
5	1.013	1.198	0.742	1.743
Mean	1.618	1.761	1.585	3.272

Table 18: Mean squared error of the spread predictions relative to the average tranche spread. Model 2. MSE1

MSE for Model 3

Tranche	Gauss	NIG	double-t	Gumbel
1	0.159	0.187	0.158	0.158
2	3.426	3.449	3.434	3.461
3	2.185	2.352	2.197	2.185
4	1.471	2.206	1.522	1.493
5	0.614	0.576	0.614	0.623
Mean	1.571	1.754	1.585	1.584

Table 19: Mean squared error of the spread predictions relative to the average tranche spread. Model 3. MSE1

MSE for Finger Regression Model

Regression Model of Finger (2009):

1. Calculate the first principal component for the normalized tranche price daily changes.
2. Regress it against the changes on the iTraxx index, plus a constant.

Tranche	Regression
1	0.159
2	3.467
3	2.187
4	1.464
5	0.606
Mean	1.576

Table 20: Mean squared error of the spread predictions relative to the average tranche spread. Finger's regression model. MSE1