Localising Forward Intensities for Multiperiod Default

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Poisson process and default time

▸ Homogen. Pois. ▸ Non-homogen. Pois.

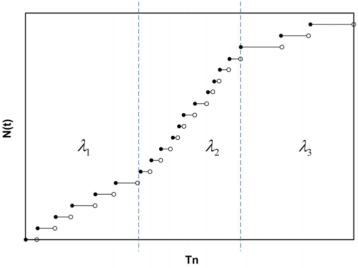


Figure 1: Poisson process $N(t)$ with intensity λ_t.

- Time of default τ_D: first jump time of $N(t)$

Survival probability

- For known path λ_s, survival probability in $[t, t+\tau]$

$$\mathsf{P}\left(\tau_D > t+\tau\right) = \exp\left\{-\int_t^{t+\tau} \lambda_s \, ds\right\} \qquad (1)$$

- $\lambda_s = \lambda(X_s)$ stochastic determined by state variable X_s.
  Filtration $\mathcal{F}_t$, simulating path of X_s, DSW

$$\mathsf{P}\left(\tau_D > t+\tau | \mathcal{F}_t\right) = \mathsf{E}\left[\exp\left\{-\int_t^{t+\tau} \lambda(X_s, \theta_t) \, ds\right\} | X_t\right] \qquad (2)$$

- Simulation on high dimensional X_s is quite challenging

Forward default intensity, $\lambda_t(s)$

- Hazard function where survival time is evaluated at a fixed horizon

$$\lambda_t(s) \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\mathrm{P}\left(t+s<\tau_D \leq t+s+\Delta t \,|\tau_D \geq t+s\right)}{\Delta t} \qquad (3)$$

$$= F_t'(s)/\{1-F_t(s)\},$$

where $F_t(s) = 1 - \mathrm{P}\left(\tau_D > t+s \,|\mathcal{F}_t\right)$

- Avoid modelling X_s, specify $\lambda_t(s) = \lambda(s, X_t)$

$$\mathrm{P}\left(\tau_D > t+\tau \,|\mathcal{F}_t\right) = \exp\left\{-\int_t^{t+\tau} \lambda_t(s)\, ds\right\}, \qquad (4)$$

Specifications of λ_s

λ_s	$=$	$\lambda(X_s;\theta_t)$	, Duffie et al. (2007)
$\lambda_t(s)$	$=$	$\lambda(\theta_s;X_t)$	, Duan et al. (2012)
$\lambda_t(s)$	$=$	$\lambda(\theta_{s,t};X_t)$	, Our approach

Table 1: The specifications of the default intensity.

$$X_{it} = (x_{it,1}, x_{it,2}, \dots, x_{it,p}) = (W_t, U_{it})$$

W_t – Macroeconomic factors (common) ▸ W_t

U_{it} – Firm specific attributes ▸ U_{it}

Doubly stochastic Poisson process

Default (with intensity λ_t) and other exits (ϕ_t) governed by two independent doubly stochastic Poisson process

Conditional probability to survive and to default in $[t, t+\tau]$

$$\mathsf{E}\left[\exp\left\{-\int_t^{t+\tau}(\lambda_s+\phi_s)ds\right\}|X_t\right]$$

$$\mathsf{E}\left[\int_t^{t+\tau}\exp\left\{-\int_t^{s}(\lambda_u+\phi_u)du\right\}\lambda_s\, ds|X_t\right]$$

detail

Problem: λ_s and ϕ_s unknown

Solution: simulating X_s, $\lambda_s = \lambda_1(X_s, \theta_{1,t})$, $\phi_s = \lambda_2(X_s, \theta_{2,t})$ or specifying forward intensities

Forward intensities

Combined exit: $\lambda_t + \phi_t$

Reparameterize $\lambda_{it}(\tau)$ as $f_{it}(\tau)$ and forward combined exit intensity $g_{it}(\tau)$

Duan et al. (2012), $f_{it}(\tau)$ and $g_{it}(\tau)$ are parameterized with $f_{it}(\tau) > 0$ and $g_{it}(\tau) \geq f_{it}(\tau)$:

$$\begin{aligned} f_{it}(\tau) &= \exp\left\{\alpha_0(\tau) + \alpha_1(\tau)x_{it,1} + \ldots + \alpha_p(\tau)x_{it,p}\right\} \\ g_{it}(\tau) &= f_{it}(\tau) + \exp\left\{\beta_0(\tau) + \beta_1(\tau)x_{it,1} + \ldots + \beta_p(\tau)x_{it,p}\right\} \end{aligned} \tag{5}$$

Is this a satisfactory calibration technique ?

What can go wrong ?

For the horizon τ,

- forward intensities (5) are time homogeneous, i.e. at each t

  $f_{it}(\tau)$ and $g_{it}(\tau)$ follow the same structural equation,
  $\alpha_j(\tau)$ and $\beta_j(\tau)$ are constant over time

- Are the parameters constant ?
  If not, why and where they deviate ?

What can go wrong ?

Figure 2: Rolling windows (length = 6 years), $\hat{\alpha}_{12}(\tau)$ and $\hat{\beta}_{12}(\tau)$, with $\tau = 0, 1, \ldots, 36$.

Homogeneous interval

Adaptively selecting a data-driven window length

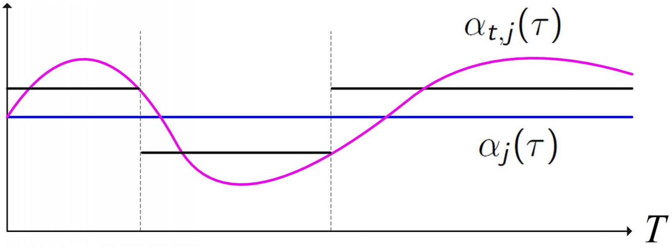


Figure 3: Time varying parameters approximated by piecewise constants.

Outline

1. Motivation $\checkmark$
2. Forward Intensities Approach
3. Local Change Point detection
4. Empirical Result
5. Conclusion

Combined exit: default & other exit

τ_C as combined exit time, survival probability in $[t, t+\tau]$

$$\mathsf{P}\left(\tau_C > t+\tau|\mathcal{F}_t\right) \stackrel{\text{def}}{=} \mathsf{E}\left[\exp\left\{-\int_t^{t+\tau}(\lambda_s+\phi_s)ds\right\}|X_t\right] \qquad (6)$$

Forward combined exit intensity $g_t(s)$

$$\begin{aligned}\mathsf{P}\left(\tau_C > t+\tau\,|\mathcal{F}_t\right) &\stackrel{\text{def}}{=} \exp\left\{-\int_t^{t+\tau} g_t(s)ds\right\} \qquad (7)\\ &= \exp\left\{-\psi_t(\tau)\tau\right\}\end{aligned}$$

with $\psi_t(\tau) \stackrel{\text{def}}{=} -\frac{\log\{1-G_t(\tau)\}}{s}$, $\quad G_t(\tau) = 1 - \mathsf{P}\left\{\tau_C > t+\tau\,|\mathcal{F}_t\right\}$

Combined exit

Forward combined exit intensity (hazard rate) for firm i

$$g_{it}(\tau) \stackrel{\text{def}}{=} \frac{G'_{it}(\tau)}{1 - G_{it}(\tau)} = \psi_{it}(\tau) + \psi'_{it}(\tau)\tau \tag{8}$$

detail

Therefore

$$\psi_{it}(\tau)\tau = \int_0^\tau g_{it}(s)ds$$

detail

Forward default intensity: combined exit

No combined exit till time s, default probability over $[t, t+\tau]$

$$\int_0^{\tau} \exp\left\{-\psi_{it}(s)s\right\} f_{it}(s)\, ds$$

with forward default intensity $f_{it}(s)$ is defined as

$$\begin{aligned} &\overset{\text{def}}{=} \quad e^{-\psi_{it}(s)s} \lim_{\Delta t \to 0} \frac{\mathsf{P}\left(t+s < \tau_{Di} = \tau_{Ci} \leq t+s+\Delta t \,|\tau_{Di} = \tau_{Ci} \geq t+s\right)}{\Delta t} \\ &= \quad e^{-\psi_{it}(s)s} \lim_{\Delta t \to 0} \frac{\mathsf{E}\left[\int_{t+s}^{t+s+\Delta t} \exp\left\{-\int_t^u (\lambda_{iv} + \phi_{iv})dv\right\}\lambda_{iu} du \,|\tau_{Di} = \tau_{Ci} \geq t+s\right]}{\Delta t} \end{aligned}$$

Note that $\tau_{Ci} < \tau_{Di}$

Recall: forward intensities

Duan et al. (2012), $f_{it}(\tau)$ and $g_{it}(\tau)$ are parameterized with $f_{it}(\tau) > 0$ and $g_{it}(\tau) \geq f_{it}(\tau)$:

$$\begin{aligned} f_{it}(\tau) &= \exp\left\{\alpha_0(\tau) + \alpha_1(\tau)x_{it,1} + \ldots + \alpha_p(\tau)x_{it,p}\right\} \\ g_{it}(\tau) &= f_{it}(\tau) + \exp\left\{\beta_0(\tau) + \beta_1(\tau)x_{it,1} + \ldots + \beta_p(\tau)x_{it,p}\right\} \end{aligned}$$

Note: $\tau = 0$ obtain the spot intensity of Duffie et al. (2007)

Localising the forward intensities

- Given (5) for each t one might look for a homogeneous interval I in which forward intensities are adequately described

- Longer estimation period – reduced variability, enlarge bias

  LPA finds a balance between parameter variability and modelling bias

- Estimation windows with potentially varying length. Find the longest stable (homogeneity) interval

Interval selection

Given time t, go back and split time series into K intervals,

$$
\begin{array}{ccccccc}
I_K & \supset \cdots \supset & I_k & \supset \cdots \supset & I_1 & \supset & I_0 \\
\widetilde{\theta}_K & \cdots & \widetilde{\theta}_k & \cdots & \widetilde{\theta}_1 & & \widetilde{\theta}_0
\end{array}
$$

for $t \in I_k$, $I_k = [t - m_{k+1} + 1, t]$, with length $|I_k| = m_k$, estimates are obtained using log-likelihood

$$
\begin{aligned}
&\widetilde{\theta}_k = \widetilde{\theta}_{I_k} = \left(\widetilde{\alpha}_k, \widetilde{\beta}_k\right)^\top \\
&\widetilde{\alpha}_k = \left\{\widetilde{\alpha}_{I_k}(0), \ldots, \widetilde{\alpha}_{I_k}(\tau - 1)\right\}; \quad \widetilde{\alpha}_{I_k}(s) = \left(\widetilde{\alpha}_{I_k,0}(s), \ldots, \widetilde{\alpha}_{I_k,p}(s)\right)^\top \\
&\widetilde{\beta}_k = \left\{\widetilde{\beta}_{I_k}(0), \ldots, \widetilde{\beta}_{I_k}(\tau - 1)\right\}; \quad \widetilde{\beta}_{I_k}(s) = \left(\widetilde{\beta}_{I_k,0}(s), \ldots, \widetilde{\beta}_{I_k,p}(s)\right)^\top
\end{aligned}
$$

MLE

Maximum likelihood estimates (MLEs) of $\theta_k = (\alpha_k, \beta_k)^\top$

$$\widetilde{\theta}_k = \arg \max_{\theta \in \Theta} L_{k,\tau}(\alpha_k, \beta_k) \tag{9}$$

where $L_{k,\tau}(\alpha_k, \beta_k)$ is likelihood for interval I_k evaluated at τ

$$L_{k,\tau}(\alpha_k, \beta_k) = \prod_{i=1}^{N} \prod_{\substack{t=0 \\ t \in I_k}}^{T-1} L_{\tau,i,t}(\alpha_k, \beta_k) \tag{10}$$

Likelihood

where sample period from 0 to T for each I_k
N is number of companies at a point in time t

MLE: Decomposable

The likelihood is decomposable into separate $\alpha_k(\tau)$ and $\beta_k(\tau)$ corresponding to different τ's represented by s,

$$L\left\{\alpha_k(s)\right\} = \prod_{i=1}^{N}\prod_{t=0}^{T-s-1} L_{i,t}\left\{\alpha_k(s)\right\} \tag{11}$$

$$L\left\{\beta_k(s)\right\} = \prod_{i=1}^{N}\prod_{t=0}^{T-s-1} L_{i,t}\left\{\beta_k(s)\right\} \tag{12}$$

where $s = 0, 1, \ldots, \tau - 1$ ▸ Likelihood

MLE: Decomposable

Back

$$
\begin{aligned}
& L_{i,t}\left\{\alpha_k(s)\right\} \\
& \quad = \quad 1_{\{t_{0i}\le t,\ \tau_{Ci}>t+s+1\}} \exp\left\{-f_{it}(s)\Delta t\right\} \\
& \quad\quad\quad +1_{\{t_{0i}\le t,\ \tau_{Di}=\tau_{Ci}\le t+s+1\}} \left[1-\exp\left\{-f_{it}(s)\Delta t\right\}\right] \\
& \quad\quad\quad +1_{\{t_{0i}\le t,\tau_{Di}\neq\tau_{Ci},\tau_{Ci}\le t+s+1\}} \exp\left\{-f_{it}(s)\Delta t\right\} \\
& \quad\quad\quad +1_{\{t_{0i}>t\}} + 1_{\{\tau_{Ci}\le t+s+1\}}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
& L_{i,t}\left\{\beta_k(s)\right\} \\
& \quad = \quad 1_{\{t_{0i}\le t,\ \tau_{Ci}>t+s+1\}} \exp\left\{-\left[g_{it}(s)-f_{it}(s)\right]\Delta t\right\} \\
& \quad\quad\quad +1_{\{t_{0i}\le t,\ \tau_{Di}=\tau_{Ci}\le t+s+1\}} \\
& \quad\quad\quad +1_{\{t_{0i}\le t,\tau_{Di}\neq\tau_{Ci},\tau_{Ci}\le t+s+1\}} \left[1-\exp\left\{-\left[g_{it}(s)-f_{it}(s)\right]\Delta t\right\}\right] \\
& \quad\quad\quad +1_{\{t_{0i}>t\}} + 1_{\{\tau_{Ci}\le t+s+1\}}
\end{aligned} \tag{14}
$$

where $\quad g_{it}(s)-f_{it}(s)=\exp\left\{\beta_0(s)+\beta_1(s)x_{it,1}+\ldots+\beta_p(s)x_{it,p}\right\}$

MLE: Decomposable

Grouping observation into

$$X^0 = \left(x_1^0, \ldots, x_{N_0}^0\right)^\top, X^1 = \left(x_1^1, \ldots, x_{N_1}^1\right)^\top,$$
$$X^2 = \left(x_1^2, \ldots, x_{N_2}^2\right)^\top,$$

where X^0, X^1, and X^2 contain all firm-month observation that survive, default, and exit due to other reason, respectively.

The N_0, N_1, and N_2 are number of observation in each category.

MLE: Decomposable, $\Delta t = 1/12$

Put log at each indicator function ▸ Forward PD ▸ Cum. Forward PD

$$
\begin{aligned}
\log L\left\{\alpha(s)\right\} &= -\sum_{i=1}^{N_0} \exp(x_i^0 \alpha)\Delta t \\
&\quad + \sum_{i=1}^{N_1} \log\left[1 - \exp\{-\exp(x_i^1 \alpha)\Delta t\}\right] - \sum_{i=1}^{N_2} \exp(x_i^2 \alpha)\Delta t, \\
\log L\left\{\beta(s)\right\} &= -\sum_{i=1}^{N_0} \exp(x_i^0 \beta)\Delta t \\
&\quad + \sum_{i=1}^{N_1} \log\left[1 - \exp\{-\exp(x_i^1 \beta)\Delta t\}\right].
\end{aligned}
$$

Sequential test, fixed τ Note

H_0 : Parameter homogeneity within I_k
H_1 : Change point within I_k

Test statistic

$$T_{k,\tau} = \left| L_{I_k}(\widetilde{\theta}_k) - L_{I_k}(\widehat{\theta}_{k-1}) \right|^r, \quad k = 1, \ldots, K \tag{15}$$

$\mathfrak{z}_{k,\tau}$ – Critical values

If $T_{k,\tau} > \mathfrak{z}_{k,\tau}$, accepts I_{k-1} as homogeneous, $\widehat{\theta}_k = \widehat{\theta}_{k-1} = \widetilde{\theta}_{k-1}$
Otherwise, accepts I_k as homogeneous, $\widehat{\theta}_k = \widetilde{\theta}_k$

Critical value, $\mathfrak{z}_{k,\tau}$

'Propagation' condition (under H_0)

$$\mathsf{E}_{\theta^*} \left| L_{k,\tau} \left\{ \widetilde{\theta}_k, \widehat{\theta}_k \right\} \right|^r \leq \frac{k\,\rho}{K} \mathcal{R}_r\left(\theta^*\right), \quad \forall k \leq K$$

ρ and r are two hyper-parameters ▸ Hyper-par.

'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)

'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

Constant risk bound $\mathcal{R}_r\left(\theta^*\right)$ w.r.t. true parameter θ^* ▸ Risk Bound

Adaptive estimation

- Compare $T_{k,\tau}$ at every step k with $\mathfrak{z}_{k,\tau}$
- Data window index of the *interval of homogeneity* - $\widehat{k}$
- Adaptive estimate

$$\widehat{\theta} = \widetilde{\theta}_{\widehat{k}}, \quad \widehat{k} = \max_{k \leq K} \{k : T_{\ell,\tau} \leq \mathfrak{z}_{\ell,\tau}, \ell \leq k\}$$

Data and Variables

2000 U.S. public firms from Feb 1991 to Dec 2011.

Macroeconomic factors (W_t) ▸ Back

- One year simple return on S&P500 index ($X_{t,1}$)
- 3-months US Treasury bill rate ($X_{t,2}$)

Firm-specific attribute (U_{it})

Level: one-year average of the measure

Trend: current value - level

Data and Variables

Firm-specific attribute (U_{it}) Back

- Volatility-adjusted leverage
  - Distance-to-Default (DTD): level ($X_{it,3}$), trend ($X_{it,4}$) Detail
- Liquidity – CASH/Total Asset: level ($X_{it,5}$), trend ($X_{it,6}$)
- Profitability – Net Income/Total Asset: level ($X_{it,7}$), trend ($X_{it,8}$)
- Relative size – log(firm's equity/average equity of S&P500's firms): level ($X_{it,9}$), trend ($X_{it,10}$)
- Market-to-book asset ratio ($X_{it,11}$)
- One-year idiosyncratic volatility ($X_{it,12}$) Back Detail, $X_{it,12} = \sigma_{it}$

Set up

- True parameters θ^* are generated as average over 35 moving windows (length: 15 years)

- Subset interval $I_k = \{5, 6, 8, 10, 12, 15\}$ years (monthly-based)

- Monte Carlo simulation to generate critical value $\mathfrak{z}_{k,\tau}$ for $\tau = \{1, 3, 6, 12, 24, 36\}$ months horizons

  Accuracy Ratio (AR) – discriminative power

Estimates: Macroeconomic

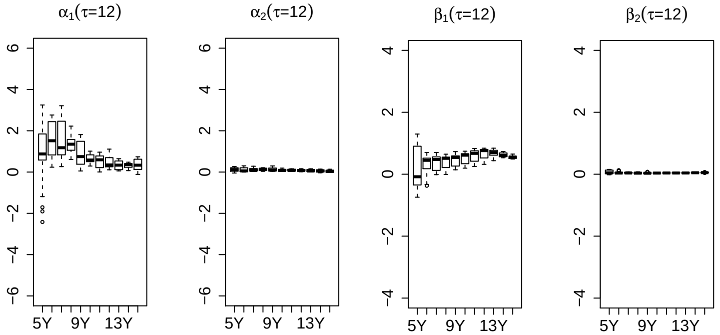


Figure 4: Box-plots of estimates, $\tau = 12$, of default (two left) and other exits (two right) over 35 windows (length: $5, 6, \ldots, 15$ years).

Estimates: Firm specific

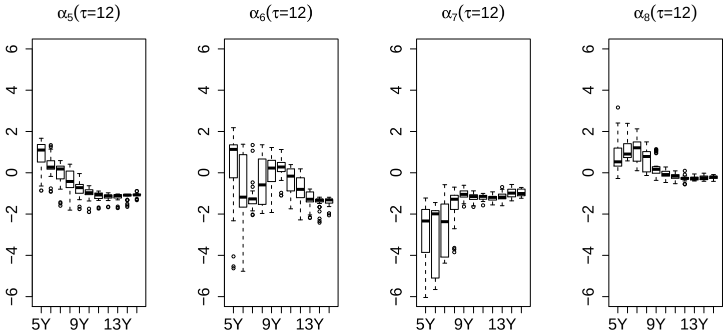
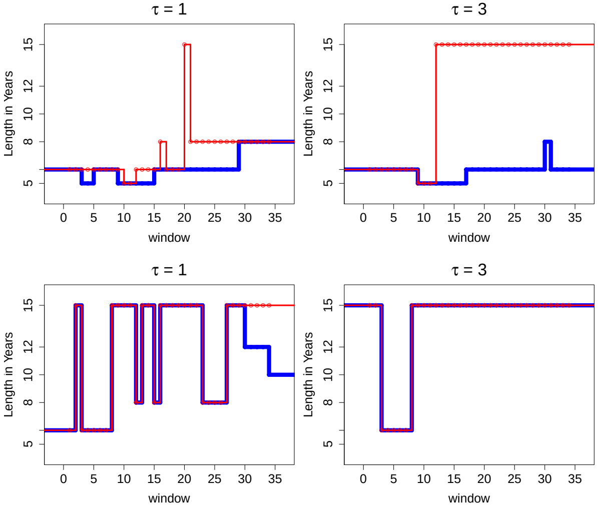


Figure 5: Box-plots of estimates, $\tau = 12$, of default over 35 windows (length: $5, 6, \ldots, 15$ years).

Estimates

- Robust to I_k
  - Macroeconomic: 3-months US Treasury interest rate
  - Firm specific: DTD, company size, market-to-book ratio

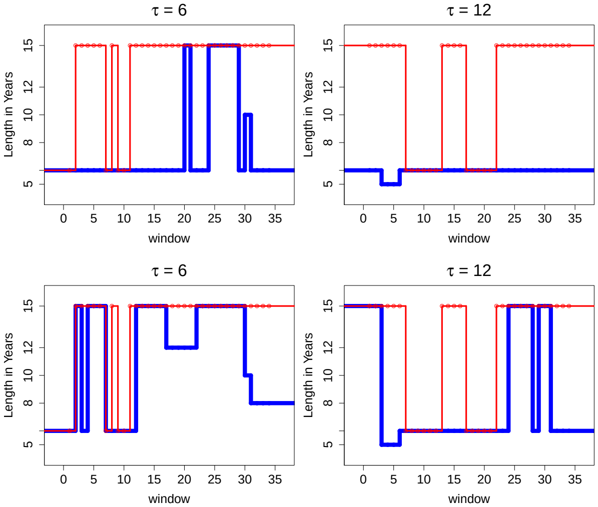
- Sensitive to I_k
  - Macroeconomic: 1-year return of S&P500
  - Firm specific: Liquidity, profitability
  - Intercept



$\rho = 0.50$

$\rho = 0.75$

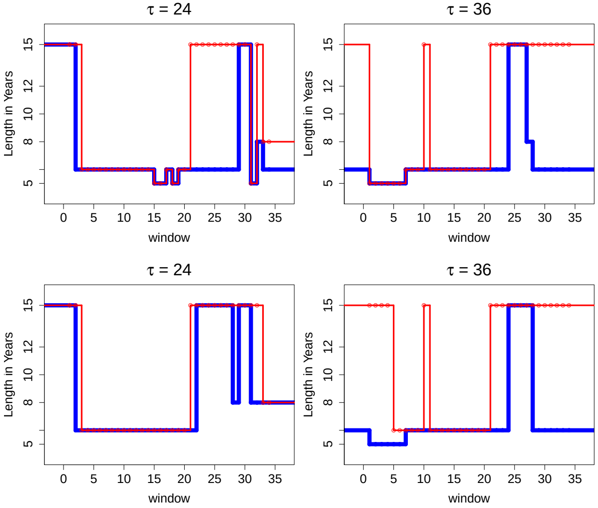
Table 2: Interval of homogeneity, $r = \{0.5, 1\}$, $\tau = \{1, 3\}$ months.



$\rho = 0.50$

$\rho = 0.75$

Table 3: Interval of homogeneity, $r = \{0.5, 1\}$, $\tau = \{6, 12\}$ months.



$\rho = 0.50$

$\rho = 0.75$

Table 4: Interval of homogeneity, $r = \{0.5, 1\}$, $\tau = \{24, 36\}$ months.

Accuracy Ratio, $\rho = 0.50$

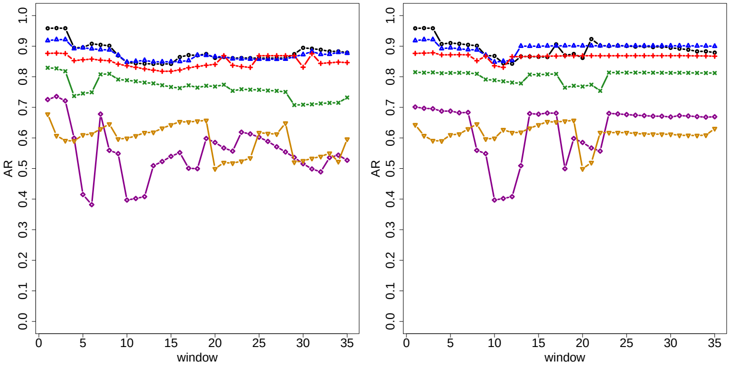


Figure 6: AR over windows, $r = 0.5$ (left), $r = 1$ (right), for $\tau = \{1, 3, 6, 12, 24, 36\}$ months horizons.

Accuracy Ratio, $\rho = 0.75$

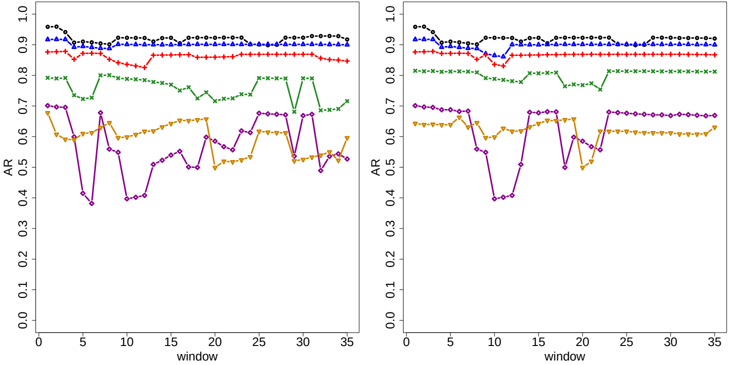


Figure 7: AR over windows, $r = 0.5$ (left), $r = 1$ (right), for $\tau = \{1, 3, 6, 12, 24, 36\}$ months horizons.

window	$\tau=1$ global	$\tau=1$ local $r=0.5,\rho=0.5$	$\tau=1$ local $r=0.5,\rho=0.75$	$\tau=1$ local $r=1,\rho=0.5$	$\tau=1$ local $r=1,\rho=0.75$	$\tau=3$ global	$\tau=3$ local $r=0.5,\rho=0.5$	$\tau=3$ local $r=0.5,\rho=0.75$	$\tau=3$ local $r=1,\rho=0.5$	$\tau=3$ local $r=1,\rho=0.75$	$\tau=6$ global	$\tau=6$ local $r=0.5,\rho=0.5$	$\tau=6$ local $r=0.5,\rho=0.75$	$\tau=6$ local $r=1,\rho=0.5$	$\tau=6$ local $r=1,\rho=0.75$
1		$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
2		$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
$\vdots$															
34			$\surd$			$\surd$					$\star$		$\star$		
35	$\surd$					$\surd$					$\star$		$\star$		
	global	local				global	local				global	local			
	$\surd$ (14)	$\surd$ (7)				$\surd$ (32)	$\surd$ (3)				$\surd$ (13)	$\surd$ (13)			
	$\star$ (14)	$\star$ (14)				$\star$ (0)	$\star$ (0)				$\star$ (9)	$\star$ (9)			

Table 5: AR-based performance for horizon 1, 3, and 6 months. Mark $\surd$ denotes the corresponding approach results in higher AR whereas $\star$ denotes equal accuracy for both.

window	$\tau = 12$					$\tau = 24$					$\tau = 36$				
	global	local				global	local				global	local			
		$r = 0.5, \rho = 0.5$	$r = 0.5, \rho = 0.75$	$r = 1, \rho = 0.5$	$r = 1, \rho = 0.75$		$r = 0.5, \rho = 0.5$	$r = 0.5, \rho = 0.75$	$r = 1, \rho = 0.5$	$r = 1, \rho = 0.75$		$r = 0.5, \rho = 0.5$	$r = 0.5, \rho = 0.75$	$r = 1, \rho = 0.5$	$r = 1, \rho = 0.75$
1		$\surd$	$\surd$	$\surd$	$\surd$			$\surd$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
2		$\surd$	$\surd$	$\surd$	$\surd$			$\surd$	$\surd$	$\surd$					$\surd$
$\vdots$															
34				$\surd$	$\surd$				$\surd$	$\surd$	$\star$			$\star$	$\star$
35				$\surd$	$\surd$	$\star$			$\star$	$\star$				$\surd$	$\surd$
	global	local				global	local				global	local			
	$\surd$ (17)	$\surd$ (18)				$\surd$ (10)	$\surd$ (23)				$\surd$ (3)	$\surd$ (26)			
	$\star$ (0)	$\star$ (0)				$\star$ (2)	$\star$ (2)				$\star$ (6)	$\star$ (6)			

Table 6: AR-based performance for horizon 12, 24, and 36 months. Mark $\surd$ denotes the corresponding approach results in higher AR whereas $\star$ denotes equal accuracy for both.

Conclusion

- Employing all past observation (as benchmark) results in better accuracy prediction for short horizon (1 and 3 months)
- Local approach performs with the same accuracy as the benchmark for six months horizon
- The accuracy prediction is improved for the longer horizon $(12, 24, 36$ months$)$

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Duffie, Saita, Wang (2007)

Surv. Prob.

Conditional probability of default (PD) within τ years ahead

$$\mathrm{E}\left[\int_t^{t+\tau} \exp\left\{-\int_t^s (\lambda_u + \phi_u)\, du\right\}\lambda_s\, ds \,|X_t\right]$$

$\{X_t : t \geq 0\}$ be time-homogeneous Markov process in $\mathbb{R}^p, p \geq 1$
$\lambda_t = \wedge_1(X_t)$ and $\phi_t = \wedge_2(X_t)$
$\wedge$ is non-negative real-valued measurable function on $\mathbb{R}^p$

State variable X_t governing the Poisson intensities are assumed to follow a specific high-dimensional VAR process

Deducing PD multiperiod ahead from repeating one-period ahead prediction

Poisson process

Back

Let D_i are times between jumps (events), $\{D_i\}_{i=1}^{n}$ i.i.d. $\exp(\lambda)$

$$T_n = \sum_{i=1}^{n} D_i, \quad T_0 = 0$$

Poisson process with intensity λ:

$$N(t) = \sup\{n \geq 0 : T_n \leq t\} \text{ for } t \geq 0$$

Filtration

Number of jumps in $[t, t+\tau] \sim \mathrm{Pois}(\lambda\tau)$

$$\mathrm{P}\left[N(t+\tau) - N(t) = d\right] = \frac{e^{-(\lambda\tau)}(\lambda\tau)^d}{d!}$$

Poisson distribution

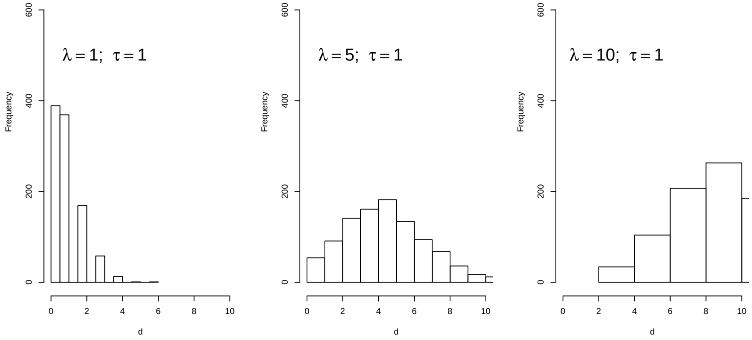


Figure 8: Distribution of number of evants in $[t, t + \tau]$ follow Poisson distribution. Sample size $n = 1000$.

Non-homogeneous Poisson process

Back

Intensity λ_t may change over the time

$$\mathsf{E}\left[N(\tau)|\lambda_s, t \le s \le t+\tau\,\right] = \int_t^{t+\tau} \lambda_s \, ds$$

Number of jumps in $[t, t+\tau] \sim \mathrm{Pois}\left(\int_t^{t+\tau} \lambda_s \, ds\right)$

$$\mathsf{P}\left[N(t+\tau) - N(t) = d\,\right] = \frac{e^{-\int_t^{t+\tau} \lambda_s \, ds} \left(\int_t^{t+\tau} \lambda_s \, ds\right)^d}{d!}$$

Filtration

τ_{Di} is default time of firm i

$$\mathrm{P}\left(\tau_{Di} > t + \tau | \mathcal{F}_t\right) = \mathrm{E}\left[1_{\{\tau_{Di} \, > \, t+\tau\}} | \mathcal{F}_t\right] \tag{16}$$

Let $X_t = (W_t, U_t)$, W is common factor and U is firm-specific
$\{\mathcal{F}_t : t \geq 0\}$ is filtration, where $\mathcal{F}_t$ is σ-algebra generated by

$$\{(U_\tau, D_\tau) : \tau \leq \min(t, \tau_D)\} \cup \{W_\tau : \tau \leq t\}$$

with D be Poisson process with intensity $\lambda(X_t)$

▸ Poisson process, $D_\tau = N(\tau)$

Conditioning on observable smaller filtration

$$\mathrm{P}\left(\tau_{Di} > t + \tau | \mathcal{F}_t\right) \stackrel{\text{def}}{=} \mathrm{E}\left[1_{\{\tau_{Di} \, > \, t+\tau\}} | X_t\right] = \mathrm{P}\left(\tau_{Di} > t + \tau | X_t\right)$$

Filtration

back

Let $X_t = (W_t, U_t)$, W is common factor and U is firm-specific
$\{\mathcal{F}_t : t \geq 0\}$ is filtration, where $\mathcal{F}_t$ is σ-algebra generated by

$$\{(U_\tau, D_\tau, O_\tau) : \tau \leq \min(t, \tau_D, \tau_O)\} \cup \{W_\tau : \tau \leq t\}$$

with (D, O) be doubly stochastic Poisson process with intensity $\lambda(X_t)$ for default and $\phi(X_t)$ for other exit

τ_{Di} is default time of firm i as stopping time

$$\tau_{Di} = \inf\{t : D_t > 0, O_t = 0\}$$

Forward intensity at τ

Back

$$
\begin{aligned}
G_{it}(\tau) &= 1 - \exp\left\{-\psi_{it}(\tau)\tau\right\} \\
\\
G'_{it}(\tau) &= -\exp\left\{-\psi_{it}(\tau)\tau\right\}\left\{-\psi'_{it}(\tau)\tau - \psi_{it}(\tau)\right\} \\
&= \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi'_{it}(\tau)\tau + \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi_{it}(\tau)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{G'_{it}(\tau)}{1 - G_{it}(\tau)} &= \frac{\exp\left\{-\psi_{it}(\tau)\tau\right\}\psi_{it}(\tau) + \exp\left\{-\psi_{it}(\tau)\tau\right\}\psi'_{it}(\tau)\tau}{\exp\left\{-\psi_{it}(\tau)\tau\right\}} \\
&= \psi_{it}(\tau) + \psi'_{it}(\tau)\tau
\end{aligned}
$$

Forward intensity at τ

Back

$$g_{it}(\tau) = \psi_{it}(\tau) + \psi'_{it}(\tau)\tau$$

Therefore

$$\begin{aligned}
\int_0^\tau g_{it}(s)ds &= \int_0^\tau \psi_{it}(s)ds + \int_0^\tau \psi'_{it}(s)s\ ds \\
&= \int_0^\tau \psi_{it}(s)ds + \psi_{it}(\tau)\tau - \int_0^\tau \psi_{it}(s)ds \\
&= \psi_{it}(\tau)\tau
\end{aligned}$$

Likelihood

Back

In I_k and t
use the status info $\{\text{survive}, \text{default}, \text{other exit}\}$ of firm i at $t+\tau$

$$
\begin{aligned}
L_{\tau,i,t}(\alpha_k, \beta_k) \\
= \; & \mathbf{1}_{\{t_{0i} \le t,\ \tau_{Ci} > t+\tau\}} \mathrm{P}_t(\tau_{Ci} > t+\tau) \\
& + \mathbf{1}_{\{t_{0i} \le t,\ \tau_{Di} = \tau_{Ci} \le t+\tau\}} \mathrm{P}_t(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \le t+\tau) \\
& + \mathbf{1}_{\{t_{0i} \le t, \tau_{Di} \ne \tau_{Ci}, \tau_{Ci} \le t+\tau\}} \mathrm{P}_t(\tau_{Ci}; \tau_{Di} \ne \tau_{Ci} \& \tau_{Ci} \le t+\tau) \\
& + \mathbf{1}_{\{t_{0i} > t\}} + \mathbf{1}_{\{\tau_{Ci} \le t\}}
\end{aligned}
$$

with $\mathrm{P}_t(\tau_{Ci}) = \mathrm{P}(\tau_{Ci} | \mathcal{F}_t)$ and t_{0i} be the first month that firm i appeared in the sample

Pseudo-Likelihood

Back

with $\Delta t = 1/12$, approximate integral by sum

$$\mathrm{P}_t\left(\tau_{Ci} > t+\tau\right) = \exp\left\{-\sum_{s=0}^{\tau-1} g_{it}(s)\Delta t\right\}$$

$$\mathrm{P}_t\left(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \le t+\tau\right)$$

$$= \begin{cases} 1-\exp\left\{-f_{it}(0)\Delta t\right\} & \text{if} \quad \tau_{Ci} = t+1, \\ \\ \left[1-\exp\left\{-f_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\}\right] & \\ \times \exp\left\{-\sum_{s=0}^{\tau_{Ci}-t-2} g_{it}(s)\Delta t\right\} & \text{if} \quad t+1 < \tau_{Ci} \le t+\tau \end{cases}$$

Pseudo-Likelihood

Back

$$
\begin{aligned}
&\mathrm{P}_t\left(\tau_{Ci};\tau_{Di}\neq\tau_{Ci}\&\tau_{Ci}\leq t+\tau\right)\\
&=\begin{cases}
\exp\left\{-f_{it}(0)\Delta t\right\}-\exp\left\{-g_{it}(0)\Delta t\right\} & \text{if}\quad \tau_{Ci}=t+1,\\
\\
\exp\left\{-f_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\} & \\
-\exp\left\{-g_{it}\left(\tau_{Ci}-t-1\right)\Delta t\right\} & \\
\times\exp\left\{-\sum_{s=0}^{\tau_{Ci}-t-2} g_{it}(s)\Delta t\right\} & \text{if}\quad t+1<\tau_{Ci}\leq t+\tau
\end{cases}
\end{aligned}
$$

Forward PD

Provided by estimate $\widetilde{\theta}$, ▸ Decomposable Log-Lik.

In $[t+\tau, t+\tau+1]$ with discretized time interval $\Delta t = 1/12$

(i) Forward probability of default (PD)

$$\mathsf{P}_t(t+\tau < \tau_{Di} = \tau_{Ci} \le t+\tau+1) = e^{-\psi_{it}(\tau)\tau\Delta t}\left\{1 - e^{-f_{it}(\tau)\Delta t}\right\}$$

(ii) Forward combined exit probability

$$\mathsf{P}_t(t+\tau < \tau_{Ci} \le t+\tau+1) = e^{-\psi_{it}(\tau)\tau\Delta t}\left\{1 - e^{-g_{it}(\tau)\Delta t}\right\}$$

Forward PD

In interval $[t, t+\tau]$, ▸ Decomposable Log-Lik.

(iii) Cumulative PD

$$\mathrm{P}_t(t < \tau_{Di} = \tau_{Ci} \leq t + \tau) = \sum_{s=0}^{\tau-1} e^{-\psi_{it}(s)s\Delta t} \left\{1 - e^{-f_{it}(s)\Delta t}\right\}$$

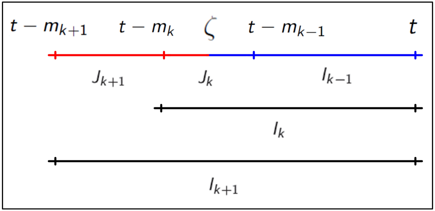
(iv) Spot combined exit intensity

$$\psi_{it}(\tau) = \frac{1}{\tau} \left\{\psi_{it}(\tau - 1)(\tau - 1) + g_{it}(\tau - 1)\right\}$$

No need to specify $\psi_{it}(0)$ since it is irrelevant

Sequential test ($k = 1, \ldots, K$), fixed τ Back

H_0 : parameter homogeneity within I_k
H_1 : change point within I_k

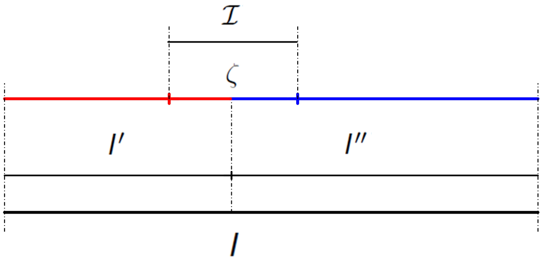


$$T_{k,\tau} = \sup_{\zeta \in J_k} \left[L_{A_{k,\zeta,\tau}} \left\{ \widetilde{\theta}_{A_{k,\zeta}} \right\} + L_{B_{k,\zeta,\tau}} \left\{ \widetilde{\theta}_{B_{k,\zeta}} \right\} - L_{I_{k+1},\tau} \left\{ \widetilde{\theta}_{I_{k+1}} \right\} \right],$$ Detail

with $J_k = I_k \setminus I_{k-1}$, $A_{k,\zeta} = [t - m_{k+1}, \zeta + \tau]$ and $B_{k,\zeta} = (\zeta, t + \tau]$
$I_k = [t - m_k, t + \tau]$ and $I_{k-1} = [t - m_{k-1}, t + \tau]$

Test statistics

Back



$\mathcal{I}$: tested interval possibly contain change point
$I = [I', I'')$: larger testing interval

Test statistics

Back

H_0 : homogeneity within $\mathcal{I}$ vs. H_1 : change point within $\mathcal{I}$
LRT Statistics, $L(\cdot)$ is log likelihood function

$$\begin{aligned} T_{\mathcal{I},\zeta} &= \max_{\theta',\theta''} \left\{ L_{I''}(\theta'') + L_{I'}(\theta') \right\} - \max_{\theta} L_I(\theta) \\ &= L_{I'}(\widetilde{\theta}_{I'}) + L_{I''}(\widetilde{\theta}_{I''}) - L_I(\widetilde{\theta}_I) \end{aligned}$$

Reject H_0 if $T_{\mathcal{I},\zeta} \geq \mathfrak{z}$
Thus,

$$T_{\mathcal{I}} = \max_{\zeta \in \mathcal{I}} T_{\mathcal{I},\zeta}$$

Let $\mathcal{I} = I_k \setminus I_{k-1}$,

$$\widehat{\boldsymbol{\theta}} = \widetilde{\boldsymbol{\theta}}_{\widehat{k}}, \quad \widehat{k} = \max_{k \leq K} \{ k : T_\ell \leq \mathfrak{z}_\ell, \ell \leq k \}$$

LRT: Poisson distribution

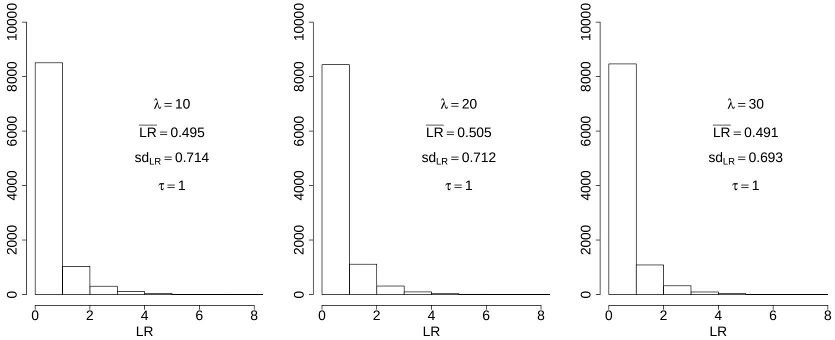


Figure 9: Monte Carlo simulation, similar result for $\lambda = 1, 2, \ldots, 9$

LRT: Exponential distribution

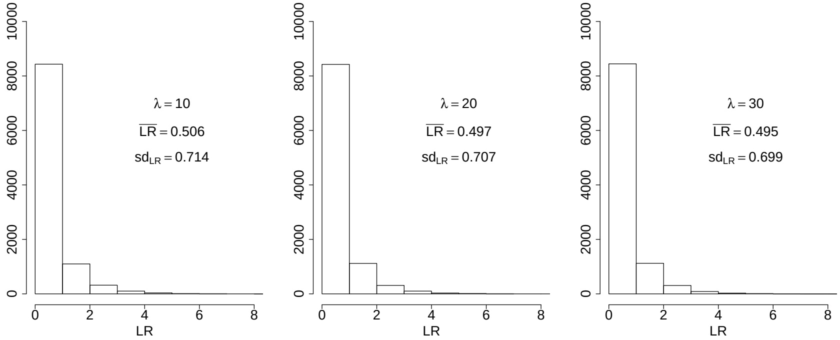


Figure 10: Monte Carlo simulation, similar result for $\lambda = 1, 2, \ldots, 9$

Hyper parameters

Back

- The role of ρ is similar to the significance level of a test
- The r denotes the power of loss function

$$\mathsf{E}_{\theta^*} L^r_{k,\tau}\left\{\widetilde{\theta}_k, \widehat{\theta}_k\right\} \to \mathsf{P}_{\theta^*}\left\{\widetilde{\theta}_k \neq \widehat{\theta}_k\right\}, \quad r \to 0.$$

- The $\mathfrak{z}_{1,\tau}; \ldots; \mathfrak{z}_{K-1,\tau}$ enter implicitely in the propagation condition: if false alarm event $\left\{\widetilde{\theta}_k \neq \widehat{\theta}_k\right\}$ happen too often, it is indication that some $\mathfrak{z}_{1,\tau}; \ldots; \mathfrak{z}_{k-1,\tau}$ are too small
- Note: propagation condition relies on artificial parametric model P_{θ^*} instead of the true model P

Parametric risk bound

▸ Propagation

$$
\begin{aligned}
\mathrm{E}_{\theta^*} \left| L_K(\widetilde{\theta}_K, \theta^*) \right|^r &= \mathcal{R}_r(\theta^*) \\
&= -\int_{\mathfrak{z} \geq 0} \mathfrak{z}^r \, d\, \mathrm{P}_{\theta^*} \left\{ \left| L_K(\widetilde{\theta}_K, \theta^*) \right| > \mathfrak{z} \right\} \\
&= r \int_0^\infty \mathfrak{z}^{r-1} \, \mathrm{P}_{\theta^*} \left\{ \left| L_K(\widetilde{\theta}_K, \theta^*) \right| > \mathfrak{z} \right\} d\mathfrak{z} \\
&= r \int_0^\infty \mathfrak{z}^{r-1} \, \mathrm{P}_{\theta^*} \left\{ \left| L_K(\widetilde{\theta}_K, \theta^*) \right| > \mathfrak{z}, \widetilde{\theta}_K \in \mathcal{E}(\mathfrak{z}) \right\} d\mathfrak{z} \\
&\quad + r \int_0^\infty \mathfrak{z}^{r-1} \, \mathrm{P}_{\theta^*} \left\{ \left| L_K(\widetilde{\theta}_K, \theta^*) \right| > \mathfrak{z}, \widetilde{\theta}_K \notin \mathcal{E}(\mathfrak{z}) \right\} d\mathfrak{z} \\
&\leq 2r \int_0^\infty \mathfrak{z}^{r-1} e^{-\mathfrak{z}} \, d\mathfrak{z} < \infty
\end{aligned}
$$

Note: $\mathcal{E}(\mathfrak{z}) \stackrel{\mathrm{def}}{=} \left\{ \theta^* : L_K(\widetilde{\theta}_K) - L_K(\theta^*) \leq \mathfrak{z} \right\}$

Distance-to-Default (DTD), Merton

firms are financed by equity (E) and one single pure discount bond with maturity time T and principal Db (book value of the debt).
Firm's asset value $V_{A,t}$ follow Geometric Brownian Motion (GBM)

$$dV_{A,t} = \mu V_{A,t} dt + \sigma_A V_{A,t} dB_t \tag{17}$$

μ and σ_A are instantaneous drift and volatility, B is standard Wiener process

Black-Scholes model

$$V_{E,t} = V_{A,t}\Phi(d_{1,t}) - Db\, e^{-r(T-t)}\Phi(d_{2,t}) \tag{18}$$

with

$$d_{1,t} = \frac{\log(V_{A,t}/Db) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{(T-t)}}, \quad d_{2,t} = d_{1,t} - \sigma_A\sqrt{T-t} \tag{19}$$

where $V_{E,t}$ is market value of equity at time t, $(T-t)$ is time to expiration (of call option V_A), and r is risk-free interest rate

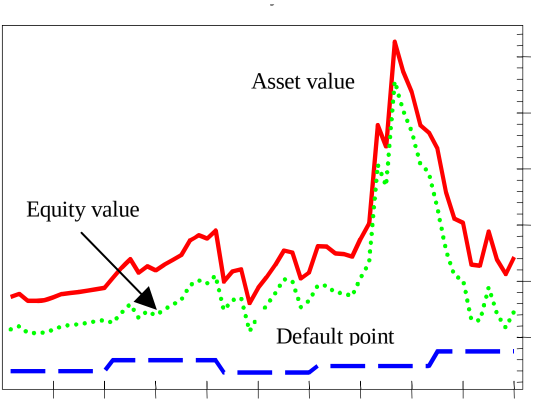


Figure 11: Market value of asset, equity, and book value of liabilities (default point)

DTD, Merton & KMV

▸ Variable

Probability of Default (PD)

$$\mathrm{PD}_t = \mathrm{P}(V_{A,t+T} \leq Db_t | V_{A,t}) = \Phi(DTD_t)$$

with

$$DTD_t = \frac{\log(V_{A,t}/Db) + (\mu - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{(T-t)}} \tag{20}$$

μ cannot be estimated with reasonable precision unless for very long time span data

KMV's DTD avoids using μ

$$DTD_t = \frac{\log(V_{A,t}/Db)}{\sigma_A\sqrt{(T-t)}} \tag{21}$$

Distance-to-Default (DTD)

▸ Variable

KMV typically set $(T-t)$ to one year and default point

$$Db = Db_{ST} + 0.5 Db_{LT} \tag{22}$$

where ST is hort term and LT is long term

Problem: Financial firm typically have large amount of liabilities that are neither classified as ST nor LT

Duan (2012) modified KMV default point as

$$Db = Db_{ST} + 0.5 Db_{LT} + \delta Db_{other} \tag{23}$$

Idiosyncratic Volatility

▸ Variable

Over the preceeding 12 months

$$R_{it} = \beta R_t + \varepsilon_{it}, \quad \varepsilon_{it} \sim \mathrm{N}(0, \sigma_{it}^2) \tag{24}$$

R_{it} is stock return of firm i
R_t is value-wieghted CRSP monthly return
σ_{it} is one-year idiosyncratic volatility

Following Shumway (2001), σ_{it} is missing if there are less than 12 monthly returns