FASTEC- FActorizable Sparse Tail Event Curves

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Motivation

- Many data come as curves or bunch of time series
- Common structure analysis:
  - Spread: between $\{q(\tau), q(1-\tau)\}$ τ-range; Changes of τ-range: expanding, shrinking, shifting, shifting with expanding/shrinking
  - Tail: τ is close to 0 or 1. Tail event curves (TEC)
- Sparsity: common structure is reduced to a few factors
- FASTEC: FActorisable Sparse Tail Event Curve

FASTEC construction

- Data: $\{(\boldsymbol{X}_i, \boldsymbol{Y}_i)\}_{i=1}^n$ in $\mathbb{R}^{p+m}$ i.i.d.
- Linear model for τ-quantile curve of Y_j, $j = 1, ..., m$, $0 < \tau < 1$:

$$q_j(\tau|\boldsymbol{X}_i) = \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j}(\tau), \tag{1}$$

where coefficients for j response: $\boldsymbol{\Gamma}_{*j} \in \mathbb{R}^p$

- Sparse factorisation: $f_k^\tau(\boldsymbol{X}_i) = \boldsymbol{\varphi}_k^\top(\tau)\boldsymbol{X}_i$ factors

$$q_j(\tau|\boldsymbol{X}_i) = \sum_{k=1}^{r} \psi_{j,k}(\tau) f_k^\tau(\boldsymbol{X}_i), \tag{2}$$

where
$\boldsymbol{\Gamma}_{*j}(\tau) = (\sum_{k=1}^r \psi_{j,k}(\tau)\varphi_{k,1}(\tau), ..., \sum_{k=1}^r \psi_{j,k}(\tau)\varphi_{k,p}(\tau))$

FASTEC examples

- CAViaR: Y_{ij} log returns at i day and institution j; $\boldsymbol{X}_i$: $\cup_{j=1}^{m}(|Y_{i-1,j}|, Y_{i-1,j}^{-})$ is of $p = 2m$ dimension, $j = 1, ..., m$, $i = 1, ..., n$;
- Temperature data: Y_{ij}: temperature at i day and j weather station; $\boldsymbol{X}_i = (b_1(t_i), ..., b_p(t_i))$, where b_1,...,b_p: B-spline basis, $t = i/n$, $i = 1, ..., n$;
- Further application: image analysis, joint analysis of many images and find the common patterns

Temperature Data

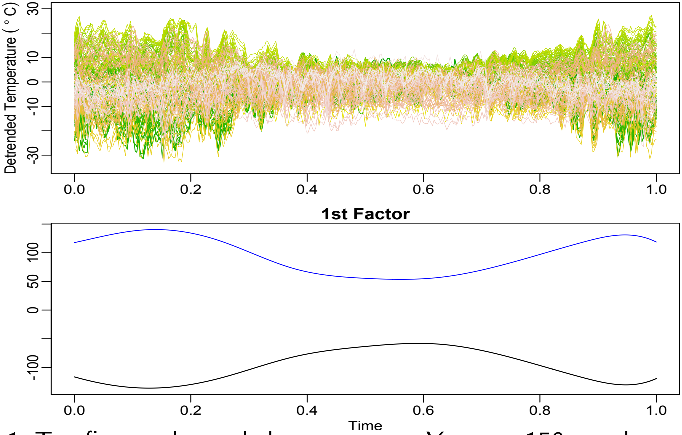


Figure 1: Top figure: detrended temperature Y_{ij}, $m = 159$ weather station, $t = i/n$ time point in year 2008, $n = 365$; bottom figure: quantile factors 1 $f_1^{0.01}(\boldsymbol{X}_i)$ and $f_1^{0.99}(\boldsymbol{X}_i)$; $p = n^{0.4} \approx 11$. ▸ Detrending

Financial Data

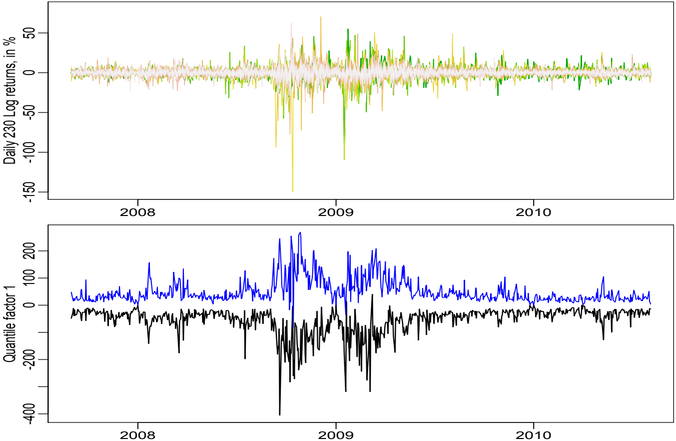


Figure 2: Top figure: log returns Y_{ij}, $n = 765$ ranging from Aug. 2007-Aug. 2010. $m = 230$ firm index. $p = 460$ covariate dimension; bottom figure: quantile factors 1 $f_1^{0.01}(\boldsymbol{X}_i)$, $f_1^{0.99}(\boldsymbol{X}_i)$.

Fluid Data

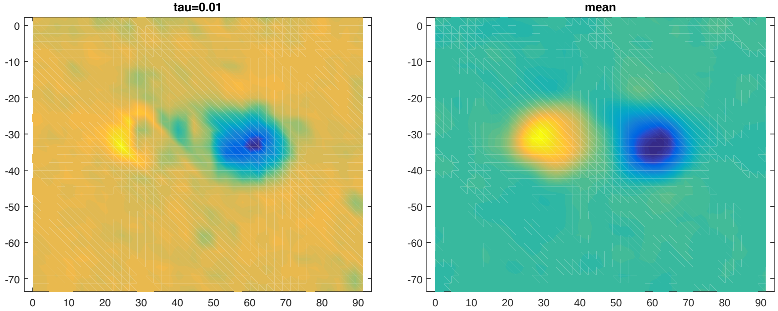


Figure 3: 2D vortex vorticity distribution. Darker color: higher vorticity. Yellow: clockwise; blue: counter-clockwise.

Spread gestalt

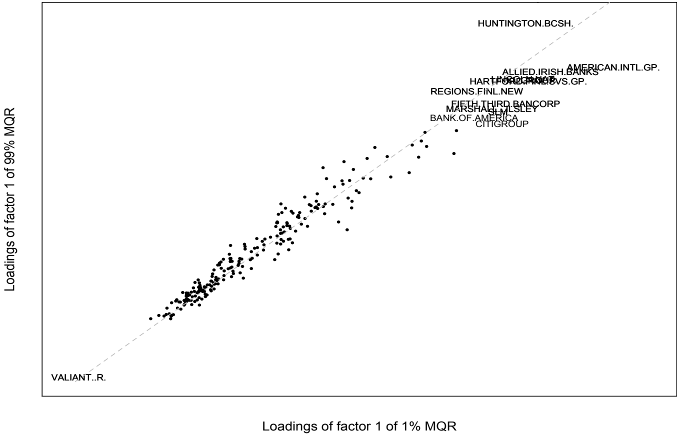


Figure 4: Loadings $(\psi_{j,1}(0.01), \psi_{j,1}(0.99))$ on factors 1 for 230 firms. Close distance indicates similar τ-range pattern.

Tail behavior

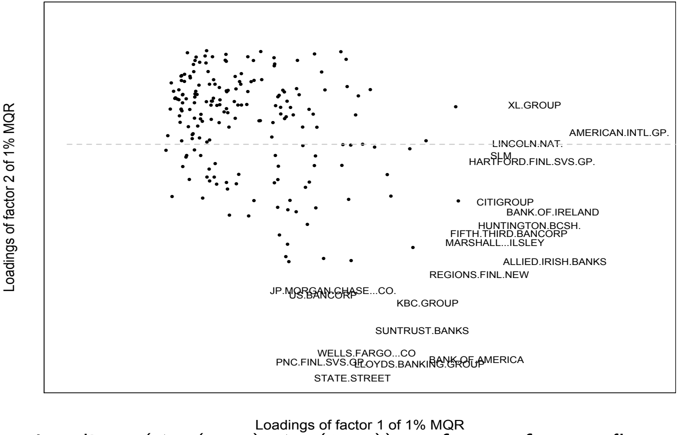


Figure 5: Loadings $(\psi_{j,1}(0.01), \psi_{j,2}(0.01))$ on factors for 230 firms. Close distance implies similar τ-quantile behavior.

Challenges

- Implementation of FASTEC needs regularized multivariate quantile regression (MQR)
  - Estimation
  - Proper model tuning
  - Non-asymptotic error bounds
- Applications: High-dimensional joint Value-at-Risk analysis, common temperature risk
- Dimension reduction

Outline

1. Motivation ✓
2. High-dimensional multivariate quantile regression (MQR)
3. Algorithm and convergence analysis (FASTEC)
4. Oracle inequalities
5. Numerical analysis
6. Application 1: Sparse Asymmetric Multivariate Conditional Value-at-Risk (SAMCVaR) Model
7. Application 2: Chinese temperature data at year 2008

Implementing FASTEC: MQR formulation

Recall $q_j(\tau|\boldsymbol{X}_i) = \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j}(\tau)$, for $\boldsymbol{\Gamma} = [\boldsymbol{\Gamma}_{*1}, ..., \boldsymbol{\Gamma}_{*m}]$,

$$L(\boldsymbol{\Gamma}) \stackrel{\text{def}}{=} \underbrace{(nm)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho_\tau (Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j})}_{(A)} + \lambda \underbrace{\|\boldsymbol{\Gamma}\|_*}_{(B)}, \tag{3}$$

$$\widehat{\boldsymbol{\Gamma}}_{\lambda,\tau} \stackrel{\text{def}}{=} \arg \min_{\boldsymbol{\Gamma} \in \mathbb{R}^{p \times m}} L(\boldsymbol{\Gamma}) \tag{4}$$

$\rho_\tau(u) = |\mathbf{I}(u \le 0) - \tau||u|$. $\boldsymbol{\Gamma}_{*j}$: jth column of $\boldsymbol{\Gamma}$. ▸ Shape ρ_τ

- (A): quantile regression fitting quality. Ferguson (1967), Koenker and Bassett (1978), Koenker and Portnoy (1990)

Recall $q_j(\tau|\boldsymbol{X}_i) = \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j}(\tau)$, for $\boldsymbol{\Gamma} \in \mathbb{R}^{p\times m}$,

$$L(\boldsymbol{\Gamma}) \stackrel{\text{def}}{=} \underbrace{(nm)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_\tau(Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j})}_{(A)} + \lambda \underbrace{\|\boldsymbol{\Gamma}\|_*}_{(B)}, \tag{5}$$

$$\widehat{\boldsymbol{\Gamma}}_{\lambda,\tau} \stackrel{\text{def}}{=} \arg \min_{\boldsymbol{\Gamma}\in\mathbb{R}^{p\times m}} L(\boldsymbol{\Gamma}) \tag{6}$$

- (B): nuclear norm $\|\boldsymbol{\Gamma}\|_* = \sum_{k=1}^{\text{rank}(\boldsymbol{\Gamma})} \sigma_k(\boldsymbol{\Gamma})$ prompts 0 for singular values, $\text{rank}(\boldsymbol{\Gamma})$= number of nonzero singular values
- $\lambda = \lambda_{n,p,m,\mathbf{X},\tau} > 0$ converges to 0 as $n \to \infty$

Multivariate regression

- $\{(\boldsymbol{X}_i, \boldsymbol{Y}_i)\}_{i=1}^n$ in $\mathbb{R}^{p+m}$ i.i.d. $\boldsymbol{Y}_i = \boldsymbol{\Gamma}^\top \boldsymbol{X}_i + \varepsilon_i$, $\mathsf{E}\varepsilon_i = 0$. Factor number $= \mathrm{rank}(\boldsymbol{\Gamma})$
- Izenman (1975), Reinsel and Velu (1998): Reduced-rank model. $\varepsilon_i \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$

$$\min_{\substack{\mathbf{S} \in \mathbb{R}^{p \times m} \\ \mathrm{rank}(\mathbf{S}) = r < \min\{p, m\}}} \mathrm{tr}\left\{(\mathbf{Y} - \mathbf{X}\mathbf{S})\boldsymbol{\Omega}(\mathbf{Y} - \mathbf{X}\mathbf{S})^\top\right\}.$$

$\boldsymbol{\Omega} = \boldsymbol{\Sigma}_\varepsilon^{-1}$: efficient estimator; $\boldsymbol{\Omega} = \mathbf{I}_m$: consistent estimator

Multivariate regression

- $\min_{\mathbf{S} \in \mathbb{R}^{p \times m}} \|\mathbf{Y} - \mathbf{X}\mathbf{S}\|_{\mathrm{F}} + \lambda \operatorname{rank}(\mathbf{S})$ Bunea et al. (2011). $\| \cdot \|_{\mathrm{F}}$: Frobenius norm. Not convex.
- $\min_{\mathbf{S} \in \mathbb{R}^{p \times m}} \|\mathbf{Y} - \mathbf{X}\mathbf{S}\|_{\mathrm{F}} + \lambda \|\mathbf{S}\|_{*}$, $\|\mathbf{S}\|_{*}$ Yuan et al. (2007), Negahban and Wainwright (2011). Convex and easy to work with non-smooth convex empirical loss

Estimating Γ

$$L(\boldsymbol{\Gamma}) = \underbrace{(nm)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho_\tau(Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j})}_{(A)} + \lambda \underbrace{\|\boldsymbol{\Gamma}\|_*}_{(B)},$$

- Convex optimization problem
- (A) and (B) are non-smooth

Estimating Γ

$$L(\boldsymbol{\Gamma}) = \underbrace{(nm)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_\tau(Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j})}_{(A)} + \lambda \underbrace{\|\boldsymbol{\Gamma}\|_*}_{(B)},$$

- Smoothed fast iterative shrinkage-thresholding algorithm (SFISTA)
  - Smoothing (A) by $f_\kappa(\boldsymbol{\Gamma})$ with Lipschitz gradient, where $\kappa > 0$ controls trade-off "smoothness" v.s. "quality of approximation"

Smoothing by Nesterov (2005)

Estimating Γ

$$L(\boldsymbol{\Gamma}) = \underbrace{(nm)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho_\tau(Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j})}_{(A)} + \lambda \underbrace{\|\boldsymbol{\Gamma}\|_*}_{(B)},$$

- Smoothed fast iterative shrinkage-thresholding algorithm (SFISTA)
  - Proximity operator of (B) has a closed form $S_{\|\cdot\|_*,\lambda}$
    Proximity operator
  - Fast iterative shrinkage-thresholding algorithm (FISTA), Beck and Toubelle (2009)

Fast iterative shrinkage-thresholding algorithm (FISTA)

$$\boldsymbol{\Gamma}_t = S_{\frac{\lambda}{M}}(\boldsymbol{\Omega}_t - \tfrac{1}{M}\nabla_{\boldsymbol{\Gamma}} f_\kappa(\boldsymbol{\Omega}_t))$$

Soft-thresholding singular values

Step size: $\delta_{t+1} = \frac{1+\sqrt{1+4\delta_t^2}}{2}$

$$\boldsymbol{\Omega}_{t+1} = \boldsymbol{\Gamma}_t + \tfrac{\delta_t - 1}{\delta_{t+1}}(\boldsymbol{\Gamma}_t - \boldsymbol{\Gamma}_{t-1})$$

Theorem (Convergence analysis of SFISTA)

Let $\{\boldsymbol{\Gamma}_t\}_{t=0}^T$ *be the SFISTA sequence, and* $\widehat{\boldsymbol{\Gamma}}_{\tau,\lambda}$ *minimize* (5) *for* $0 < \tau < 1$ *and* $\lambda > 0$*. Then for any* t *and* $\epsilon > 0$,

$$\left| L(\boldsymbol{\Gamma}_t) - L(\widehat{\boldsymbol{\Gamma}}_{\tau,\lambda}) \right| \leq \underbrace{\frac{\epsilon\{\tau \vee (1-\tau)\}^2}{2}}_{\text{Loss from smoothing}} + \underbrace{\frac{4mn\|\boldsymbol{\Gamma}_0 - \widehat{\boldsymbol{\Gamma}}_{\tau,\lambda}\|_{\mathrm{F}}^2 \|\mathbf{X}\|^2}{(t+1)^2\epsilon}}_{\text{convergence of FISTA}}. \tag{7}$$

Requiring $L(\boldsymbol{\Gamma}_t) - L(\widehat{\boldsymbol{\Gamma}}_{\tau,\lambda}) \leq \epsilon$ *(e.g.* $\epsilon = 10^{-6}$*) yields*

$$t \geq 2\frac{\sqrt{mn}\|\widehat{\boldsymbol{\Gamma}}_{\tau,\lambda} - \boldsymbol{\Gamma}_0\|_{\mathrm{F}}\|\mathbf{X}\|}{\epsilon\left[1 - \frac{\{\tau\vee(1-\tau)\}^2}{2}\right]}. \tag{8}$$

$\|\mathbf{X}\|$*: spectral norm (largest singular value) of design matrix* $\mathbf{X}$.

▸ Proof

FASTEC estimation

- High-dimensional setting: $p, m \to \infty$ with n, $m = \dim(\boldsymbol{Y}_i)$; $p = \dim(\boldsymbol{X}_i)$
- Sparsity in factor: $\operatorname{rank}(\boldsymbol{\Gamma})$ is finite and fixed
- Quality measures:
  - Prediction error: $\|\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}\|^2_{L_2(\Pi)} \stackrel{\text{def}}{=} m^{-1}\mathsf{E}\|(\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma})^\top \boldsymbol{X}\|_2^2$, where Π is the distribution for $\boldsymbol{X}$
  - Frobenius error: $\|\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}\|^2_{\mathrm{F}} \stackrel{\text{def}}{=} \operatorname{tr}\{(\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma})(\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma})^\top\}$
  - Nuclear error: $\|\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}\|_* = \sum_{k=1}^{\operatorname{rank}(\boldsymbol{\Gamma})} \sigma_k(\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma})$

Estimation noise

Tuning parameter λ depends on:

$$\Delta_\tau \stackrel{\text{def}}{=} \left\|(mn)^{-1}\mathbf{X}^\top \mathbf{W}_\tau\right\|$$

$(\mathbf{W}_\tau)_{ij} = \mathbf{I}\{Y_{ij} - \boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{*j} \leq 0\} - \tau \sim \textit{Bernoulli}(\tau)$

Lemma

Under Assumptions 1 and 2, ▸ Assumption

$$n^{-1}\|\mathbf{X}^\top \mathbf{W}_\tau\| \leq C^* \sqrt{\sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}})\{\tau \vee (1-\tau)\}}\sqrt{\frac{p+m}{n}}, \qquad (9)$$

with probability greater than $1 - 3e^{-(p+m)\log 8} - \gamma_n$, *where* $\Sigma_{\boldsymbol{X}}$ *is the covariance matrix for* $\boldsymbol{X}_i$, $\gamma_n \to 0, C^* = 4\sqrt{2\frac{c_2}{C'}\log 8}$, C' *and* c_2 *are absolute constants.*

▸ γ_n

Nonasymptotic Risk Bounds

Theorem

Under regularity conditions and

Assumption

$$\lambda = 2C^* \sqrt{\sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}})\{\tau \vee (1-\tau)\}} \sqrt{\frac{p+m}{n}},$$

where C^* *and* $\Sigma_{\boldsymbol{X}}$ *are defined in previous page. Then*

$$\|\widehat{\boldsymbol{\Gamma}}_\tau - \boldsymbol{\Gamma}_\tau\|_{L_2(\Pi)} \leq \frac{C_0}{\underline{f}\sqrt{m}} \sqrt{\frac{\sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}})}{\sigma_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{X}})}} \sqrt{\tau \vee (1-\tau)} \sqrt{r} \sqrt{\frac{p+m}{n}}, \quad (10)$$

with probability greater than $1 - \gamma_n - 9(p+m)^{-2} - 3e^{-(p+m)\log 8}$ *and* $p+m > 3$, *where* $C_0 = 16\sqrt{2}\left\{\left(\sqrt{\frac{2}{C'}} + 4\right)\sqrt{c_2} \vee 4\sqrt{2\frac{c_2}{C'}\log 8}\right\}$,

- Dimensionality:
  - When p, m fixed: the estimator converges in rate $n^{-1/2}$
  - Oracle property: performance depends on unknown number of parameters $r(p+m)$
- Design: condition number $\sigma_{\max}(\mathbf{\Sigma}_{\boldsymbol{X}})/\sigma_{\min}(\mathbf{\Sigma}_{\boldsymbol{X}})$, where $\mathbf{\Sigma}_{\boldsymbol{X}}$ is the covariance for $\boldsymbol{X}$
- Conditional densities:
  - $\underline{f} = \inf_{j \leq m} \inf_{\boldsymbol{x}} f_{Y_{ij}|\boldsymbol{X}_i}(\boldsymbol{x}^\top \mathbf{\Gamma}_{*j}|\boldsymbol{x})$
  - Difficult to estimate at τ close to 0 or 1

Frobenius norm and nuclear norm bounds differ to the prediction error bound by a factor $\sqrt{m}$ and a constant

Tuning

- Δ_τ has the same distribution as

$$\Lambda_\tau = (nm)^{-1}\|\mathbf{X}^\top \widetilde{\mathbf{W}}_\tau\|, \qquad (11)$$

where $\widetilde{W}_{ij,\tau} = \mathbf{I}(U_{ij} \leq 0) - \tau$, $\{U_{ij}\}$ for $i = 1, ..., n$ and $j = 1, ..., m$ are i.i.d. $\mathrm{U}(0, 1)$

- Λ_τ is pivotal (independent of unknown $\mathbf{\Gamma}$) conditioning on $\boldsymbol{X}$

Tuning

- Bound estimation noise with α quantile of Λ, for small $0 < \alpha < 1$:

$$\lambda_\tau = 2 \cdot \Lambda_\tau(1 - \alpha|\mathbf{X}), \tag{12}$$

where $\Lambda_\tau(1 - \alpha|\mathbf{X}) \stackrel{\text{def}}{=} (1 - \alpha)$-quantile of Λ_τ conditional on $\mathbf{X}$ is computed via simulation (Λ_τ is pivotal)

- By symmetry, $\lambda_\tau = \lambda_{1-\tau}$
- Pivotal principle: QR-Lasso Belloni and Chernozuhkov (2011) and $\sqrt{\text{Lasso}}$ Belloni, Chernozuhkov and Wang (2011)

Simulation: symmetric situation

- $m = p = 500$, $n = 500$. Iteration=500.
- $\boldsymbol{X}_i$ i.i.d. $N(0, \boldsymbol{\Sigma})$ with $\Sigma_{ij} = 0.5^{|i-j|}$.

$$\boldsymbol{Y}_i = \boldsymbol{\Gamma}^\top \boldsymbol{X}_i + \varepsilon_i, \ \varepsilon_i \sim N(0, \mathbf{I}_m) \text{ i.i.d. } \varepsilon_i \perp \boldsymbol{X}$$

- $\boldsymbol{\Gamma}$ generation: sampling entries from i.i.d. $N(0, 1)$
    1. Model LS (less sparse): The last 375 singular values of $\boldsymbol{\Gamma}$ are 0, $r = \operatorname{rank}(\boldsymbol{\Gamma}) = 125$
    2. Model MS (moderately sparse): Set the first 10 singular values to 30 and the rest 0, $r = 10$
    3. Model ES (extremely sparse): Set the first singular value to 20 and the rest 0, $r = 1$

Simulation: asymmetric situation

- Simulate Y_{ij} with asymmetric conditional quantiles
- $m = p = 500$, $n = 500$. Iteration=500.
- Generating $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ with $\mathrm{rank}(\mathbf{\Gamma}_1) = 2$ and $\mathrm{rank}(\mathbf{\Gamma}_2) = r_2$:
- Model AES (Asymmetric Extremely Sparse): $r_2 = 2$
- Model AMS (Asymmetric Moderately Sparse): $r_2 = 10$

Generating $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$

Simulation: asymmetric situation

- $\boldsymbol{X}_i = \Phi(\tilde{\boldsymbol{X}}_i)$ where $\tilde{\boldsymbol{X}}_i$ i.i.d. $N(0, \boldsymbol{\Sigma})$ with $\Sigma_{ij} = 0.5^{|i-j|}$. $\Phi(\cdot)$: cdf of $N(0, 1)$
- $\{U_{ij}\}$ i.i.d. $U[0, 1]$, $i = 1, ..., n$, $j = 1, ..., m$

$$Y_{ij} = \Phi^{-1}(U_{ij})\boldsymbol{X}_i^\top \{\boldsymbol{\Gamma}_{1,*j}\mathbf{I}(U_{ij} < 0.5) + \boldsymbol{\Gamma}_{2,*j}\mathbf{I}(U_{ij} \geq 0.5)\}$$

- Quantiles of Y_{ij} given $\boldsymbol{X}$:

$$q_j(\tau|\boldsymbol{X}) = \Phi^{-1}(\tau)\boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{1,*j}, \quad \tau < 0.5;$$
$$q_j(\tau|\boldsymbol{X}) = \Phi^{-1}(\tau)\boldsymbol{X}_i^\top \boldsymbol{\Gamma}_{2,*j}, \quad \tau \geq 0.5.$$

- Given $\boldsymbol{X}_i$, Y_{ij} independent in j

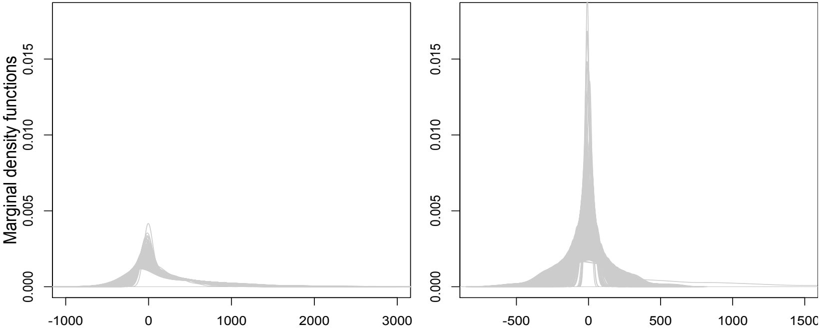


Figure 6: 500 marginal densities (kernel estimators) of $\boldsymbol{Y}_i$. Left figure: AMS shows asymmetry as the right tail corresponds to higher rank $\boldsymbol{\Gamma}_2$; right: AES shows symmetry as the rank of $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are equal.

Performance

- Prediction error: $\|\widehat{\mathbf{\Gamma}} - \mathbf{\Gamma}\|^2_{L_2(\Pi)}$
  1. V shape: tail quantiles have larger error
  2. For more sparse model: larger λ
- Frobenius error and nuclear norm error show similar patterns as prediction error
- Estimated number of nonzero singular values

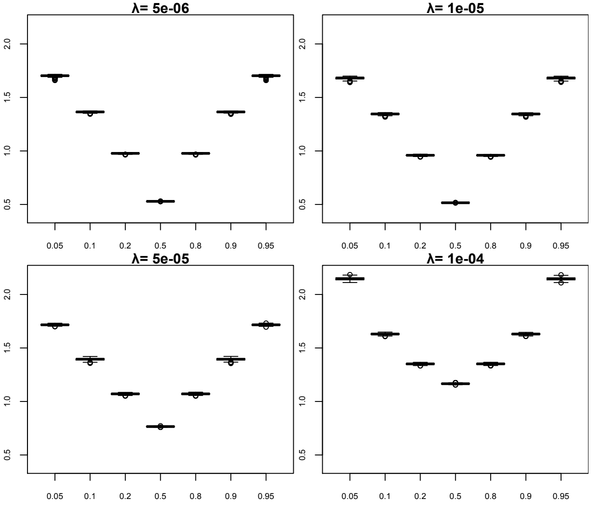


Figure 7: Model LS Prediction Error box plots. Symmetric "V" shape is observed for different choices of λ. Model MS and ES perform similarly.

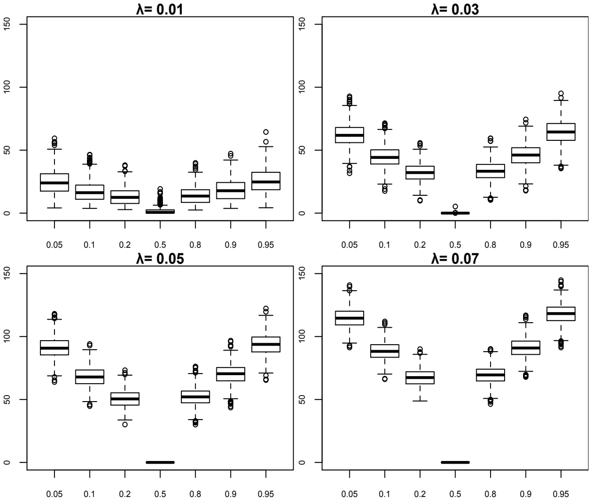


Figure 8: Model AES Prediction Error box plots. Prediction errors present symmetric "V" shape since $\mathrm{rank}(\mathbf{\Gamma}_1) = \mathrm{rank}(\mathbf{\Gamma}_2)$.

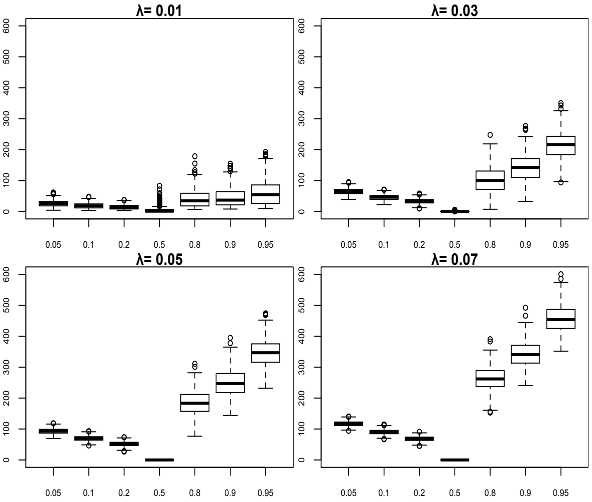


Figure 9: Model AMS Prediction Error box plots. For $\tau > 0.5$ the prediction errors are higher as $\mathrm{rank}(\mathbf{\Gamma}_1) < \mathrm{rank}(\mathbf{\Gamma}_2)$.

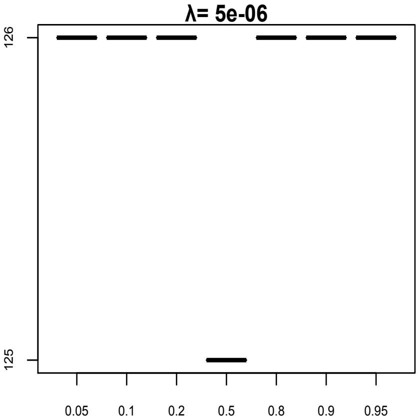


Figure 10: Model LS Estimated number of nonzero singular values box plot. True number of singular value is 125. The result is the same for other choice of λ with certain threshold.

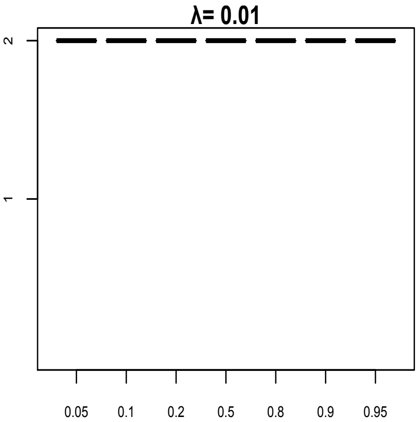


Figure 11: Model AES Estimated number of nonzero singular values box plot. The true number is 2. The result is the same for other choice of λ with certain threshold.

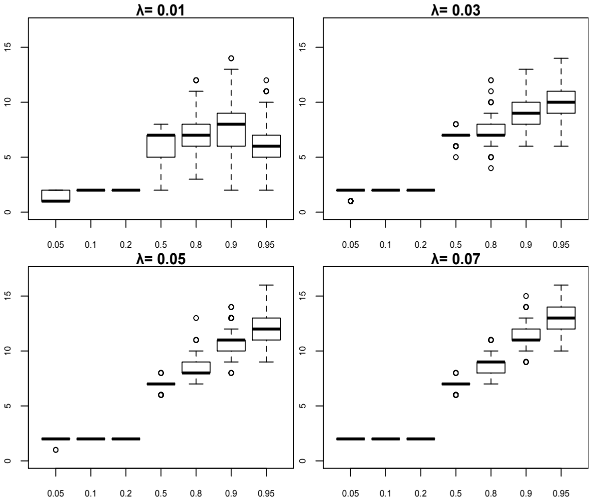


Figure 12: Model AMS Estimated number of nonzero singular values box plots. The true number is 2 for $\tau < 0.5$ and 10 for $\tau \geq 0.5$.

Sparse Asymmetric Multivariate Conditional Value-at-Risk (SAMCVaR)

$$q_{t,j}(\tau|\mathcal{F}_{t-1}) = \boldsymbol{X}_{t-1}^\top \boldsymbol{\Gamma}_{*j},$$
$$\boldsymbol{X}_{t-1} = (|Y_{t-1,1}|, ..., |Y_{t-1,m}|, Y_{t-1,1}^-, ..., Y_{t-1,m}^-)^\top \in \mathbb{R}^{2m},$$

where $Y^- = \max\{-Y, 0\}$

- Engle and Manganelli (2004): Conditional Autoregressive Value-at-Risk (CAViaR)
- White et al. (2008): Univariate Multi-Quantile CAViaR (MQ-CAViaR)
- White et al. (2015) "VAR for VaR": estimate bivariate VAR due to computational burden

Factorization

- Factorisation: $r = \mathrm{rank}(\boldsymbol{\Gamma})$,

$$q_{t,j}(\tau|\mathcal{F}_{t-1}) = \sum_{k=1}^{r} \psi_{j,k}(\tau) f_k^{\tau}(\boldsymbol{X}_t) \tag{13}$$

$$f_k^{\tau}(\boldsymbol{X}_t) = \sum_{l=1}^{m} \varphi_{1,k,l}(\tau)|Y_{t-1,l}| + \sum_{l=1}^{m} \varphi_{2,k,l}(\tau) Y_{t-1,l}^{-}, \tag{14}$$

- Flow from component j to f_k^{τ} :

$$\frac{\partial f_k^{\tau}}{\partial(|Y_j|, Y_j^{-})} = \{\varphi_{|\cdot|,k,j}(\tau), \varphi_{-,k,j}(\tau)\}.$$

Sensitivity of j quantile to $f_k(\tau)$: $\dfrac{\partial q_j(\tau|\boldsymbol{X})}{\partial f_k^{\tau}} = \psi_{j,k}(\tau).$

Goals

- Leverage effect: $Y^{-}_{t-1,j} > 0$ implies the increase in $\sigma_{t,j}$. Black (1976) and Engle and Ng (1993)
  - Is leverage effect symmetric? i.e., $|\varphi_{-,k,j}(\tau)| = |\varphi_{-,k,j}(1-\tau)|$?
- Risk sensitivity analysis with τ-range: plot of $\{\psi_{j,1}(\tau), \psi_{j,1}(1-\tau)\}$
- Tail analysis at τ: plot of $\{\psi_{j,1}(\tau), \psi_{j,2}(\tau)\}$

Data

- Data period: August 31, 2007 to August 5, 2010. 766 daily closing price for each stock in the sample.

	Banks	Financial Services	Insurances	Total
EU	47	22	27	96
North America	25	17	28	70
Asia	47	14	3	64
Total	119	53	58	$m = 230$

Table 1: Financial firms summary.

- $p = 2m = 460$ (2 transformations of lag return $|Y_{t-1,j}|$, $Y^{-}_{t-1,j}$)
- Downloaded from Simone Manganelli's website
- Using tuning method introduced previously: $\lambda = 0.0247$ for both $\tau = 1\%$ and 99%

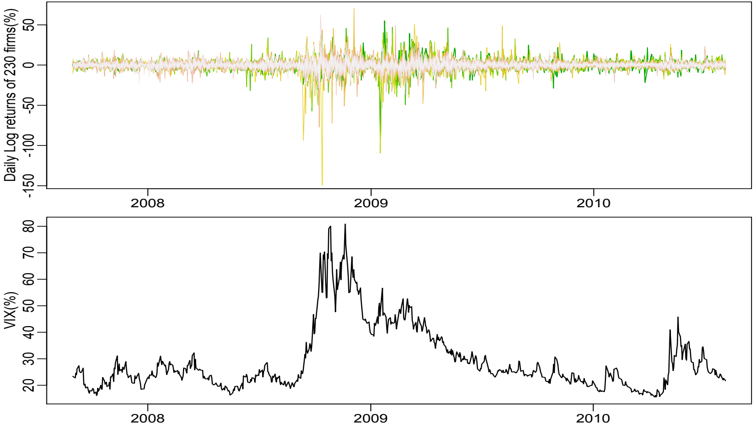


Figure 13: Time series plots of log returns Y_{ij}, i ranging from Aug. 31, 2007-Aug. 5, 2010. $n = 765$. $j = 1, ..., 230$ firm. The lower figure shows the time series of VIX. FASTECSAMCVaR

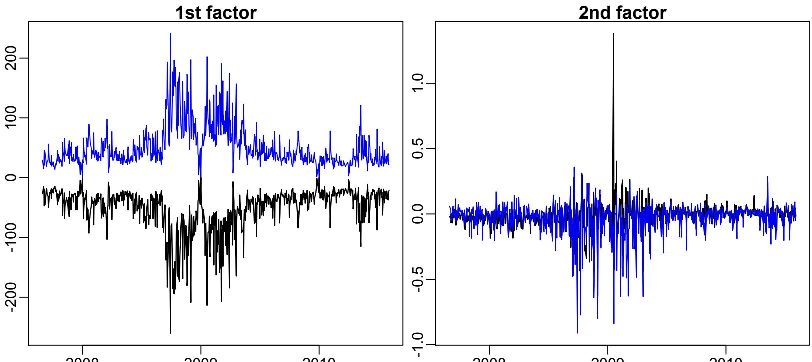


Figure 14: Time series of first factors $f_1^{0.01}$, $f_1^{0.99}$ (left) and second factors $f_2^{0.01}$, $f_2^{0.99}$ (right). Large deviation periods of first factors correspond to that of VIX. The magnitude of factor 2 is much smaller than that of factor 1. FASTECSAMCVaR

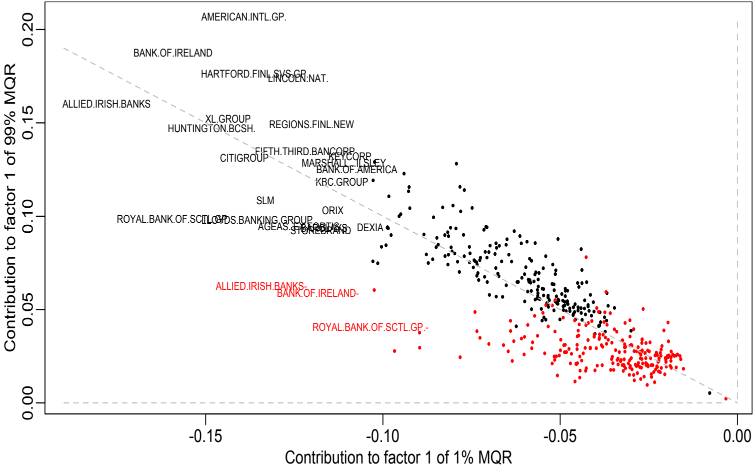


Figure 15: Scatter plot of $(\varphi_{|\cdot|,1,j}(0.01), \varphi_{|\cdot|,1,j}(0.99))$ and $(\varphi_{-,1,j}(0.01), \varphi_{-,1,j}(0.99))$ for $j = 1, ..., 230$. $Y^{-}_{t-1,j}$ relates more to left dispersion, while $|Y_{t-1,j}|$ contributes symmetrically. FASTECSAMCVaR

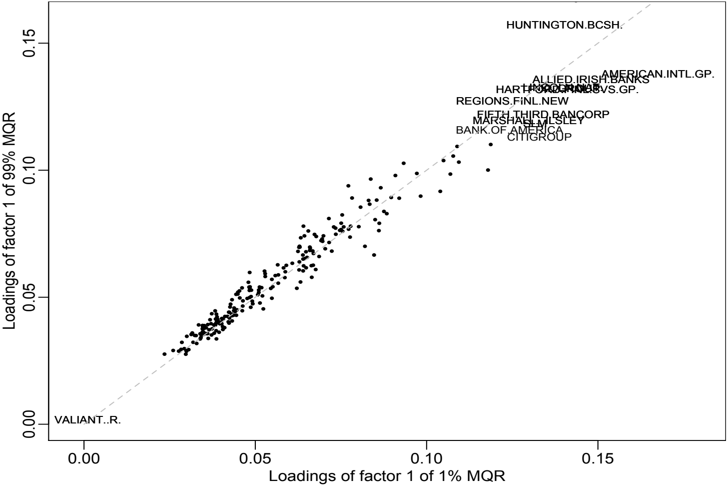


Figure 16: Scatter plot of loadings $(\psi_{j,1}(0.01), \psi_{j,1}(0.99))$ for $j = 1, ..., 230$ firms. Firms on the northeast corner are more associated to the extreme event of the market. FASTECSAMCVaR

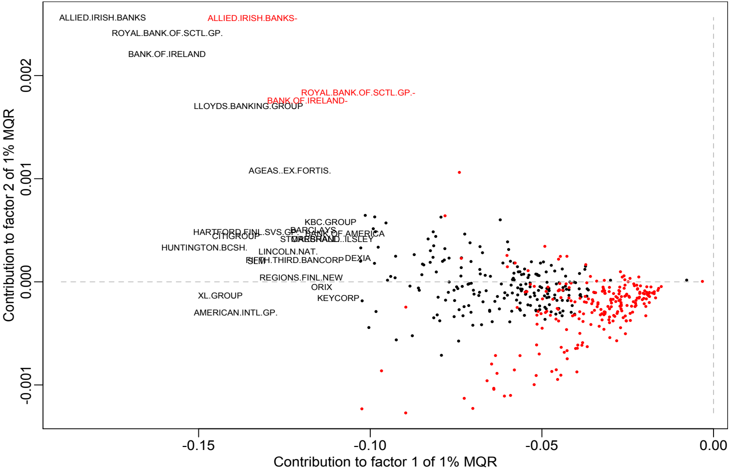


Figure 17: Scatter plot of $(\varphi_{|\cdot|,1,j}(0.01), \varphi_{|\cdot|,2,j}(0.01))$ and $(\varphi_{-,1,j}(0.01), \varphi_{-,2,j}(0.01))$ for $j = 1, ..., 230$. Northwest corner: associated with the peak of $f_2^{1\%}$ beginning of 2009. FASTECSAMCVaR

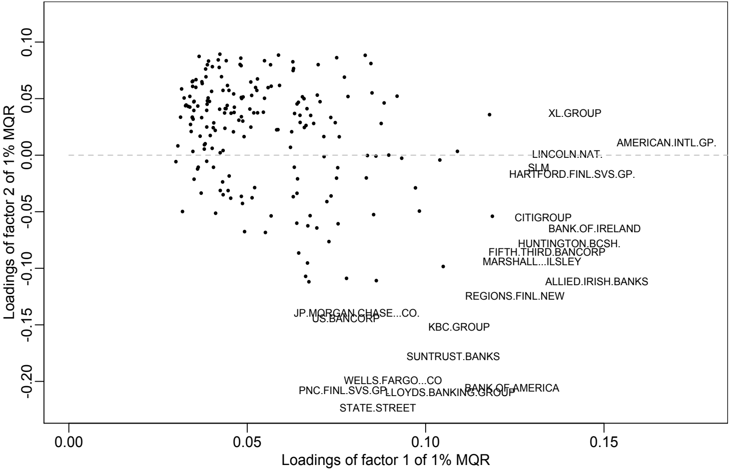


Figure 18: Scatter plot of loadings $(\psi_{j,1}(0.01), \psi_{j,2}(0.01))$ for $j = 1, ..., 230$. Southeast direction: serious downward movement. FASTECSAMCVaR

FASTEC: SAMCVaR

- Figure 6-7
  - leverage effect: $Y_{t-1,j}^{-}0$ leads to left τ-range expansion
  - $|Y_{t-1,j}|$ contributes symmetrically to τ-range
- Figure 6-8 Large loadings $(\psi_{j,1}(0.01), \psi_{j,1}(0.99))$, large τ-range
- Figure 6-9: Allied Irish Bank, RBS, Bank of Ireland contribute to the peak of $f_2^{1\%}$ beginning of 2009
- Figure 6-10 Southeast direction: serious downward movement

2008 Chinese winter storm

- January 25-February 6, 2008
- Loss: 151.65 billion RMB
- Fatalities: 129
- Areas affected: Hubei, Hunan, Zhejiang, Guizhou, Guangdong, Jiangxi, Guangxi, Fujian, Henan, Shandong, Jiangsu, Anhui, Shanghai, Chongqing, Shanxi, Sichuan

Figure 19: Source: Wikipedia.

Chinese Temperature Data

- Temperature data from $m = 159$ weather stations in China in year 2008, downloaded from Research Data Center of CRC 649

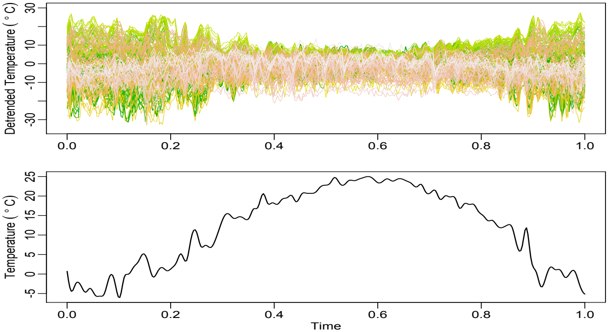


Figure 20: Upper: detrended temperature $Y_j(t)$ and yearly trend by smoothing spline. j: weather station, $t \in [0, 1]$ time point in year 2008. Lower: trend. Detrending FASTECChinaTemper2008

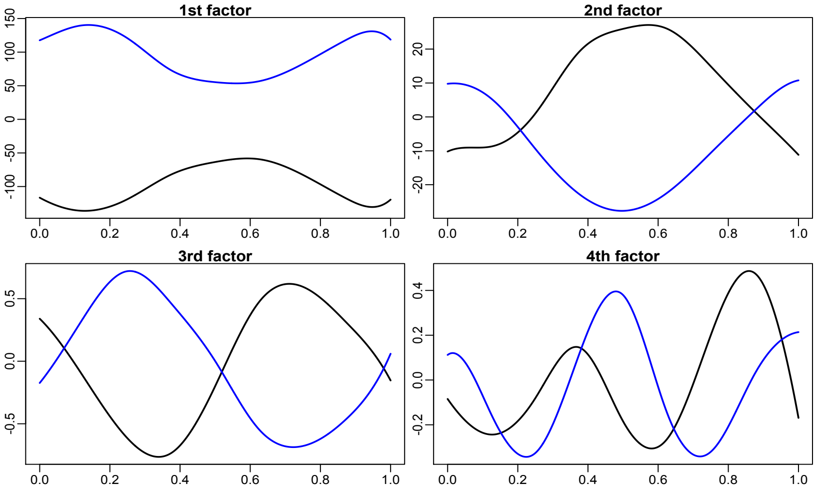


Figure 21: The first 4 factor curves. $\tau = 90\%$. $\tau = 10\%$.
FASTECChinaTemper2008

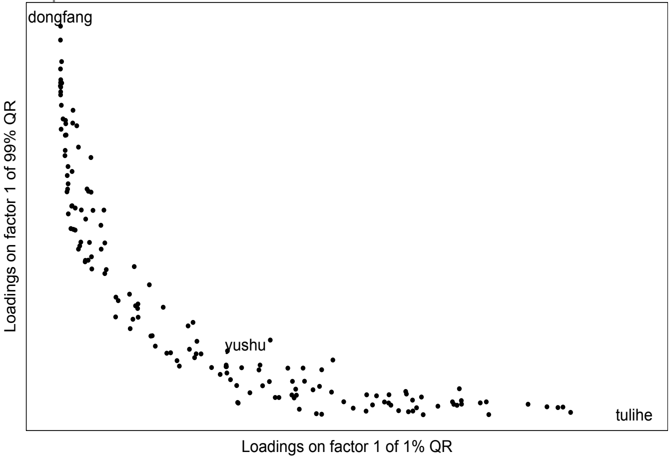
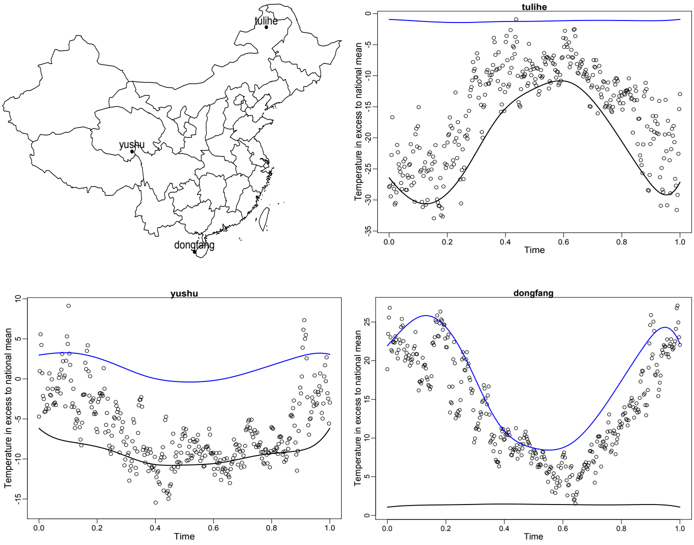


Figure 22: Scatter plot of factor loadings of weather station j, $j = 1, ..., 159$, demonstrates a "L"-like shape: stations associated with factor 1 of 1% have almost no association with that of 99%.

FASTECChinaTemper2008


tulihe
yushu
dongfang
tulihe
Temperature in excess to national mean
Time
0.0
0.2
0.4
0.6
0.8
1.0
0
-5
-10
-15
-20
-25
-30
-35
yushu
Temperature in excess to national mean
Time
10
5
0
-5
-10
-15
dongfang
Temperature in excess to national mean
Time
25
20
15
10
5
0

Temperature analysis

- The algorithm classifies the northern and southern temperature patterns
- "L" like shape in Figure 22: stations associated with factor 1 of 1% have almost no association with that of 99%
- Stations in the middle cannot be explained by either northern or southern temperature pattern
- Yuchu: region avg. 4000 meters high above sea level, highland climate with reverse seasonality

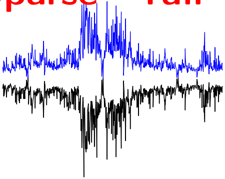
Summary and Extensions

- Conditional quantiles are useful for studying tail events and spread of dispersion
- Nuclear norm regularized multivariate quantile regression
- Algorithm and oracle properties are derived

Further research directions:

- Expectile regression, support vector machine (non-smooth loss)
- Confidence intervals for singular values
- Nonconvex penalty. e.g. nonconvex adaptive nuclear norm $\|\mathbf{\Gamma}\|_* = \sum_{i=1}^{p \wedge m} w_i d_i(\mathbf{\Gamma})$ by Chen, Dong and Chan (2013, Biometrika)

FASTEC- FActorizable Sparse Tail Event Curves



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http://www.case.hu-berlin.de
https://www.stat.wisc.edu

Check function

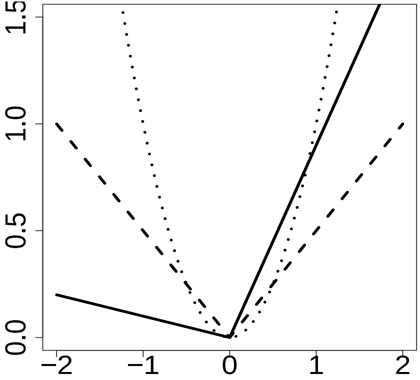

Figure 23: Solid line: $\tau = 0.9$. Dashed line: $\tau = 0.5$. Dotted line: square loss u^2 (OLS regression). LQRcheck

Loss function

Nonsmooth loss function: $\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_\tau\left(Y_{ij}-\boldsymbol{X}_i^\top\boldsymbol{\Gamma}_{*j}\right)+\lambda\|\boldsymbol{\Gamma}\|_*$

Introduce dual variables

$$\max_{\Theta_{ij}\in[\tau-1,\tau]}\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})$$

$$\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{ij}(Y_{ij}-\boldsymbol{X}_i^\top\boldsymbol{\Gamma}_{*j})$$

Smoothing by Nesterov (2005)

$$f_\kappa(\boldsymbol{\Gamma})=\max_{\Theta_{ij}\in[\tau-1,\tau]}\left\{\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})-\frac{\kappa}{2}\|\boldsymbol{\Theta}\|_F^2\right\}$$

$$\nabla_{\boldsymbol{\Gamma}}f_\kappa(\boldsymbol{\Gamma})=-(mn)^{-1}\mathbf{X}^\top[[(\kappa mn)^{-1}(\mathbf{Y}-\mathbf{X}\boldsymbol{\Gamma})]]_\tau$$

Lipschitz constant $M=(\kappa m^2n^2)^{-1}\|\mathbf{X}\|^2$

$\kappa=\epsilon/2mn$

Project on low rank space

$$S_\lambda(\boldsymbol{\Gamma})\stackrel{\text{def}}{=}\mathbf{U}(\mathbf{D}-\lambda\mathbf{I})_+\mathbf{V}^\top$$

Algorithm
$[[\cdot]]_\tau$ and Theorem of Nesterov

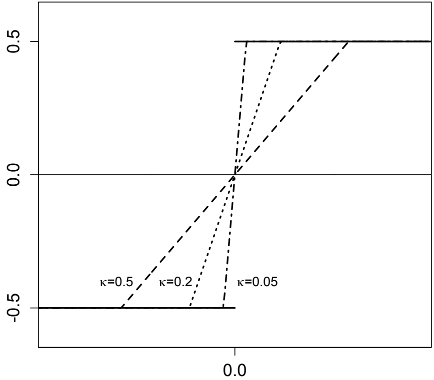

Figure 24: $\mathbf{X} = 1$, $m = p = n = 1$. Solid line: $\psi_\tau(u) = \tau - \mathbf{I}(u \leq 0)$ with $\tau = 0.5$; Dashed, dotted, dot-dash line: smoothing gradient $[[\kappa^{-1}(\mathbf{Y} - \mathbf{X}\mathbf{\Gamma})]]_\tau$, $\kappa = 0.5, 0.2, 0.05$. ▸ Algorithm

Nonsmooth loss function: $\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho_\tau\left(Y_{ij}-\boldsymbol{X}_i^\top\boldsymbol{\Gamma}_{*j}\right)+\lambda\|\boldsymbol{\Gamma}\|_*$

Introduce dual variables

$\max_{\Theta_{ij}\in[\tau-1,\tau]}\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})$

$\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{ij}(Y_{ij}-\boldsymbol{X}_i^\top\boldsymbol{\Gamma}_{*j})$

Smoothing by Nesterov (2005)

$f_\kappa(\boldsymbol{\Gamma})=\max_{\Theta_{ij}\in[\tau-1,\tau]}\left\{\ell(\boldsymbol{\Gamma},\boldsymbol{\Theta})-\frac{\kappa}{2}\|\boldsymbol{\Theta}\|_F^2\right\}$

$\nabla_{\boldsymbol{\Gamma}}f_\kappa(\boldsymbol{\Gamma})=-(mn)^{-1}\mathbf{X}^\top[[(\kappa mn)^{-1}(\mathbf{Y}-\mathbf{X}\boldsymbol{\Gamma})]]_\tau$

Lipschitz constant $M=(\kappa m^2n^2)^{-1}\|\mathbf{X}\|^2$

$\kappa=\epsilon/2mn$

Project on low rank space

$S_\lambda(\boldsymbol{\Gamma})\stackrel{\text{def}}{=}\mathbf{U}(\mathbf{D}-\lambda\mathbf{I})_+\mathbf{V}^\top$

▸ Algorithm

▸ Proximity operator

Detrending of Chinese temperature data

- Chapter 4 of Ramsay and Silverman (2005): smooth discretized data with smoothing spline
- Estimation of mean function and smoothing are done jointly by minimizing

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left[Y_{ij}-\hat{\mu}(t_i)\right]^2+\eta\int[D^2\hat{\mu}(s)]^2 ds \qquad (15)$$

where $\eta > 0$ is a smoothing parameter selected by cross-validation and $\hat{\mu}$ is fitted by cubic spline basis

▸ Introduction-temperature data ▸ Application-temperature data

$$[[a_{ij}]]_\tau = \begin{cases} \tau, & \text{if } a_{ij} \geq \tau; \\ a_{ij}, & \text{if } \tau - 1 < a_{ij} < \tau; \\ \tau - 1, & \text{if } a_{ij} \leq \tau - 1. \end{cases}$$

Theorem

For any $\kappa > 0$, $f_\kappa(\mathbf{\Gamma})$ is well-defined, convex and continuously-differentiable function in $\mathbf{\Gamma}$ with the gradient $\nabla f_\kappa(\mathbf{\Gamma}) = -(mn)^{-1}\mathbf{X}^\top \mathbf{\Theta}^*(\mathbf{\Gamma}) \in \mathbb{R}^{p \times m}$, where $\mathbf{\Theta}^*(\mathbf{\Gamma})$ is the optimal solution to $\max_{\Theta_{ij} \in [\tau-1,\tau]} \left\{(mn)^{-1}\ell(\mathbf{\Gamma}, \mathbf{\Theta}) - \frac{\kappa}{2}\|\mathbf{\Theta}\|_{\mathrm{F}}^2\right\}$, namely

$$\mathbf{\Theta}^*(\mathbf{\Gamma}) = [[(\kappa mn)^{-1}(\mathbf{Y} - \mathbf{X}\mathbf{\Gamma})]]_\tau. \tag{16}$$

Moreover, the gradient $\nabla f_\kappa(\mathbf{\Gamma})$ is Lipschitz continuous with the Lipschitz constant $M = (\kappa m^2 n^2)^{-1}\|X\|^2$.

▸ Smoothing the loss

Definition (Proximity Operator)

Let $\mathcal{X} = \mathbb{R}^{p\times n}$ with inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}(\mathbf{A}^\top \mathbf{B})$ and $\|\cdot\|$ be the induced norm. $f : \mathcal{X} \to \mathbb{R}$ a lower semicontinuous convex function. The proximity operator of f, $S_f : \mathcal{X} \to \mathcal{X}$:

$$S_f(\mathbf{Y}) \stackrel{\text{def}}{=} \arg\min_{\mathbf{X}\in\mathcal{X}} \left\{ f(\mathbf{X}) + \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|^2 \right\}, \forall \mathbf{Y} \in \mathcal{X}.$$

Theorem (Cai et al. (2010))

SVD: $\mathbf{Y} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. *The proximity operator* $S_\lambda(\cdot)$ *of* $\lambda\|\cdot\|_*$ *is*

$$S_\lambda(\mathbf{Y}) \stackrel{\text{def}}{=} \mathbf{U}(\mathbf{D} - \lambda\mathbf{I})_+\mathbf{V}^\top, \tag{17}$$

▸ Estimating Γ ▸ Smoothing the loss

Proof.

- $\ell(\mathbf{\Gamma}) = (nm)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho_\tau(Y_{ij} - \boldsymbol{X}_i^\top \mathbf{\Gamma}_{*j})$
- $L(\mathbf{\Gamma}) \stackrel{\text{def}}{=} \ell(\mathbf{\Gamma}) + \lambda \|\mathbf{\Gamma}\|_*$
- $\widetilde{L}(\mathbf{\Gamma}) = f_\kappa(\mathbf{\Gamma}) + \lambda \|\mathbf{\Gamma}\|_*$
- $f_\kappa(\mathbf{\Gamma}) = \min_{\mathbf{\Theta} \in [\tau-1,\tau]^{n\times m}} \ell(\mathbf{\Gamma}, \mathbf{\Theta}) - \frac{\kappa}{2} \|\mathbf{\Theta}\|_{\mathrm{F}}^2$

$$L(\mathbf{\Gamma}_t) - L(\widehat{\mathbf{\Gamma}}) = L(\mathbf{\Gamma}_t) - \widetilde{L}(\mathbf{\Gamma}_t) + \widetilde{L}(\mathbf{\Gamma}_t) - \widetilde{L}(\widehat{\mathbf{\Gamma}}) + L(\widehat{\mathbf{\Gamma}}) - \widetilde{L}(\widehat{\mathbf{\Gamma}}).$$

1. $\widetilde{L}(\mathbf{\Gamma}) \leq L(\mathbf{\Gamma}) \leq \widetilde{L}(\mathbf{\Gamma}) + \kappa \max_{\mathbf{\Theta} \in [\tau-1,\tau]^{n\times m}} \frac{\|\mathbf{\Theta}\|_{\mathrm{F}}^2}{2} \leq \widetilde{L}(\mathbf{\Gamma}) + \kappa \mu(\tau)^2 \frac{nm}{2}$
2. BT(2009): $\left|\widetilde{L}(\mathbf{\Gamma}_t) - \widetilde{L}(\widehat{\mathbf{\Gamma}})\right| \leq \frac{2M\|\mathbf{\Gamma}_0 - \widehat{\mathbf{\Gamma}}\|_{\mathrm{F}}^2}{(t+1)^2}$, $M = (\kappa m^2 n^2)^{-1} \|\mathbf{X}\|^2$: Lipschitz constant of $\nabla f_\kappa(\mathbf{\Gamma})$,

□

▸ Convergence of SFISTA ▸ Proof of oracle property

Nonasymptotic risk bounds

Generalization of support using projections:

- SVD: $\mathbf{A} = \sum_{j=1}^{r} \sigma(\mathbf{A})\mathbf{u}_j\mathbf{v}_j^\top$ for matrix $\mathbf{A}$
- $\mathbf{U}_r = [\mathbf{u}_1, ..., \mathbf{u}_r]$, $\mathbf{V} = [\mathbf{v}_1, ..., \mathbf{v}_r]$.
  $P_{\mathbf{A},1} = \mathbf{U}_r\mathbf{U}_r^\top$; $P_{\mathbf{A},2} = \mathbf{V}_r\mathbf{V}_r^\top$ are orthogonal projections
- $\mathcal{P}_{\mathbf{A}}(\mathbf{S}) \stackrel{\text{def}}{=} \mathbf{S} - P_{\mathbf{A},1}^\perp \mathbf{S} P_{\mathbf{A},2}^\perp$; $\mathcal{P}_{\mathbf{A}}^\perp(\mathbf{S}) \stackrel{\text{def}}{=} P_{\mathbf{A},1}^\perp \mathbf{S} P_{\mathbf{A},2}^\perp$

The cone

$$\mathcal{K}(\mathbf{\Gamma}, c_0) \stackrel{\text{def}}{=} \left\{ \mathbf{S} \in \mathbb{R}^{p\times m} : \|\mathcal{P}_{\mathbf{\Gamma}}^\perp(\mathbf{S})\|_* \leq c_0 \|\mathcal{P}_{\mathbf{\Gamma}}(\mathbf{S})\|_* \right\}. \tag{18}$$

The norm: $\|\mathbf{S}\|_{L_2(\Pi)}^2 \stackrel{\text{def}}{=} m^{-1} \mathsf{E}_\Pi \|\mathbf{S}^\top \boldsymbol{X}_i\|_2^2$

▸ Nonasymptotic Risk Bounds

Assumption (Sampling setting)

Samples $(\boldsymbol{X}_1, \boldsymbol{Y}_1), ..., (\boldsymbol{X}_n, \boldsymbol{Y}_n)$ *are i.i.d. copies of* $(\boldsymbol{X}, \boldsymbol{Y})$ *random vectors in* $\mathbb{R}^{p+m}$*.* $F^{-1}_{Y_{ij}|\boldsymbol{X}_i}(\tau|\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{\Gamma}_{*j}(\tau)$*. Conditioning on* $\boldsymbol{X}_i$*,* Y_{ij} *is independent in* j*.*

Assumption (Covariance matrix condition)

Let the covariance matrix of $\boldsymbol{X}$ *be* $\boldsymbol{\Sigma}_{\boldsymbol{X}}$*, assume that* $0 < \sigma_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{X}}) < \sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}}) < \infty$*. Moreover, assume the sample covariance matrix of covariates* $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{X}} = \frac{1}{n}\mathbf{X}^\top\mathbf{X}$ *satisfies*

$$\mathrm{P}\left[\sigma_{\min}(\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{X}}) \geq c_1 \sigma_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{X}}), \sigma_{\max}(\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{X}}) \leq c_2 \sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}})\right] \geq 1 - \gamma_n. \quad (19)$$

Covariates come from a joint p-Gaussian distribution $N(0, \boldsymbol{\Sigma}_{\boldsymbol{X}})$:
$c_1 = 1/9$, $c_2 = 9$ and $\gamma_n = 4\exp(-n/2)$ from Wainwright (2009)

▸ Nonasymptotic Risk Bounds

▸ Estimation noise

Assumption (Conditional density condition)

There exist $\underline{f} > 0$ *and* $\bar{f}' < \infty$ *such that* $|\frac{\partial}{\partial y_j} f_{Y_{ij}|\boldsymbol{X}_i}(y_i|\boldsymbol{x})| \leq \bar{f}'$ *and* $\inf_{j \leq m} \inf_{\boldsymbol{x}} f_{Y_{ij}|\boldsymbol{X}_i}(\boldsymbol{x}^\top \boldsymbol{\Gamma}_{*j}|\boldsymbol{x}) \geq \underline{f}$, *where* $f_{Y_{ij}|\boldsymbol{X}_i}$ *is the conditional density function of* Y_{ij} *on* $\boldsymbol{X}_i$.

Assumption (Restricted eigenvalue)

For a given probability distribution Π *for* $\boldsymbol{X}$,

$$\beta_{\boldsymbol{\Gamma},3} \stackrel{\text{def}}{=} \inf\left\{\beta > 0 : \beta \|\mathcal{P}_{\boldsymbol{\Gamma}}(\boldsymbol{\Delta})\|_{\mathrm{F}} \leq \|\boldsymbol{\Delta}\|_{L_2(\Pi)},\ \forall \boldsymbol{\Delta} \in \mathcal{K}(\boldsymbol{\Gamma}, 3)\right\} > 0. \tag{20}$$

A rough lower bound: $\beta_{\boldsymbol{\Gamma},3} \geq m^{-1/2} \sqrt{\sigma_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{X}})}$ ▸ Nonasymptotic Risk Bounds

Assumption (Restricted nonlinearity)

$$\nu \stackrel{\text{def}}{=} \frac{3}{8}\frac{\underline{f}}{\bar{f}'} \inf_{\substack{\boldsymbol{\Delta}\in\mathcal{K}(\boldsymbol{\Gamma},3)\\ \boldsymbol{\Delta}\neq\mathbf{0}}} \frac{\|\boldsymbol{\Delta}\|^3_{L_2(\Pi)}}{m^{-1}\sum_{j=1}^m \mathsf{E}[|\boldsymbol{X}_i^\top\boldsymbol{\Delta}_{*j}|^3]}, \tag{21}$$

$$\nu > \frac{C'_\tau}{\underline{f}\sqrt{m}}\sqrt{\frac{\sigma_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{X}})}{\sigma_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{X}})}}\sqrt{\tau\vee(1-\tau)}\left(\sqrt{\frac{\log(p+m)}{n}}+\sqrt{\frac{p+m}{n}}\right)\sqrt{r}. \tag{22}$$

Section 2.5 of Belloni and Chernozuhkov (2011) calculate ν for various data generating processes

▸ Nonasymptotic Risk Bounds

Asymmetric situation: Γ generation

1. Basis vectors $\{v_1, v_2\}$ and $\{u_1, ..., u_{r_2}\}$ in $\mathbb{R}^p$. Components in v_j and u_k follow $U[0, 1]$
2. $\mathbf{\Gamma}_{1,*j} = a_{1,j} v_1 + a_{2,j} v_2$, $a_{1,j}, a_{2,j} \sim U[0, 1]$ i.i.d.;
   $\mathbf{\Gamma}_{2,*j} = b_{1,j} u_1 + ... + b_{r_2,j} u_{r_2}$, $b_{1,j}$,...,$b_{r_2,j} \sim U[0, 1]$ i.i.d.

▸ Asymmetric Models

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