ICARE - localising Conditional AutoRegressive Expectiles

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Motivation

- Risk Exposure
  - Measure tail events
  - Conditional autoregressive expectile (CARE) model

- Time-varying parameters
  - Time-varying parameters in CARE
  - Interval length reflects the structural changes in economy
Objectives

- Localising CARE Models
  - Local parametric approach (LPA)
  - Balance between modelling bias and parameter variability

- Tail Risk Dynamics
  - Estimation windows with varying lengths
  - Time-varying expectile parameters
Econometrics and Risk Management

Econometrics
- Modelling bias vs. parameter variability
- Interval length and economic variables

Risk Management
- Parameter dynamics and structural changes
- Measuring tail risk

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Risk Exposure

An investor observes daily DAX returns from 20050103 to 20141231 and estimates the underlying risk exposure via percentiles (e.g., 1% and 5%) over a one-year time horizon.

Modelling strategies

(a) Data windows fixed on an ad hoc basis

(b) Adaptively selected data intervals: time-varying parameters
Portfolio Protection

An investor decides about the daily allocation into a stock portfolio (DAX). Goal: a proportion of the initial portfolio value (100) is preserved at the end of a horizon, i.e., the target floor equals 90.

Decision at day $t$: multiple of the difference between the portfolio value and the discounted floor up to $t$ is invested into the stock portfolio (DAX), the rest into a riskless asset.

Multiplier $m$ selection: constant or time-varying (ICARE)
Research Questions

How to account for time-varying parameters in tail event risk measures estimation?

What are the typical data interval lengths assessing risk more accurately, i.e., striking a balance between bias and variability?

How well does the ICARE technique perform in practice?
Outline

1. Motivation ✓
2. Conditional Autoregressive Expectile (CARE)
3. Local Parametric Approach (LPA)
4. Empirical Results
5. Applications
6. Conclusions
Conditional Autoregressive Expectile (CARE)

Conditional Autoregressive Expectile

- Taylor (2008), Kuan et al. (2009), Engle and Manganelli (2004)
- Random variable $Y$ (e.g. returns), identically distributed, $y_t, \ t = 1, \ldots, n$
- CARE specification conditional on information set $\mathcal{F}_{t-1}$

\[
y_t = e_{t,\tau} + \varepsilon_{t,\tau} \quad \varepsilon_{t,\tau} \sim \text{AND} \left(0, \sigma^2_{\varepsilon,\tau}, \tau\right)
\]

\[
e_{t,\tau} = \alpha_{0,\tau} + \alpha_{1,\tau} y_{t-1} + \alpha_{2,\tau} \left(y_{t-1}^+\right)^2 + \alpha_{3,\tau} \left(y_{t-1}^-\right)^2
\]

- Expectile $e_{t,\tau}$ at $\tau \in (0, 1)$, $\theta_{\tau} = \left\{\alpha_{0,\tau}, \alpha_{1,\tau}, \alpha_{2,\tau}, \alpha_{3,\tau}, \sigma^2_{\varepsilon,\tau}\right\}^T$
- Returns: $y_{t-1}^+ = \max \left\{y_{t-1}, 0\right\}$, $y_{t-1}^- = \min \left\{y_{t-1}, 0\right\}$
Parameter Estimation

- Data calibration with time-varying intervals
- Observed returns $\mathcal{Y} = \{y_1, \ldots, y_n\}$
- Quasi maximum likelihood estimate (QMLE)

$$\tilde{\theta}_{I,\tau} = \arg \max_{\theta_{\tau} \in \Theta} \ell_I (\mathcal{Y}; \theta_{\tau})$$

$\ell_I (\cdot)$ - quasi log likelihood

- $I = [t_0 - \nu, t_0]$ - interval of $(\nu + 1)$ observations at $t_0$
- $\ell_I (\cdot)$ - quasi log likelihood
Estimation Quality

- Mercurio and Spokoiny (2004), Spokoiny (2009)
- Quality of estimating true parameter vector $\theta^*_\tau$ by QMLE $\tilde{\theta}_{I,\tau}$ in terms of Kullback-Leibler divergence; $R_r(\theta^*_\tau)$ - risk bound

$$
E_{\theta^*_\tau} \left| \ell_I(Y; \tilde{\theta}_{I,\tau}) - \ell_I(Y; \theta^*_\tau) \right|^r \leq R_r(\theta^*_\tau)
$$

- ’Modest’ risk, $r = 0.5$ (shorter intervals of homogeneity)
- ’Conservative’ risk, $r = 1$ (longer intervals of homogeneity)

_Solomon Kullback and Richard A. Leibler_ on BBI:

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Local Parametric Approach (LPA)

- LPA, Spokoiny (1998, 2009)
  - Time series parameters can be locally approximated
  - Finding the *interval of homogeneity* ➤ Details
  - Balance between modelling bias and parameter variability

- Time series literature
  - GARCH(1, 1) models - Čižek et al. (2009)
  - Realized volatility - Chen et al. (2010)
  - Multiplicative Error Models - Härdle et al. (2015)
Interval Selection

\( (K + 1) \) nested intervals with length \( n_k = |l_k| \)

\[
\begin{align*}
I_0 & \subset I_1 \subset \cdots \subset I_k \subset \cdots \subset I_K \\
\tilde{\theta}_0 & \subset \tilde{\theta}_1 \subset \cdots \subset \tilde{\theta}_k \subset \cdots \subset \tilde{\theta}_K
\end{align*}
\]

Example: Daily index returns

Fix \( t_0, l_k = [t_0 - n_k, t_0], n_k = [n_0 c^k], c > 1 \)

\( \{n_k\}_{k=0}^{11} = \{20 \text{ days}, 25 \text{ days}, \ldots, 250 \text{ days}\}, c = 1.25 \)
Local Change Point Detection

 Fix $t_0$, sequential test ($k = 1, \ldots, K$)

$H_0$ : parameter homogeneity within $I_k$
$H_1$ : $\exists$ change point within $J_k = I_k \setminus I_{k-1}$

$$T_{k,\tau} = \sup_{s \in J_k} \left\{ \ell_{A_k,s} \left( \mathcal{Y}, \tilde{\theta}_{A_k,s,\tau} \right) + \ell_{B_k,s} \left( \mathcal{Y}, \tilde{\theta}_{B_k,s,\tau} \right) - \ell_{l_{k+1}} \left( \mathcal{Y}, \tilde{\theta}_{l_{k+1},\tau} \right) \right\}$$

with $A_{k,s} = [t_0 - n_{k+1}, s]$ and $B_{k,s} = (s, t_0]$
Critical Values, $\hat{z}_{k,\tau}$

- Simulate $\hat{z}_k$ - homogeneity of the interval sequence $I_1, \ldots, I_k$
- 'Propagation' condition

$$E_{\theta^*_{\tau}} \left| \ell_{I_k} (Y; \widehat{\theta}_{I_k,\tau}) - \ell_{I_k} (Y; \widehat{\theta}_\tau) \right| \leq \rho_k R_r (\theta^*_{\tau})$$

$$\rho_k = \frac{\rho_k}{K}$$ for a given significance level $\rho$

- Check $\hat{z}_{k,\tau}$ for (six) different $\theta^*_{\tau}$

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Critical Values, $\delta_{k,\tau}$

Figure 1: Simulated critical values across different parameter constellations for the modest case $r = 0.5$, $\tau = 0.05$ and $\tau = 0.01$
Critical Values, $\hat{z}_{k,\tau}$

Figure 2: Simulated critical values across different parameter constellations for the conservative case $r = 1$, $\tau = 0.05$ and $\tau = 0.01$
Adaptive Estimation

- Compare $T_{k,\tau}$ at every step $k$ with $\delta_{k,\tau}$
- Data window index of the *interval of homogeneity* - $\hat{k}$

- Adaptive estimate

$$\hat{\theta}_\tau = \tilde{\theta}_{l_{\hat{k},\tau}}, \quad \hat{k} = \max_{k \leq K} \{ k : T_{\ell,\tau} \leq \delta_{\ell,\tau}, \ell \leq k \}$$
Data

- **Series**
  - DAX, FTSE 100 and S&P 500 returns
    20050103-20141231 (2608 days)
  - Research Data Center (RDC) - Datastream

- **Setup**
  - Expectile levels: $\tau = 0.05$ and $\tau = 0.01$
  - Modest ($r = 0.5$) and conservative ($r = 1$) risk cases
  - $\{n_k\}_{k=0}^{11} = \{20 \text{ days, 25 days, \ldots, 250 days}\}$

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Adaptive Estimation

Figure 3: Estimated length \( n^*_k \) of intervals of homogeneity from 20060103-20141231 for the modest risk case \( r = 0.5 \), at expectile level \( \tau = 0.05 \). The red line presents the one-month smoothed values.
Figure 4: Estimated length $\hat{n}_k$ of intervals of homogeneity from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.05$. The red line presents the one-month smoothed values.

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Adaptive Estimation

Figure 5: Estimated length $n_k$ of intervals of homogeneity from 20060103-20141231 for the modest risk case $r = 0.5$, at expectile level $\tau = 0.01$. The red line presents the one-month smoothed values.
Adaptive Estimation

Figure 6: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.01$. The red line presents the one-month smoothed values.
Risk Exposure

Figure 7: DAX index returns (*) and adaptively estimated expectile \( e_{t,\tau} \) \((r = 1 \text{ and } \tau = 0.05)\) from 20060103-20141231
Figure 8: DAX index returns (•), adaptively estimated expectile $e_{t,\tau}$ and expected shortfall $ES_{e_{t,\tau}}$ ($r = 1$ and $\tau = 0.05$) from 20060103-20141231.
Portfolio Protection

- Portfolio protection strategy
  - Aim: Guarantee a proportion level of wealth at the investment horizon.
  - The investor can reduce the downside risk as well as participating in gains of risky assets.

Example

Decision at day $t$: multiple of the difference between the portfolio value and the discounted floor up to $t$ is invested into the stock portfolio (DAX), the rest into a riskless asset.
**Portfolio Protection**

- **Crucial ingredient:** the multiplier $m$
  - $m$: the proportion value invested into risky assets
  - The larger $m$, the more risky exposure

- **How to select the multiplier?**
  - Standard constant value
  - Based on tail risk measure, VaR or ES

- **Multiplier selection - Hamidi et al. (2014), ICARE**
  
  $m_{t,\tau} = |ES_{et,\tau}|^{-1}$

  - Practice: threshold range for $m_{t,\tau}$, [1, 12]
Multiplier Dynamics

Figure 9: Time-varying multiplier $m_{t, \tau}$ for DAX index returns based on lCARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231
Performance

- One-year rolling details
- CAViaR-based rolling details
- Target floor

Figure 10: Portfolio value: (a) DAX index (black), (b) $m = 5$, (c) one-year rolling, (d) CAViaR one-year rolling ($\alpha = 0.065$), (e) $m_{t,\tau}$ - ICARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231.

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Performance

<table>
<thead>
<tr>
<th></th>
<th>Return (%)</th>
<th>Volatility (%)</th>
<th>VaR 99%</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>8.79</td>
<td>22.54</td>
<td>-4.24</td>
<td>0.24</td>
<td>10.33</td>
<td>0.02</td>
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<tr>
<td>ICARE</td>
<td>7.36</td>
<td>13.60</td>
<td>-2.31</td>
<td>0.52</td>
<td>9.16</td>
<td>0.03</td>
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<tr>
<td>Rolling one-year</td>
<td>5.70</td>
<td>10.18</td>
<td>-1.59</td>
<td>0.17</td>
<td>10.05</td>
<td>0.04</td>
</tr>
<tr>
<td>CAViaR rolling</td>
<td>0.01</td>
<td>7.35</td>
<td>-1.43</td>
<td>-0.90</td>
<td>13.04</td>
<td>0.00</td>
</tr>
<tr>
<td>Multiplier 1</td>
<td>3.51</td>
<td>2.25</td>
<td>-0.41</td>
<td>0.20</td>
<td>10.05</td>
<td>0.10</td>
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<td>Multiplier 2</td>
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<td>4.50</td>
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<td>10.00</td>
<td>0.06</td>
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<td>Multiplier 3</td>
<td>4.41</td>
<td>6.74</td>
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<td>0.17</td>
<td>9.90</td>
<td>0.04</td>
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<td>Multiplier 4</td>
<td>4.78</td>
<td>9.00</td>
<td>-1.71</td>
<td>0.15</td>
<td>9.88</td>
<td>0.03</td>
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<td>Multiplier 5</td>
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<td>11.17</td>
<td>-2.10</td>
<td>0.11</td>
<td>9.91</td>
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<td>Multiplier 6</td>
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<td>5.36</td>
<td>-0.99</td>
<td>-0.33</td>
<td>6.48</td>
<td>0.04</td>
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<td>Multiplier 7</td>
<td>2.65</td>
<td>6.04</td>
<td>-1.08</td>
<td>-0.51</td>
<td>6.49</td>
<td>0.03</td>
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<tr>
<td>Multiplier 8</td>
<td>2.13</td>
<td>6.55</td>
<td>-1.17</td>
<td>-0.59</td>
<td>7.90</td>
<td>0.02</td>
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<tr>
<td>Multiplier 9</td>
<td>1.70</td>
<td>6.96</td>
<td>-1.25</td>
<td>-0.74</td>
<td>10.38</td>
<td>0.02</td>
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<tr>
<td>Multiplier 10</td>
<td>1.46</td>
<td>7.33</td>
<td>-1.38</td>
<td>-0.93</td>
<td>12.90</td>
<td>0.01</td>
</tr>
<tr>
<td>Multiplier 12</td>
<td>0.82</td>
<td>7.56</td>
<td>-1.47</td>
<td>-1.25</td>
<td>16.65</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Figure 11: Portfolio return moments comparison. Returns and volatility are annualized. The investment strategy is on a one-year investment basis.
Performance - Summary

- ICARE vs empirical data
  - Slightly lower return (7.36% vs 8.79%)
  - largely lower volatility (13.60% vs 22.54%)
  - Guarantee the target floor value

- ICARE vs other strategies
  - higher return than the candidates with CAViaR-based or expectile one-year rolling
  - Outperform typical constant multiplier benchmarks
Conclusions

- Localising CARE Model
  - Balance between modelling bias and parameter variability
  - Parameter dynamics

- Tail Risk Dynamics
  - Expectile levels $\tau = 0.05$ and $\tau = 0.01$
  - Expectile and Expected Shortfall

- Asset Allocation
  - Portfolio insurance on DAX at level $\tau = 0.05$
  - Outperform one-year rolling window and other benchmarks
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*Asymmetric least squares regression estimation: a nonparametric approach*
Why Expectiles? Quantile VaR

Figure 12: Distribution of returns, the 5% quantile remains unchanged under the changing tail structure

ICARE - localising Conditional AutoRegressive Expectiles
Expectile v.s. Quantile

- **Tail inference**
  - Quantile: zero-moment of tail structure - probability
    Central quantile: median
  - Expectile: first moment of tail structure
    Central expectile: mean

- Expectiles are sensitive to extreme magnitude, outliers
- Expectiles link to expected shortfall (ES) nicely

ICARE - localising Conditional AutoRegressive Expectiles
M-Quantiles

- Loss function, Breckling and Chambers (1988)

\[ z_\alpha = \arg \min_{\theta} E \rho_{\alpha,\gamma}(Y - \theta) \]

where \( \rho_{\alpha,\gamma}(u) = |\alpha - \{u < 0\}| |u|^\gamma, \ \gamma \geq 1 \)

- Quantile - ALD location estimate

\[ q_\alpha = \arg \min_{\theta} E \rho_{\alpha,1}(Y - \theta) \]

- Expectile - AND location estimate

\[ e_\alpha = \arg \min_{\theta} E \rho_{\alpha,2}(Y - \theta) \]
**Loss Function**

Figure 13: **Expectile** and **quantile** loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right).

---

LQRcheck

ICARE - localising Conditional AutoRegressive Expectiles
Expectiles and Quantiles

- M-Quantile

\[
\frac{\alpha}{1 - \alpha} = \frac{\int_{-\infty}^{e\alpha} |y - e\alpha|^{\gamma-1} dF(y)}{\int_{e\alpha}^{\infty} |y - e\alpha|^{\gamma-1} dF(y)}
\]

- Expectile - Global influence, obtained from

\[
\gamma = 2, \quad \frac{\alpha}{1 - \alpha} = \frac{\int_{-\infty}^{e\alpha} |y - e\alpha| dF(y)}{\int_{e\alpha}^{\infty} |y - e\alpha| dF(y)}
\]

- Quantile - Local influence, obtained from

\[
\gamma = 1, \quad \frac{\alpha}{1 - \alpha} = \frac{P(Y \leq q_\alpha)}{P(Y > q_\alpha)}
\]

ICARE - localising Conditional AutoRegressive Expectiles
CAViaR - Conditional Autoregressive Value at Risk by Regression Quantiles

- Asymmetric slope specification, conditional on information set $\mathcal{F}_{t-1}$ at time $t$

$$y_t = q_{t,\alpha} + \varepsilon_{t,\alpha} \quad \text{Quant}_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1}) = 0$$

$$q_{t,\alpha} = \beta_0 + \beta_1 q_{t-1,\alpha} + \beta_2 y_{t-1}^+ + \beta_3 y_{t-1}^-$$

- Quantile (VaR) $q_{t,\alpha}$ at $\alpha \in (0, 1)$, $\text{Quant}_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1})$ is the $\alpha$-quantile of $\varepsilon_{t,\alpha}$ conditional on information set $\mathcal{F}_{t-1}$
- With AND, set $\alpha = 0.065$ such that $e_{\tau\alpha} = q_\alpha$ when $\tau_\alpha = 0.05$

ICARE - localising Conditional AutoRegressive Expectiles
Asymmetric Normal Distribution (AND)

\[ f(w) = \frac{2}{\sigma} \left( \sqrt{\frac{\pi}{|\tau - 1|}} + \sqrt{\frac{\pi}{\tau}} \right)^{-1} \exp \left\{ -\rho_\tau \left( \frac{w - \mu}{\sigma} \right) \right\} \]

- Check function: \( \rho_\tau (u) = |\tau - \mathbb{1} \{u \leq 0\}| u^2 \)
- \( \text{AND} (\mu, \sigma^2, 1/2) = N(\mu, \sigma^2) \), Gerlach et al. (2012)
Figure 14: Density function for selected ANDs: (a) $\mu = 0, \tau = 0.5$ (b) $\mu = -1, \tau = 0.25$ (c) $\mu = -2, \tau = 0.05$ (d) $\mu = -3, \tau = 0.01$, with $\sigma^2_{\varepsilon, \tau} = 1$
Quasi Log Likelihood Function

- If $\varepsilon_\tau \sim \text{AND} (\mu, \sigma^2_\varepsilon, \tau)$ with pdf $f_\varepsilon (\cdot)$
  then $Y \sim \text{AND} (e_\tau + \mu, \sigma^2_\varepsilon, \tau)$

- Quasi log likelihood function for observed data
  $\mathcal{Y} = \{y_1, \ldots, y_n\}$ over a fixed interval $I$

$$
\ell_I (\mathcal{Y}; \theta_\tau) = \sum_{t \in I} \log f_\varepsilon (y_t - e_{t, \tau})
$$
Gaussian Regression

\[ Y_i = f(X_i) + \varepsilon_i, \ i = 1, \ldots, n, \] weights \( W = \{w_i\}_{i=1}^n \)

\[ L(W, \theta) = \sum_{i=1}^n \ell \{ Y_i, f_\theta(X_i) \} w_i, \] log-density \( \ell(\cdot), \) \( \tilde{\theta} = \arg \max_{\theta \in \Theta} L(W, \theta) \)

1. Local constant, \( f(X_i) \approx \theta^*, \ \varepsilon_i \sim N(0, \sigma^2) \)
   \[
   E_{\theta^*} \left| L(W, \tilde{\theta}) - L(W, \theta^*) \right|^r \leq E|\xi|^{2r}, \quad \xi \sim N(0,1)
   \]

2. Local linear, \( f(X_i) \approx \theta^*^T \Psi_i, \ \varepsilon_i \sim N(0, \sigma^2), \) basis functions \( \Psi = \{\psi_1(X_1), \ldots, \psi_p(X_p)\}, \) multivariate \( \xi \)
   \[
   E_{\theta^*} \left| L(W, \tilde{\theta}) - L(W, \theta^*) \right|^r \leq E|\xi|^{2r}, \quad \xi \sim N(0, I_p)
   \]
## Risk Bound

<table>
<thead>
<tr>
<th>Parameter Scenarios</th>
<th>$\tau = 0.05$</th>
<th>$\tau = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Mid</td>
</tr>
<tr>
<td>$r = 0.5$</td>
<td>0.24</td>
<td>0.33</td>
</tr>
<tr>
<td>$r = 1.0$</td>
<td>2.40</td>
<td>4.62</td>
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</table>

Table 1: Simulated $R_r(\theta^*_\tau)$, with expectile levels $\tau = 0.05$ and $\tau = 0.01$, for six selected parameter constellation groups.
## Parameter Scenarios

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \tau = 0.05 )</th>
<th></th>
<th>( \tau = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\alpha}_{0,\tau} )</td>
<td>Low: -0.0003, Mid: 0.0003, High: 0.0007</td>
<td>Low: -0.0003, Mid: 0.0003, High: 0.0007</td>
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<tr>
<td>( \tilde{\alpha}_{1,\tau} )</td>
<td>Low: -0.1058, Mid: -0.0306, High: 0.0524</td>
<td>Low: -0.1035, Mid: -0.0312, High: 0.0547</td>
<td></td>
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<tr>
<td>( \tilde{\alpha}_{2,\tau} )</td>
<td>Low: -0.5800, Mid: -0.5288, High: 0.2438</td>
<td>Low: -0.5808, Mid: -0.5266, High: 0.2089</td>
<td></td>
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<tr>
<td>( \tilde{\alpha}_{3,\tau} )</td>
<td>Low: 0.5050, Mid: 0.5852, High: 2.1213</td>
<td>Low: 0.5134, Mid: 0.5871, High: 2.2066</td>
<td></td>
</tr>
<tr>
<td>( \tilde{\sigma}^2_{\varepsilon,\tau} )</td>
<td>Low: 0.0001, Mid: 0.0001, High: 0.0002</td>
<td>Low: 0.0001, Mid: 0.0001, High: 0.0002</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Quartiles of estimated CARE parameters based on one-year estimation window, i.e., 250 observations, for the three stock market returns - DAX, FTSE 100, S&P 500 - from 20050103-20141231 (2608 trading days)
Selected Parameter Scenarios

Figure 15: Histogram of the selected parameter scenarios (Low, Mid and High) for adaptive estimation with \( \tau = 0.05 \) and \( \tau = 0.01 \).
Expected Shortfall

- Expectile level $\tau_\alpha$ such that $e_{\tau_\alpha} = q_\alpha$ ($\alpha$-quantile), Yao and Tong (1996), Acerbi and Tasche (2002)

$$
\tau_\alpha = \frac{\alpha \cdot q_\alpha - \int_{-\infty}^{q_\alpha} y dF(y)}{E[Y] - 2 \int_{-\infty}^{q_\alpha} y dF(y) - (1 - 2\alpha) q_\alpha}
$$

where $Y \sim \text{AND}$.

- Expected Shortfall (ES), Kuan et al. (2009)

$$
ES_{e_{\tau_\alpha}} = \left| 1 + \tau_\alpha (1 - 2\tau_\alpha)^{-1} \alpha^{-1} \right| e_{\tau_\alpha}
$$
Portfolio Protection Strategy

- Under certain confidence level, we aim to maintain:
  Estep and Kritzman (1988)

\[ V_t \geq k \times \max \left\{ F \ast e^{-rf \ast (T-t)}, \sup_{p \leq t} V_p \right\} = F_s^t \]

- \( V_t \): portfolio value at time \( t, t \in (0, T] \)
- \( F_s^t \): protection value (target floor)
- \( k \): exogenous parameter \((0, 1), \text{set } k = 0.9\)
- \( rf \): risky free rate, initial value \( F = 100\)
- Cushion value \( C_t = V_t - F_s^t \geq 0 \)

- Allocate \( G_t = m \cdot C_t \) proportion into stock portfolio (DAX),
  and the remaining \( V_t - G_t \) into riskless asset, multiplier \( m \geq 0 \).
Example: CPPI - Constant proportion portfolio insurance

Consider an insurance strategy under CPPI with constant floor $F = 100$, constant $m = 5$, and riskless asset rate $rf = 0$ (cash). The initial risky asset value is 100, and at each step goes up(down) 15.

<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>initial risky asset value $F$</td>
<td>100</td>
</tr>
<tr>
<td>proportion $k$</td>
<td>0.9</td>
</tr>
<tr>
<td>riskless rate $rf$</td>
<td>0</td>
</tr>
<tr>
<td>constant multiplier $m$</td>
<td>5</td>
</tr>
<tr>
<td>steps</td>
<td>4</td>
</tr>
</tbody>
</table>
### Table 3: Risky portfolio value and the value in low bracket denotes the asset return.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td>115 (0.13)</td>
<td>145 (0.10)</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>85 (0.15)</td>
<td></td>
<td>100 (0.15)</td>
<td>130 (0.12)</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>(-0.15)</td>
<td></td>
<td>70 (0.21)</td>
<td>85 (0.18)</td>
<td>55 (0.27)</td>
<td></td>
</tr>
<tr>
<td>(-0.18)</td>
<td></td>
<td>(-0.21)</td>
<td>40 (0.27)</td>
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</tbody>
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<tbody>
<tr>
<td>−0.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td></td>
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</tbody>
</table>

ICARE - localising Conditional AutoRegressive Expectiles
### Portfolio value and the cushion

<table>
<thead>
<tr>
<th>100</th>
<th>107.50 (17.5)</th>
<th>118.91 (28.91)</th>
<th>135.59 (45.60)</th>
<th>159.18 (69.18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10)</td>
<td>92.50 (4.71)</td>
<td>94.71 (8.24)</td>
<td>98.24 (13.61)</td>
<td>103.61 (13.61)</td>
</tr>
<tr>
<td></td>
<td>90.29 (0.61)</td>
<td>90.61 (1.15)</td>
<td>91.15 (1.15)</td>
<td>91.15 (1.15)</td>
</tr>
<tr>
<td></td>
<td>89.979 (-0.02 )</td>
<td>89.979 (0)</td>
<td>89.979 (0)</td>
<td>89.979 (0)</td>
</tr>
</tbody>
</table>

Table 4: Portfolio value and the value in low bracket denotes the cushion.
Multiplier

- Portfolio value $V_t$
  \[ V_{t+1} = V_t + G_t r_{t+1} + (V_t - G_t) rf_{t+1} \]
  with $r_t$ stock index return and $rf_t$ riskless rate

- Cushion value $C_t = V_t - F_t \geq 0$
  \[ C_{t+1} = C_t \{1 + m \cdot r_{t+1} + (1 - m) rf_{t+1}\} \]

- $\forall t \leq T$, since the value $C_t \geq 0$
  \[ m \cdot r_{t+1} + (1 - m) rf_{t+1} \geq -1 \]
Multipler

If \( rf_t \) is relatively small, and when \( r_{t+1} < 0 \), yield the upper bound on the multiplier:

**Proposition**  The guarantee is satisfied at any time of the management period with a probability equal to 1

\[
\forall t \leq T - 1, m \leq (-r_{t+1})^{-1}
\]

where \( r_{t+1}^- = \min \{ r_{t+1}, 0 \} \).

- Multipler \( m_t \), the leverage value on risky assets, is negatively related to the maximum extreme loss of risky assets.
- For example, if \( r_{t+1} = -10\% \), \( m \leq 10 \); if \( r_{t+1} = -20\% \), \( m \leq 5 \).
**Gap Risk**

- In practice, due to the discrete-time rebalancing, the nonnegative cushion value can not be guaranteed perfectly.

- Gap risk: the risk of violating the floor protection, i.e., the tiny level of probability that the cushion values are non-positive.

- How to define the gap risk:
  - control of the probability of a potential loss - VaR based multiplier
  - control of the potential loss size - ES based multiplier
Gap Risk - control of the probability of a potential loss - VaR based multiplier

Given a confidence level $1 - \alpha$, the insurance condition, i.e., portfolio value is above floor, is guaranteed, Föllmer and Leukert (1999),

$$P(C_t \geq 0, \forall t \leq T) \geq 1 - \alpha$$

Equivalently, (set time-varying multiplier)

$$P\left(m_t \leq (-r_{t+1}^-)^{-1}, \forall t \leq T - 1\right) \geq 1 - \alpha$$
Multiplier

- Gap risk: control of the probability of a potential loss
  Multiplier $m_t$ with quantile - Ameur and Prigent (2014)
  
  $$m_{t,q_\alpha} = |\text{VaR}_{\alpha}(r_{t+1})|^{-1}$$

- Gap risk: control of the potential loss size
  Multiple $m_t$ with expected shortfall - Hamidi et al. (2014)
  
  $$m_{t,\tau} = |\text{ES}_{e_{t,\tau}}|^{-1}$$

ICARE - localising Conditional AutoRegressive Expectiles
Multiplier Density

Figure 16: Kernel density estimate of the multiplier $m_{t,\tau}$ for DAX index returns based on ICARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231
CARE-based one-year rolling performance

Figure 17: Estimated expectile and expected shortfall by CARE based one-year fixed rolling window (upper panel), and the corresponding multiplier (lower panel) for DAX index returns from 20060103 to 20141231.
CAViaR-based one-year rolling

Figure 18: Estimated VaR ($\alpha = 0.065$) and expected shortfall by CAViaR-based one-year rolling (upper panel), and the corresponding multiplier (lower panel) for DAX from 20060103 to 20141231.
Figure 19: Portfolio value: (a) DAX index (black), (b) $m_{t,\tau}$ - ICARE ($r = 1$ and $\tau = 0.05$), (c) the corresponding target floor $F^s_t$, from 20060103-20141231.
Figure 20: Portfolio value: (a) DAX index, (b) $m = 3$, (c) $m = 6$, (d) $m = 9$, (e) $m = 12$ on DAX index in a bull market from 20090309-20110510 (left panel, 567 observations) and in a bear market from 20070716-20090306 (right panel, 431 observations).
Parameter Dynamics

Figure 21: Estimated $\hat{\alpha}_{1,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations.
Parameter Dynamics

Figure 22: Estimated $\alpha_{1,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations.

ICARE - localising Conditional AutoRegressive Expectiles
Parameter Distributions

Figure 23: Kernel density estimates of $\alpha_{1,0.05}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

ICARE - localising Conditional AutoRegressive Expectiles
Parameter Distributions

Figure 24: Kernel density estimates of $\alpha_{1,0.01}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations.

ICARE - localising Conditional AutoRegressive Expectiles
Parameter Dynamics

Figure 25: Estimated $\hat{\alpha}_{2,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations.
Figure 26: Estimated $\alpha_{2,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations
Parameter Dynamics

Figure 27: Estimated $\alpha_{3,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations
Parameter Dynamics

Figure 28: Estimated $\alpha_{3,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations.