Time Varying Lasso

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Motivation

Figure 1: Dynamic lasso

High dimensions

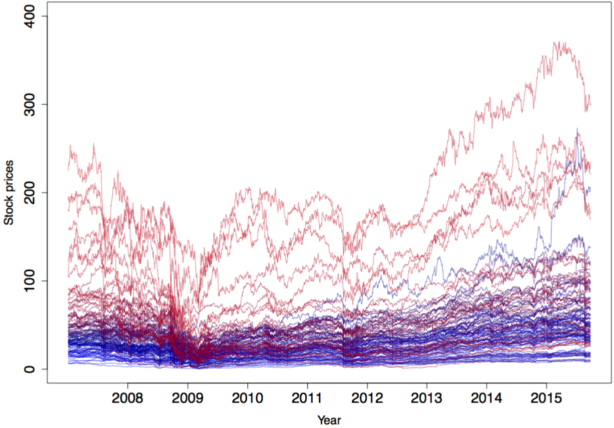


Figure 2: Daily stock prices (USD)

Financial stress

Hakkio and Keeton (2009): Key phenomena

1. Increased uncertainty about fundamental value of assets
2. Increased uncertainty about behavior of other investors
3. Increased asymmetry of information
4. Decreased willingness to hold risky assets
5. Decreased willingness to hold illiquid assets

Systemic risk

"Risk of financial instability so widespread that it impairs the functioning of a financial system to the point where economic growth and welfare suffer materially."

ECB, Financial Network and Financial Stability, 2010.

- Adrian and Brunnermeier (2011): CoVaR as a measure of systemic risk ▸ CoVaR
- Härdle et al. (2015): TENET

Financial Risk Meter

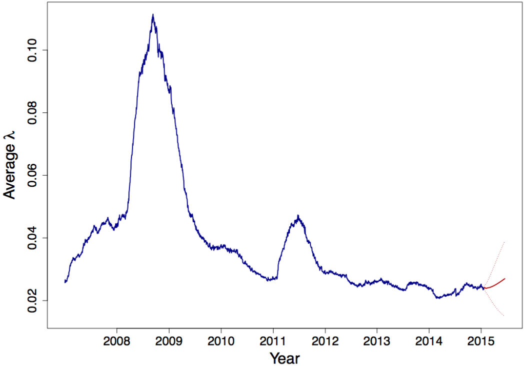


Figure 3: Time series of λ ($\tau = 0.05$) from financialriskmeter

Network Dynamics

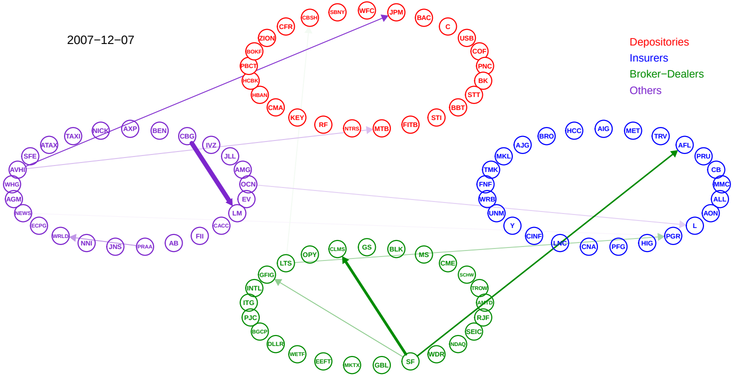


Figure 4: Financial risk network dynamics Depositories, Insurance, Broker-Dealers, Others ; $T = 266$, $\tau = 0.05$.

Challenges

- Time series of Lasso penalty parameter λ
- Prediction of λ
  - Volatility
  - Multicolinearity
  - Significant explanatory variables
- Monitoring of systemic risk

Outline

1. Motivation ✓
2. Lasso in linear regression
3. Lasso dynamics
4. Simulation
5. Real-data application

Lasso in linear regression

- Regression model

$$Y = X\beta + \varepsilon$$

with $Y = (Y_1, \dots, Y_n)^\top$, $\beta = (\beta_1, \dots, \beta_p)^\top$, $X_{(n\times p)}$, $\varepsilon_{(n\times 1)} \overset{iid}{\sim} (0, \sigma^2)$

▸ Assumptions

- Tibshirani (1996): Estimation using a **penalty** function

$$\widehat{\beta} = \arg\min_{\beta} \left\{ \sum_{i=1}^{n} \left(Y_i - X_i^\top \beta \right)^2 + \lambda \sum_{j=1}^{p} |\beta_j| \right\}$$

with $X_i = (x_{i1}, \dots, x_{ip})^\top$, tuning parameter $\lambda \geq 0$

▸ Choosing λ

Geometrical interpretation

- Quadratic objective function $(\beta - \widehat{\beta}^{OLS})^\top X^\top X(\beta - \widehat{\beta}^{OLS})$

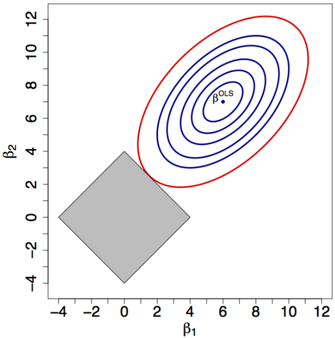


Figure 5: How does Lasso work

MVAlassocontour

Lasso in REGAR model

- Wang et al. (2007): Lasso for linear regression model with autoregressive errors (REGAR) ▸ Assumptions

$$Q(\theta) = Q(\beta, \phi) = \sum_{t=q+1}^{T} \left\{ Y_t - X_t^\top \beta - \sum_{j=1}^{q} \phi_j (Y_{t-j} - X_{t-j}^\top \beta) \right\}^2 + (T-q) \sum_{j=1}^{p} \lambda |\beta_j| + (T-q) \sum_{j=1}^{q} \gamma |\phi_j|,$$

$$\widehat{\theta} = \arg\min_{\theta} Q(\theta),\ \widehat{\theta} \geq 0$$

Lasso in AR model

- Nardi and Rinaldo (2011): Lasso for autoregressive model

  ▸ Assumptions

$$Q(\Lambda) = \frac{1}{2T} \sum_{t=1}^{T} \left\{ X_t - \sum_{j=1}^{p} \phi_j X_{t-j} \right\}^2 + \lambda \sum_{j=1}^{p} \lambda_j |\phi_j|,$$

$$\widehat{\Lambda} = \{\lambda, \{\lambda_j, j = 1, \ldots, p\}\} = \arg\min_{\Lambda} Q(\Lambda), \; \widehat{\Lambda} \geq 0$$

Penalty parameter λ

- Osborne et al. (2000): Duality interpretation ▸ Details

$$\lambda = \frac{(Y - X\widehat{\beta})^\top X\widehat{\beta}}{\| \widehat{\beta} \|_1} \tag{1}$$

- Penalty λ dependent on

  1. Residuals' size
  2. Condition number κ of $X^\top X$
  3. Cardinality of active set $q \stackrel{\text{def}}{=} \| \beta \|_0$

λ dependent on volatility σ

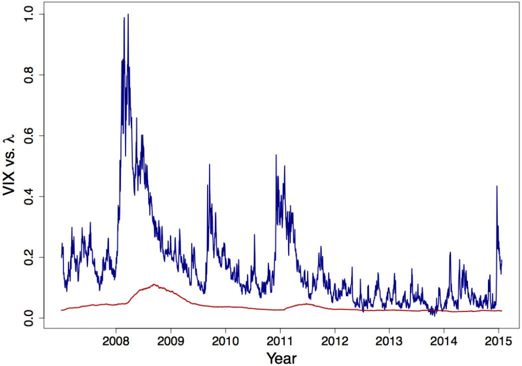


Figure 6: Implied volatility index (VIX) and λ for 20070103 - 20150925

λ dependent on active set q

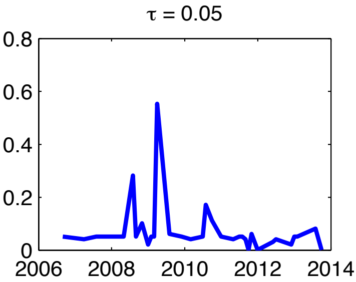


Figure 7: Time series of λ from TEDAS, Härdle et al. (2015)

How does λ change

- 100 scenarios
- Design matrix $X_{(n \times p)}$

$$\{X_i\}_{i=1}^{n} \sim N_p(0, \Sigma),$$

$n = 1000,\ p = 100$

- Covariance matrix $\Sigma_{(p \times p)}$

$$\sigma_{ij} = 0.5^{|i-j|}$$

$i, j = 1, \ldots, p$

How does λ change

1. Volatility change ($\Delta\sigma$):

$$\beta_{(100\times 1)} = (1,1,1,1,1,0,\dots,0)^\top$$

$$\varepsilon_t \sim \begin{cases} N(0,1), & \text{if } t \le 500 \\ N(0,1.21), & t > 500 \end{cases}$$

2. β change ($\Delta\beta$):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\dots,0)^\top, & \text{if } t \le 500 \\ (1,0.9,0.8,\dots,0.2,0.1,0,\dots,0)^\top, & t > 500 \end{cases}$$

$$\varepsilon_t \sim N(0,1)$$

How does λ change

3. Active set q change (Δq):

$$\beta_t = \begin{cases} (1,1,1,1,1,1,1,1,1,1,0,\dots,0)^\top, & \text{if } t \leq 500 \\ (1,1,1,1,1,0,\dots,0)^\top, & t > 500 \end{cases}$$

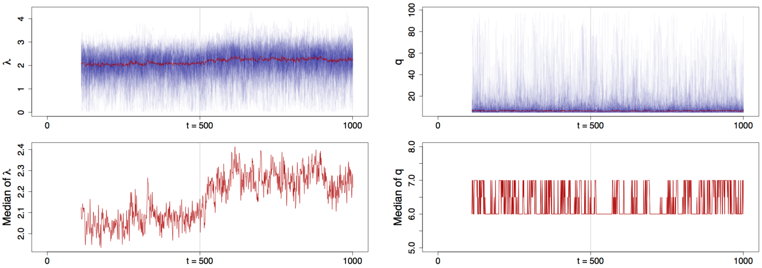
$$\varepsilon_t \sim N(0,1)$$

4. Condition number $\kappa(X^\top X)$ change $(\Delta\kappa)$:

$$\kappa_t \approx \begin{cases} 12000, (\sigma_{ij} = 0.5^{|i-j|}), & \text{if } t \leq 500 \\ 100000, (\sigma_{ij} = 0.9^{|i-j|}), & t > 500 \end{cases}$$

$$\varepsilon_t \sim N(0,1),\ \beta_{(100\times 1)} = (1,1,1,1,1,0,\dots,0)^\top$$

Simulation results ($\Delta\sigma$)



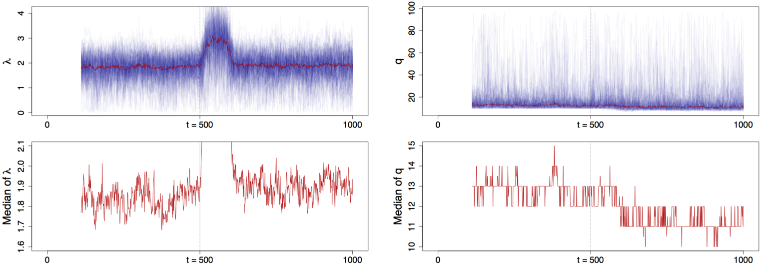
(a) Penalty parameter λ (b) Cardinality of active set q

Figure 8: Time series of λ and q with change of $\text{Var}(\varepsilon_t)$ after $t = 500$, moving windows of length 110.

More details

TVPvariance

Simulation results ($\Delta\beta$)



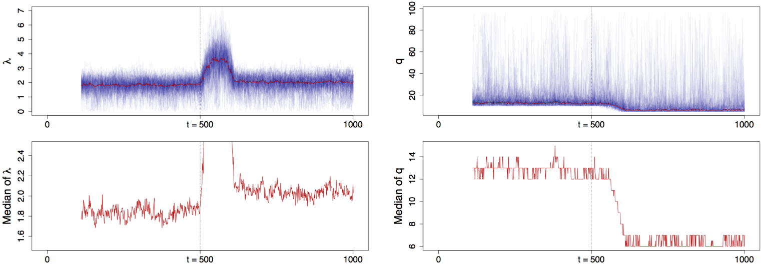
(a) Penalty parameter λ (b) Cardinality of active set q

Figure 9: Time series of λ and q with change of β_t after $t = 500$, moving windows of length 110.

▸ More details

TVPbetanorm

Simulation results (Δq)



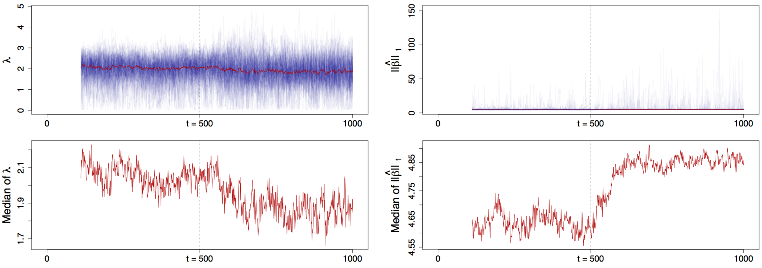
(a) Penalty parameter λ (b) Cardinality of active set q

Figure 10: Time series of λ and q with change of q after $t = 500$, moving windows of length 110.

▸ More details

TVPactiveset

Simulation results ($\Delta\kappa$)



(a) Penalty parameter λ (b) L_1-norm of $\widehat{\beta}$

Figure 11: Time series of λ and L_1-norm of $\widehat{\beta}$ with change of $\kappa(X^\top X)$ after $t = 500$, moving windows of length 110.

▸ More details

TVPdesign

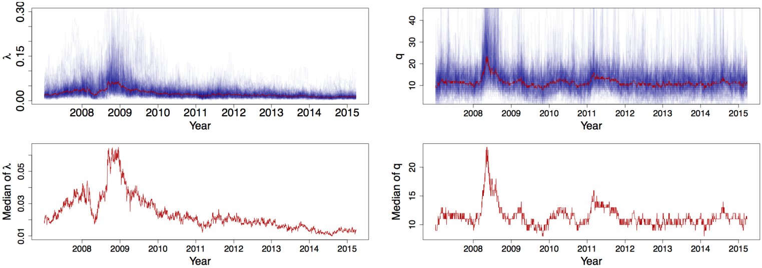
Data description

- NASDAQ: 100 US financial companies ▸ Details
- Log returns of daily adjusted stock close prices
- 6 macroprudential variables ▸ Details
- Time interval 20070103 - 20150925
- Source: Datastream, Yahoo Finance

Financial Risk Meter construction

- Each company regressed on the others and on the macroprudential variables
- LAR algorithm
- Moving windows of size $w = 126$
- λ chosen by BIC criterion

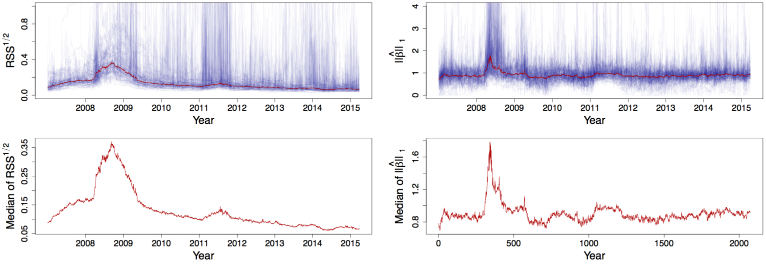
Visualization



(a) Penalty parameter λ (b) Cardinality of active set q

Figure 12: Time series of λ and q for 20070103 - 20150925.

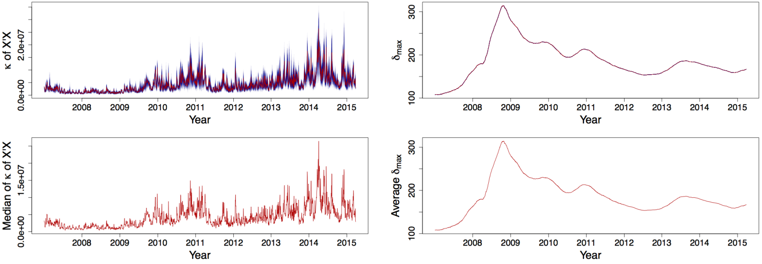
Visualization



(a) L_2-norm of residuals (b) L_1-norm of $\widehat{\beta}$

Figure 13: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ for 20070103 - 20150925.

Visualization



(a) $\kappa(X^\top X)$ (b) $\delta_{max}(X^\top X)$

Figure 14: Time series of $\kappa(X^\top X)$ and maximum eigenvalue of $X^\top X$, $\delta_{max}(X^\top X)$, for 20070103 - 20150925.

Summary

- Time series of penalty parameter λ
- Dependency of λ on
  - Residuals
  - Active set
  - Design matrix
- Evidence of variation over time

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On the "Degrees of Freedom" of the Lasso
The Annals of Statistics **35**(5): 2173-2192, 2007

CoVaR as a systemic risk measure I

Step 1. Estimate linear quantile regressions

$$X_{i,t} = \alpha_i + \gamma_i M_{t-1} + \varepsilon_{i,t},$$
$$X_{j,t} = \alpha_{j|i} + \gamma_{j|i} M_{t-1} + \beta_{j|i} X_{i,t} + \varepsilon_{j|i,t},$$

- $X_{i,t}$ is the log return of a financial institution i,
- M_{t-1} are lagged macro state variables.

Adrian and Brunnermeier (2011)

Back to "Systemic risk"

CoVaR as a systemic risk measure II

Step 2. Generate predicted values under assumption $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j|i,t}}(\tau|M_{t-1}, X_{i,t}) = 0$, $\tau = (0,1)$,

$$\widehat{\text{VaR}}^{\tau}_{i,t} = \hat{\alpha}_i + \hat{\gamma}_i M_{t-1},$$
$$\widehat{\text{CoVaR}}^{\tau}_{j|i,t} = \hat{\alpha}_{j|i} + \hat{\gamma}_{j|i} M_{t-1} + \hat{\beta}_{j|i} \widehat{\text{VaR}}^{\tau}_{i,t}.$$

Adrian and Brunnermeier (2011)

Back to "Systemic risk"

Assumptions on Lasso in linear regression

- Normalized columns of $X = (x_{ij})_{i=1,\ldots,n,\ j=1,\ldots,p}$

$$n^{-1}\sum_{i=1}^{n} x_{ij} = 0, \quad n^{-1}\sum_{i=1}^{n} x_{ij}^2 = 1$$

▸ Back to "Lasso in linear regression"

Selection of λ in Lasso

- Efron et al. (2004)
  - Least Angle Regression (LAR) algorithm
- Zou et al. (2007)
  - Bayesian Information Criterion (BIC)

$$\mathrm{BIC}(\lambda) = \frac{\| Y - X\widehat{\beta}(\lambda) \|^2}{n\widehat{\sigma}^2} + \frac{\log(n)}{n}\widehat{\mathrm{df}}(\lambda)$$

  with error variance estimator $\widehat{\sigma}^2(\lambda) = n^{-1} \| Y - X\widehat{\beta}^{OLS} \|^2$ and $\widehat{\lambda}^{BIC} = \arg\min_{\lambda} \mathrm{BIC}(\lambda)$

  - Estimator of number of effective parameters

$$\widehat{\mathrm{df}}(\lambda) = \| \widehat{\beta}(\lambda) \|_0$$

Back to "Lasso in linear regression"

REGAR model

- REGAR model

$$
\begin{aligned}
Y_t =& X_t^\top \beta + u_t, \quad t = 1, \dots, T \\
u_t =& \phi_1 u_{t-1} + \phi_2 u_{t-2} + \ldots + \phi_q u_{q-1} + \varepsilon_t,
\end{aligned}
$$

$X_t = (X_{t1}, \dots, X_{tp})^\top$, $\beta = (\beta_1, \dots, \beta_p)^\top$, $\phi = (\phi_1, \dots, \phi_q)^\top$, $\varepsilon_{(n\times 1)} \overset{iid}{\sim} (0, \sigma^2)$

▸ Back to "Lasso in REGAR model"

AR model

- AR model

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + \varepsilon_t, \quad t = 1, \ldots, T$$

$\varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$, $\text{cov}(\varepsilon_t, X_s) = 0$ for $s < t$

- $\{X_t\}$ has a MA(∞) representation

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

Back to "Lasso in AR model"

Lasso duality interpretation I

- Alternative representation

$$\widehat{\beta} = \arg\min_{\beta} \frac{1}{2}\left\{\sum_{i=1}^{n}\left(Y_i - X_i^{\top}\beta\right)^2\right\}, \text{ s. t. } \sum_{j=1}^{p}|\beta_j| \leq s \quad (2)$$

with tuning parameter $s \geq 0$

- Notation

$$f(\beta) = \tfrac{1}{2}\sum_{i=1}^{n}\left(Y_i - X_i^{\top}\beta\right)^2$$
$$g(\beta) = \sum_{j=1}^{p}|\beta_j| - s$$

Back to "Penalty parameter λ"

Lasso duality interpretation II

- For (2) as a convex programming problem the Lagrangian is

$$L(\beta, \lambda) = f(\beta) + \lambda g(\beta)$$

- Primal-dual relationship

$$\underset{\beta}{\text{minimize}} \sup_{\lambda \geq 0} L(\beta, \lambda) \geq \underset{\lambda \geq 0}{\text{maximize}} \inf_{\beta} L(\beta, \lambda)$$

- The dual function is $\inf_{\beta} L(\beta, \lambda)$ with

$$\lambda = \frac{(Y - X\widehat{\beta})^\top X\widehat{\beta}}{\| \widehat{\beta} \|_1}$$

Back to "Penalty parameter λ"

Simulation results ($\Delta\sigma$)

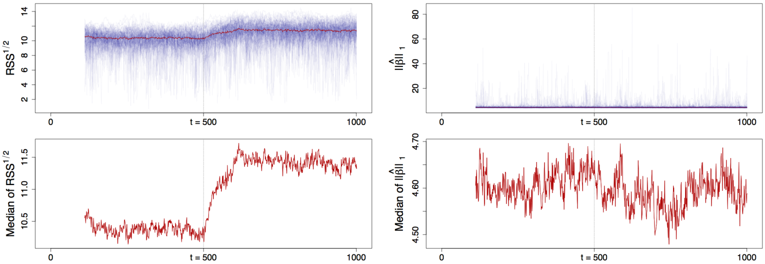

Figure 15: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of Var(ε_t) after $t = 500$, moving windows of length 110.

Back to "Simulation results ($\Delta\sigma$)"

TVPvariance

Simulation results ($\Delta\beta$)

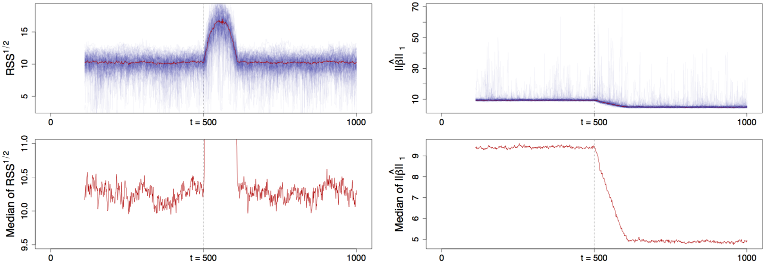

Figure 16: Time series of L_2-norm of residuals and L_1-norm of $\widehat{\beta}$ with change of β_t after $t = 500$, moving windows of length 110.

Back to "Simulation results ($\Delta\beta$)"

TVPbetanorm

Simulation results (Δq)

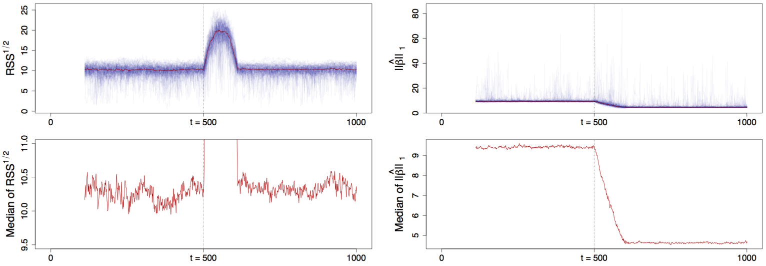

Figure 17: Time series of L_2-norm of residuals, L_1-norm of $\widehat{\beta}$ with change of q after $t = 500$, moving windows of length 110.

▸ Back to "Simulation results (Δq)"

TVPactiveset

Simulation results ($\Delta\kappa$)

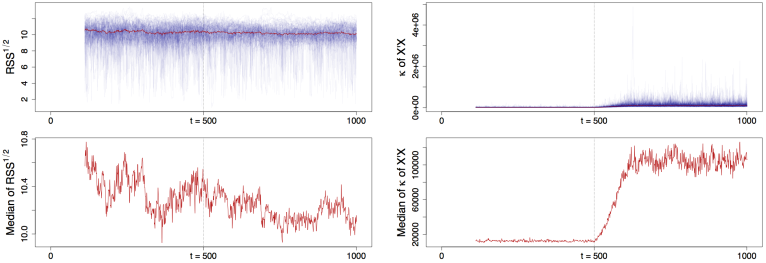

Figure 18: Time series of L_2-norm of residuals, $\kappa(X^\top X)$ with change of $\kappa(X^\top X)$ after $t = 500$, moving windows of length 110.

Back to "Simulation results ($\Delta\kappa$)"

TVPdesign

Financial companies

Back to "Data description"

WFC	Wells Fargo & Company	BAC	Bank of America Corporation
C	Citigroup Inc.	AXP	American Express Company
GS	Goldman Sachs Group, Inc. (The)	USB	U.S. Bancorp
AIG	American International Group, Inc.	MS	Morgan Stanley
BLK	BlackRock, Inc.	MET	MetLife, Inc.
PNC	PNC Financial Services Group, Inc. (The)	BK	Bank Of New York Mellon Corporation (The)
COF	Capital One Financial Corporation	SCHW	The Charles Schwab Corporation
PRU	Prudential Financial, Inc.	TRV	The Travelers Companies, Inc.
BEN	Franklin Resources, Inc.	CME	CME Group Inc.
MMC	Marsh & McLennan Companies, Inc.	STT	State Street Corporation
ALL	Allstate Corporation (The)	MHFI	McGraw Hill Financial, Inc.
BBT	BB&T Corporation	AON	Aon plc
AFL	Aflac Incorporated	ICE	Intercontinental Exchange Inc.
AMP	AMERIPRISE FINANCIAL SERVICES, INC.	CB	Chubb Corporation (The)
STI	SunTrust Banks, Inc.	TROW	T. Rowe Price Group, Inc.
AMTD	TD Ameritrade Holding Corporation	MCO	Moody's Corporation
HIG	Hartford Financial Services Group, Inc. (The)	IVZ	Invesco Plc
MTB	M&T Bank Corporation	NTRS	Northern Trust Corporation
FITB	Fifth Third Bancorp	PGR	Progressive Corporation (The)
LNC	Lincoln National Corporation	L	Loews Corporation
PFG	Principal Financial Group Inc	RF	Regions Financial Corporation
KEY	KeyCorp	AMG	Affiliated Managers Group, Inc.
CBG	CBRE Group, Inc.	EFX	Equifax, Inc.
CNA	CNA Financial Corporation	MKL	Markel Corporation
FNF	Fidelity National Financial, Inc.	HBAN	Huntington Bancshares Incorporated

Financial companies

Back to "Data description"

CINF	Cincinnati Financial Corporation	UNM	Unum Group
NDAQ	The NASDAQ OMX Group, Inc.	CMA	Comerica Incorporated
RJF	Raymond James Financial, Inc.	ETFC	E*TRADE Financial Corporation
AJG	Arthur J. Gallagher & Co.	Y	Alleghany Corporation
NYCB	New York Community Bancorp, Inc.	SEIC	SEI Investments Company
TMK	Torchmark Corporation	SIVB	SVB Financial Group
LM	Legg Mason, Inc.	SBNY	Signature Bank
WRB	W.R. Berkley Corporation	EWBC	East West Bancorp, Inc.
ZION	Zions Bancorporation	AFG	American Financial Group, Inc.
HCC	HCC Insurance Holdings, Inc.	HCBK	Hudson City Bancorp, Inc.
EV	Eaton Vance Corporation	CYN	City National Corporation
PACW	PacWest Bancorp	PBCT	People's United Financial, Inc.
DNB	Dun & Bradstreet Corporation (The)	BRO	Brown & Brown, Inc.
CFR	Cullen/Frost Bankers, Inc.	AFSI	AmTrust Financial Services, Inc.
AIZ	Assurant, Inc.	WDR	Waddell & Reed Financial, Inc.
BOKF	BOK Financial Corporation	CBSH	Commerce Bancshares, Inc.
SLM	SLM Corporation	WTM	White Mountains Insurance Group, Ltd.
ERIE	Erie Indemnity Company	ORI	Old Republic International Corporation
SNV	Synovus Financial Corp.	CACC	Credit Acceptance Corporation
UMPQ	Umpqua Holdings Corporation	SF	Stifel Financial Corporation
BPOP	Popular, Inc.	PB	Prosperity Bancshares, Inc.
FII	Federated Investors, Inc.	CNO	CNO Financial Group, Inc.
FHN	First Horizon National Corporation	MORN	Morningstar, Inc.
WBS	Webster Financial Corporation	OZRK	Bank of the Ozarks
FNFG	First Niagara Financial Group Inc.	MKTX	MarketAxess Holdings, Inc.

Macroprudential variables

1.	VIX
2.	Daily change in the 3-month Treasury maturities
3.	Change in the slope of the yield curve
4.	Change in the credit spread
5.	Daily Dow Jones U.S. Real Estate index returns
6.	Daily S&P500 index returns

Table 1: Macro state variables. Source: Adrian and Brunnermeier (2011), Datastream.

Back to "Data description"