Jointly Modelling and Robust Forecasting High-Dimensional Yield Curves

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Yield Curves Data

Figure 1: Daily yield curves of Chinese enterprise bond AAA in 2015.
Yield Curves Modelling

- Based on economic theory
  - market equilibrium: Vasicek model; CIR model
  - no-arbitrage: derivative pricing under B-S framework
  - affine-class: dynamic in maturities with time series technique

- Goodness of fit and forecasting
  - dynamic Nelson-Siegel model (Diebold and Li, 2006)
  - other generalized N-S models
Dynamic Nelson-Siegel Model

Advantages
- excellent fit to the term structure
- clear explanation on factors: level, slope and curvature
- estimation simplicity

Limitations
- specification issues
- jointly modelling across bond types and credit ratings
Motivation

Go beyond DNS

- high-dimensional curves across types and ratings
- flexible representation through high-dimensional $B$-splines
- sparse latent factors
- robust estimation via LAD regression
- risky bonds with low credit ratings
Motivation

Estimation Issues

- estimate a high-dimensional coefficient matrix
- nuclear norm penalty
  - involve a convex optimization
  - lead to a low dimensional factor model
- SVD to identify factors and loadings
- multivariate factorisable quantile/expectile regression (Chao et al. 2015; 2016)
Objectives and Contributions

- jointly modelling and robust forecasting high-dimensional yield curves
- multivariate factorisable median regression (MFMR)
- application for Chinese bond market
  - systemic liquidity and dispersion measure among curves
  - term structure and credit risk structure
  - in- and out-of-sample performance
Outline

1. Motivation
2. Model and Estimation
3. Application with Chinese Yield Curve Data
4. Concluding Remarks
Model Specification

- \( Y = (Y_{ij}) \in \mathbb{R}^{n \times m} \): multivariate curves
  - \( m \): the number of curves (across credit ratings and types)
  - \( n \): the length of observations (over time)
- \( \{X_i\}_{i=1}^n \in \mathbb{R}^p \): \( B \)-spline basis functions
- \( \max\{p, m\} \ll n \) while \( p, m \to \infty \) is allowed
- refer to Chao et al. (2016) for more assumptions
Model Specification

- Linear sparse factor structure:

\[ Y_{ij} = \sum_{k=1}^{r} \psi_{j,k} f_k(X_i) + u_{ij}, \quad (1) \]

where \( f_k(X_i) \) is the \( k \)th factor, \( r \) is the number of factors, \( \psi_{j,k} \) are the factor loadings.

- Factors are constructed by linear combination of \( X_i \):

\[ f_k(X_i) = \varphi_k^\top X_i \quad (2) \]
Model Specification

- Substituting (2) into (1):
  \[ Y_{ij} = \gamma_j^\top X_i + u_{ij}, \]  
  where \( \gamma_j = (\sum_{k=1}^{r} \psi_{j,k} \varphi_{k,1}, \ldots, \sum_{k=1}^{r} \psi_{j,k} \varphi_{k,p})^\top \)
- To estimate the coefficient matrix \( \Gamma \), where \( \gamma_j \) is the \( j \)-th column of \( \Gamma \)
Estimation

Robust estimation on $\Gamma$ via median regression

$$\hat{\Gamma} = \arg \min_{\Gamma \in \mathbb{R}^{p \times m}} \left\{ (mn)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} |Y_{ij} - X_i^\top \Gamma_j| + \lambda \|\Gamma\|_* \right\}$$

- nuclear norm $\|\Gamma\|_* = \sum_{j=1}^{\min(p,m)} \sigma_j(\Gamma)$, given the singular values of $\Gamma$: $\sigma_1(\Gamma) \geq \sigma_2(\Gamma) \geq \ldots \geq \sigma_{\min(p,m)}(\Gamma)$,
- # of nonzero singular values of $\Gamma$ is # of factors: $r$
- smooth fast iterative shrinkage thresholding algorithm
- singular value decomposition on $\Gamma$
Data

- daily yield spread in Chinese bond market
- 180 spread curves
  - maturities of 1, 2, ..., 10 years
  - enterprise bonds (9 credit ratings), chengtou bonds (5 credit ratings), company bonds (4 credit ratings)
- 733 observations from 2014.01 to 2016.12
- obtained from Wind Datafeed Service (WDS)
Factor Analysis

Figure 2: Plot of the first four factors (92.27% of the variance is explained).
Factor Loadings

Figure 3: Factor loadings for enterprise bonds of five credit ratings.
Three Factors by DNS (Treasury Bond)

Figure 4: Plot of the three factors by DNS for treasury bond.

High-Dimensional Yield Curves Modelling
Factor Analysis

Factors interpretation:
- 1st: systemic liquidity or dispersion measure among curves - 53.49%
- 2nd: level (credit risk) - 18.95%
- 3rd: slope - 14.42%
- 4th: curvature - 5.41%
Alternative Approaches

- Three factors DNS

\[ Y_{i\tau} = f_{1i} + \left\{ \frac{1 - \exp(-\lambda \tau)}{\lambda \tau} \right\} f_{2i} + \left\{ \frac{1 - \exp(-\lambda \tau)}{\lambda \tau} - \exp(-\lambda \tau) \right\} f_{3i} + u_{i\tau}, \]

where \( \tau \) denotes the maturities (for a particular bond type and credit rating).

- PCA

\[ Y_{ij} = \sum_{k=1}^{r} \psi_{kj} f_{ki} + u_{ij}, \]

where \( f_{ki}^\top = Y \gamma_k \), \( \gamma_k \) is the \( k \)-th eigenvector of \( \text{Var}(Y) \). VAR is applied to model the dynamics in factors.
Fitting Performance

Figure 5: Fitted curves by MFMR, DNS, PCA, with real observations.

High-Dimensional Yield Curves Modelling
## Fitting Performance - Whole Sample

<table>
<thead>
<tr>
<th></th>
<th>MFMR</th>
<th>DNS</th>
<th>PCA</th>
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<tr>
<td><strong>Enterprise Bonds</strong></td>
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<tr>
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<td>1.92</td>
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<td>7.95</td>
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Table 1: Fitting RMSE with the whole sample under different approaches, averaged over maturities. All numbers are of order \(10^{-2}\).
## In-Sample Fitting - Rolling Windows

<table>
<thead>
<tr>
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<th>PCA</th>
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Table 2: In-Sample RMSE with rolling windows (fixed width = 300), averaged over maturities. All numbers are of order $10^{-2}$. 
Out-of-Sample Forecasting - Rolling Windows

<table>
<thead>
<tr>
<th>Bond Type</th>
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<th>MFMR</th>
<th>DNS</th>
<th>PCA</th>
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</tbody>
</table>

Table 3: Out-of-Sample RMSE with rolling windows (fixed width = 300, one step ahead), averaged over maturities. All numbers are of order $10^{-2}$. 

High-Dimensional Yield Curves Modelling
Concluding Remarks

- jointly modelling high-dimensional spread curves
- multivariate factorisable regression with high-dimensional functional data
- latent risky factors - systemic liquidity and dispersion measure
- robust forecasting outperforms DNS
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