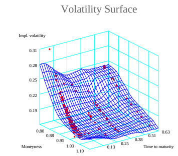
Common Functional IV Surface Analysis

Michal Benko
Wolfgang Härdle

Center for Applied Statistics and Economics
benko@wiwi.hu-berlin.de
Humboldt-Universität zu Berlin



Implied Volatility Surface

Implied Volatility (for European Options):

$$\widetilde{C}_t = C_t^{BS}(S_t, K, \tau, r, \sigma)$$

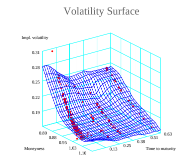
$\widetilde{C}_t$, observed market price, C_t^{BS} Black-Scholes Price

$\sigma_i(\kappa, \tau)$ is BS-IV for given

K Strike, standardization by $\kappa = K/F_\tau$ moneyness

τ Time to Maturity, i day-index

high-dimensional-problem, high data intensity



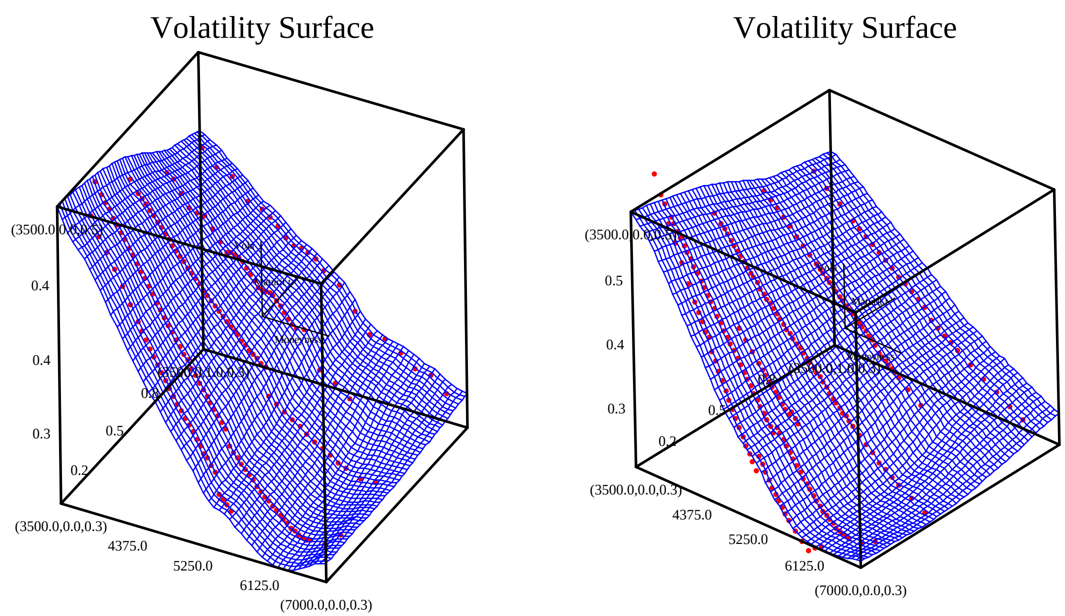
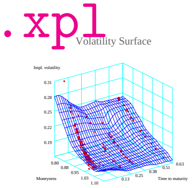


Figure 1: *Implied Volatility Surfaces:* $t_1 = 990104$ *and* $t_2 = 990201$*, ODAX*

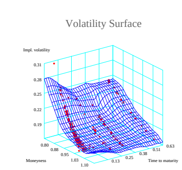
XFGiv02.xpl

<film>



Dimension Reduction Approaches

1. Parametric models
2. Factor models for 3D-surface
3. **Common Factor Model for Fixed Maturity Grid**
   Fengler, M., Härdle, W.,& Villa, P., (2003):

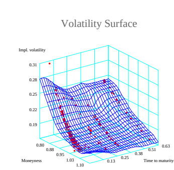


CPC model for IVS dynamics

+ CPC for IV yields the desired dimension reduction
+ trading signals and strategies can be obtained from CPC model

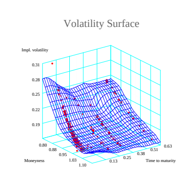
- the problem thought is of functional nature
- eigenvectors are rough

? it is possible to combine CPC & FDA



Outline

✓ 1. Motivation
2. Data Description, IV Strings
3. Functional Principal Components Analysis
4. Smooth Functional Principal Components Analysis
5. Smooth Common Principal Components
6. Conclusions, Outlook



Data Description

MD*Base, CASE HU-Berlin

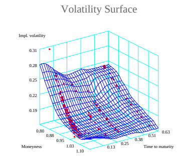
EUREX intra day ODAX, 20010201-0629, 0.25 millions contracts
(Database today: 950201 - 030225, 6.1 millions contracts)

IVs calculated by the Newton-Raphson method

r approximated by linear interpolation of EURIBOR

Transactions with $\tau < 7$D excluded

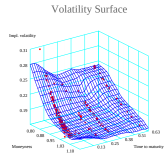
Hafner-Wallmeier correction used



IV Strings

Goal: is estimate IV functions $\sigma_i(\kappa, \tau)$
for time constant grid $\tau = 1$M, $\tau = 2$M for $\forall i$

Problem: the set of time to maturities $\{\tau_{1,i}, \ldots \tau_{d_i,i}\}$
for day i varies due to the market conventions across time.



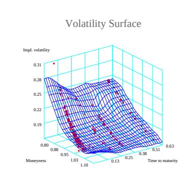
Estimation of IV Strings

$\widehat{\sigma}_i(\kappa, \tau_l),\ \kappa \in [0.9, 1.1]$ estimated using 9 Fourier basis functions for day specific τ grid., $l = 1, 2, \dots$

IV function $\widehat{\sigma}_i(\kappa, \tau^*), \tau^* \in \{1\text{M}, 2\text{M}\}$ estimated using linear interpolation of nearest neighbors.

$$\hat{\sigma}_i(\kappa, \tau^*) = \hat{\sigma}_i(\kappa, \tau^*_{i-}) \left(1 - \frac{\tau^* - \tau^*_{i-}}{\tau^*_{i+} - \tau^*_{i-}}\right) + \hat{\sigma}_i(\kappa, \tau^*_{i+}) \left(\frac{\tau^* - \tau^*_{i-}}{\tau^*_{i+} - \tau^*_{i-}}\right)$$

for period 20010102-0629:
77 IV strings for $\tau = 1\text{M}$, and 66 IV string for $\tau = 2\text{M}$



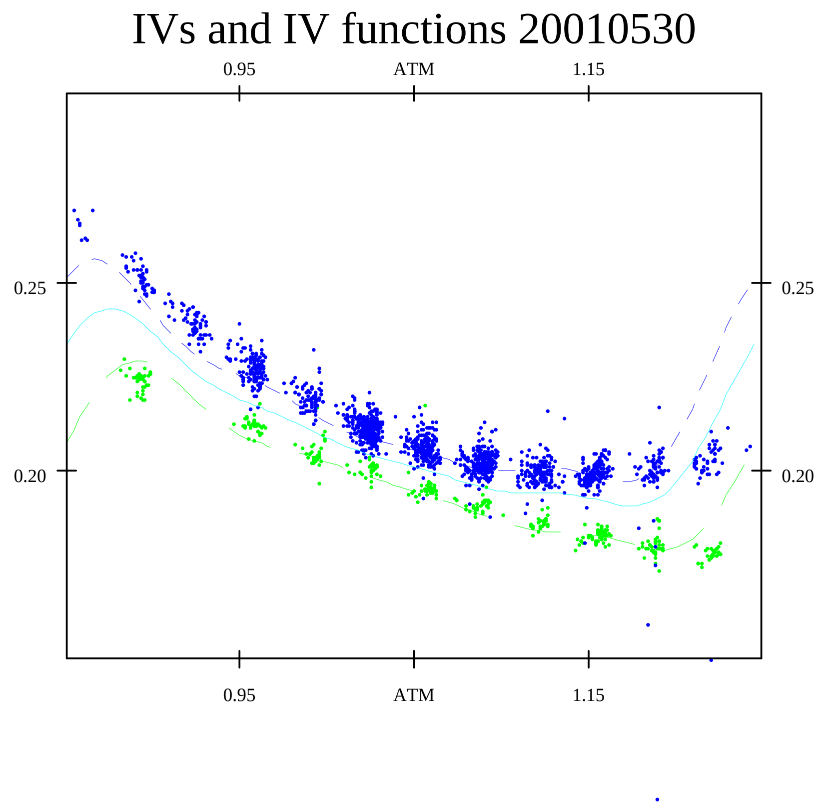
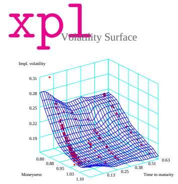


Figure 2: Example for linear interpolation of 1M (30D) IV stings, $i = 20010530$, $\tau_{i-} = 16D$, $\tau_{i+} = 50D$.

STFfda02.xpl



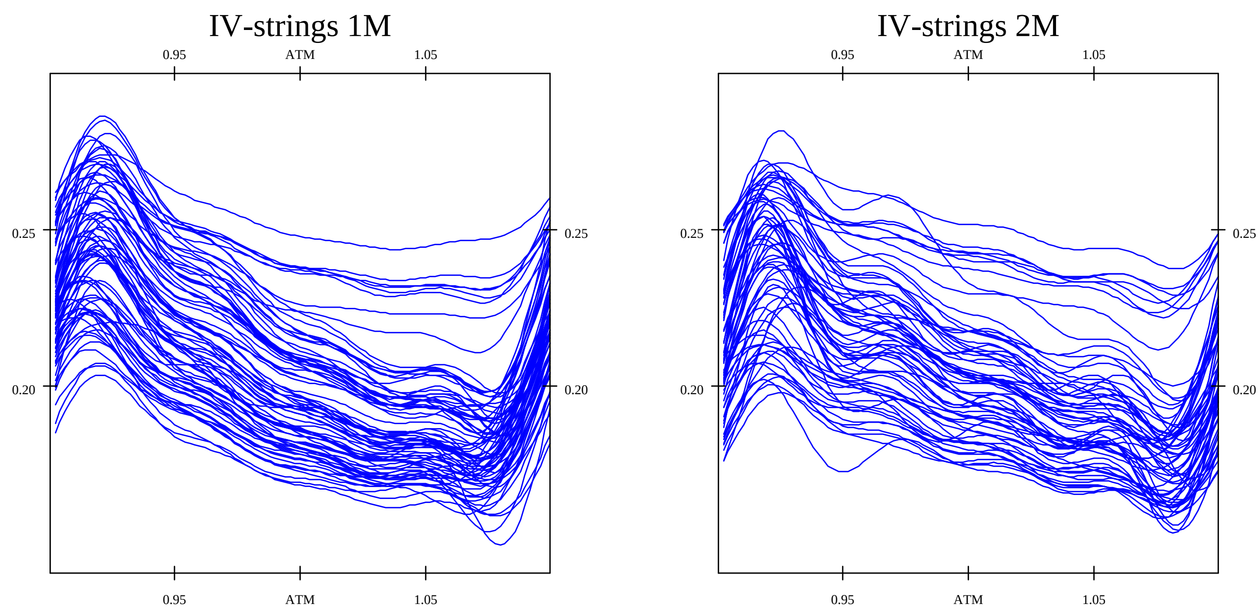
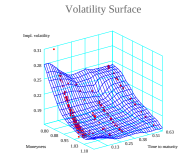


Figure 3: Functional observation using Fourier basis, $L = 9$.

STFfda03.xpl



Functional Principal Components Analysis

Analogue to multivariate PCA in functional space:

$$\underset{<\gamma_l,\gamma_k>=\delta_{lk},l\leq k}{\arg\max} \text{var}\langle \gamma_k, X \rangle \tag{1}$$

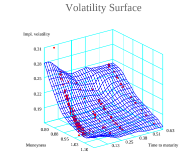
$X(t)$ - underlying stochastic process

f_k - principal components $f_k = <\gamma_k, X>$

$\gamma_k(t)$ - eigenfunctions

λ_k - eigenvalues

$$\langle u, v \rangle \stackrel{\text{def}}{=} \int u(t)v(t)dt$$



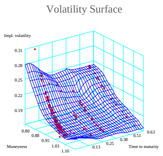
Eigenfunctions are estimated by solving the eigenequation:

$$\int \widehat{\mathsf{Cov}}(s,t)\gamma(t)dt = \lambda\gamma(s) \tag{2}$$

functional basis expansion:

$$\mathsf{Cov}(\mathbf{C})\mathbf{W}\mathbf{b} = \lambda\mathbf{b}.$$

where $\widehat{\mathsf{Cov}}(s,t)$ is empirical covariance function, $\sigma_i = \mathbf{c}_i^\top\boldsymbol{\Theta}$
$\gamma = \mathbf{b}^\top\boldsymbol{\Theta}$, $\boldsymbol{\Theta}$ vector of functional basis, $\mathbf{W}_{jk} = \langle\theta_j,\theta_k\rangle$



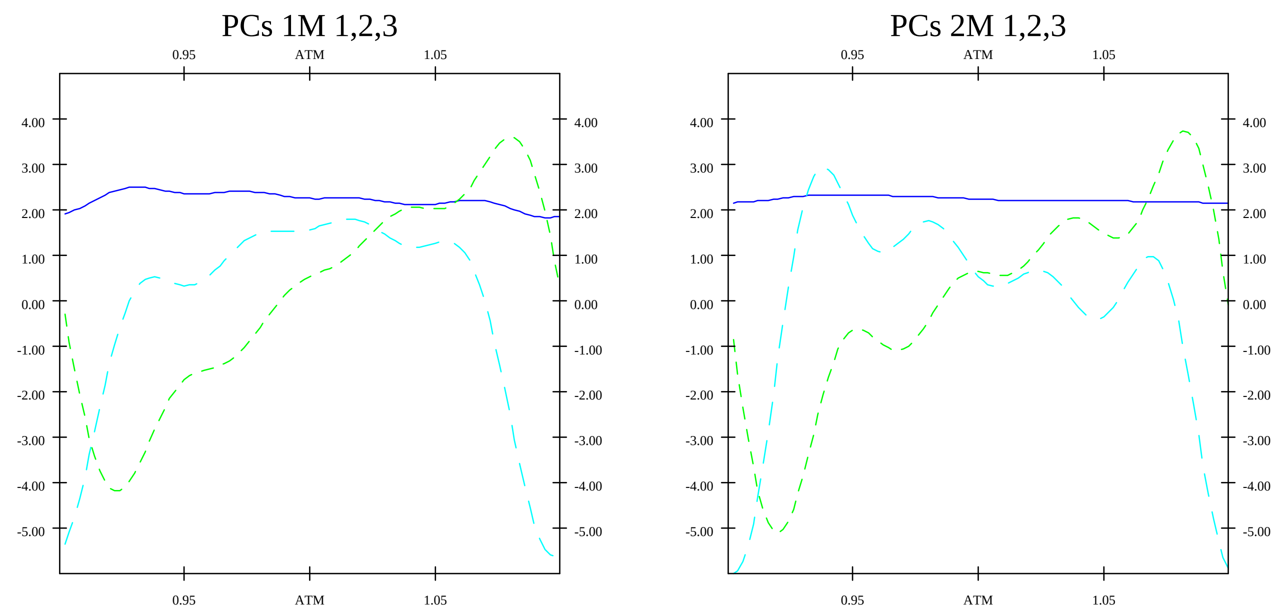
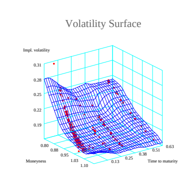


Figure 4: Weight functions FDA 1M 2M maturity, blue solid line is 1th PC function, green finely dashed 2nd PC function, cyan dashed 3th PC function

STFfda04.xpl

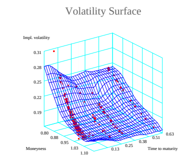


Smooth Principal Components

Problem: Eigenfunctions are rough

Idea: smooth the Eigenfunctions

Solution: Roughness penalty, Smoothed FPCA



Penalize the variance criterion for rough eigenfunction.

$$PCAPV = \frac{\int\int \gamma(s)\widehat{\mathsf{Cov}}(s,t)\gamma(t)dsdt}{\int \gamma(t)(\mathsf{I}+\alpha D^4)\gamma(t)dt}, \tag{3}$$

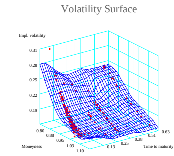
The solution is given by generalized eigenequation:

$$\int \widehat{\mathsf{Cov}}(s,t)\gamma(t)dt = \lambda(I+\alpha D^4)\gamma(s). \tag{4}$$

Using functional basis expansion we get:

$$\mathbf{W}\,\mathsf{Cov}(\mathbf{C})\mathbf{W}\mathbf{u} = \lambda(\mathbf{W}+\alpha\mathbf{K})\mathbf{u}. \tag{5}$$

$\mathbf{K}_{jk} = \langle D^2\theta_j, D^2\theta_k\rangle$, $\mathbf{K}_{jk} = \langle \theta_j, \theta_k\rangle$, D - diff. operator



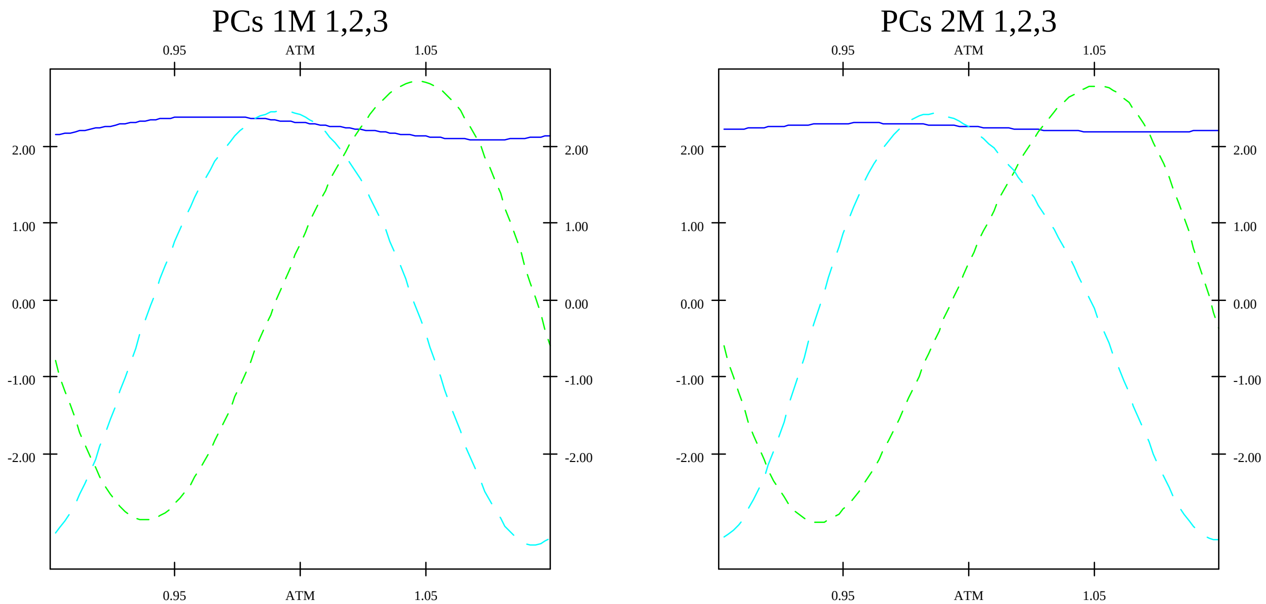
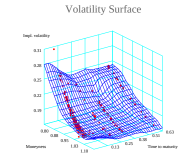


Figure 5: Smoothed weight functions, $\alpha = 10^{-7}$ blue solid line is 1th PC function, green finely dashed 2nd PC function, cyan dashed 3th PC function

STFfda05.xpl



Common Principal Components

Obtained **eigenfunctions for 1M, 2M groups are similar**

Goal: estimate eigenfunctions from both groups jointly

Idea: Use the **Common Principal Components** technique known in multivariate PCA. Flury, B., (1988)

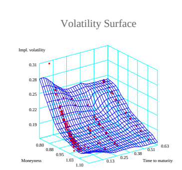
Common Principal Components - Motivation

k-sample problem in the multivariate framework:
k random vectors: $X_{(1)}, X_{(2)}, \ldots, X_{(k)} \in \mathbb{R}^p$.

$$\text{Model: } \mathbf{\Psi}_j = \mathsf{Cov}(X_{(j)}) = \Gamma \Lambda_j \Gamma^\top$$

i.e. eigenvectors same across samples, eigenvalues (variances) differs.

Let us denote sample covariance by $\mathbf{S}_j$.



Estimation

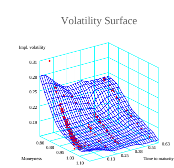
Motivated by ML Estimation, iterative FG-Algorithm maximizes:

$$\mathbf{\Phi}(\mathbf{\Gamma}) = \prod_{i=1}^{k} \left[\frac{\text{det diag}(\mathbf{\Gamma}^\top \mathbf{S_i} \mathbf{\Gamma})}{\text{det}(\mathbf{\Gamma}^\top \mathbf{S_i} \mathbf{\Gamma})} \right] \tag{6}$$

functional PCA are implemented through Spectral Analysis of Matrices

Multivariate CPC criterion (6) - FG algorithm may be employed

Remark: In the view of minimization property of FG algorithm this is reasonable approach, however obtained estimator may not be a ML estimator.



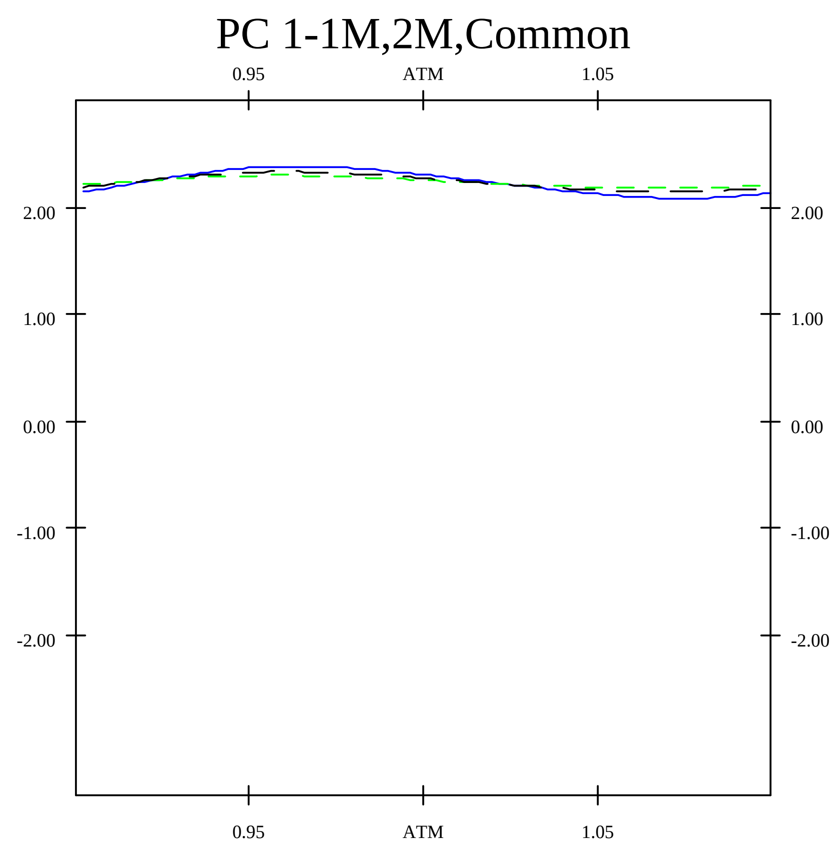
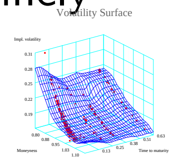


Figure 6: First eigenfunctions, $\alpha = 10^{-7}$, solid blue – 1M maturity group, finely dashed green 2M maturity group, dashed black – common eigenfunction.



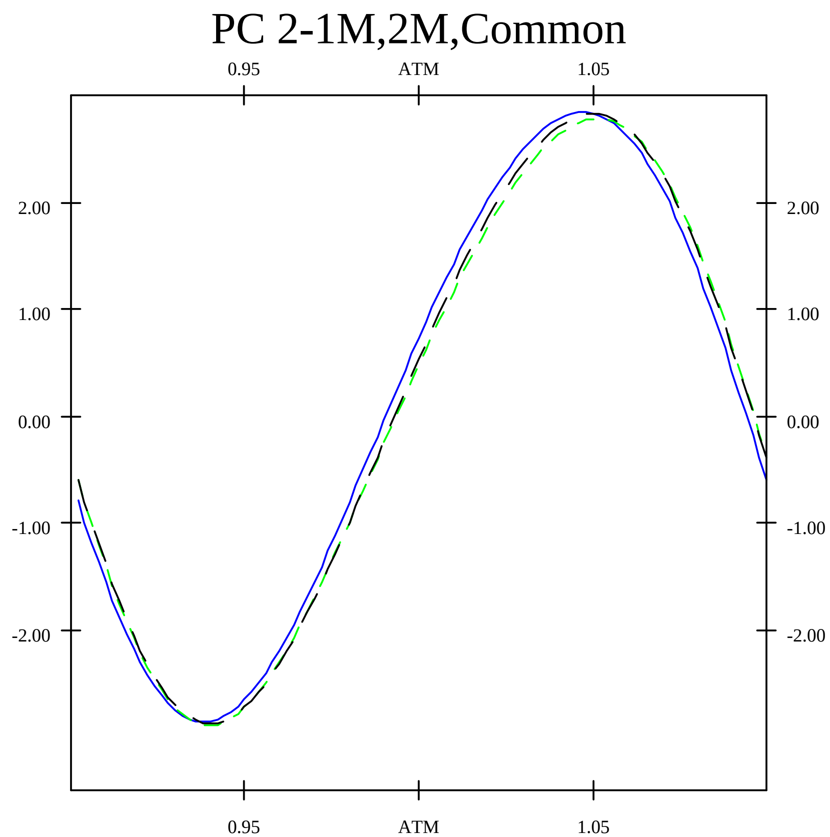
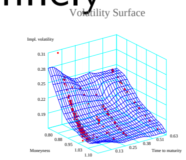


Figure 7: Second eigenfunctions, $\alpha = 10^{-7}$ solid blue – 1M maturity group, finely dashed green 2M maturity group, dashed black – common eigenfunction.



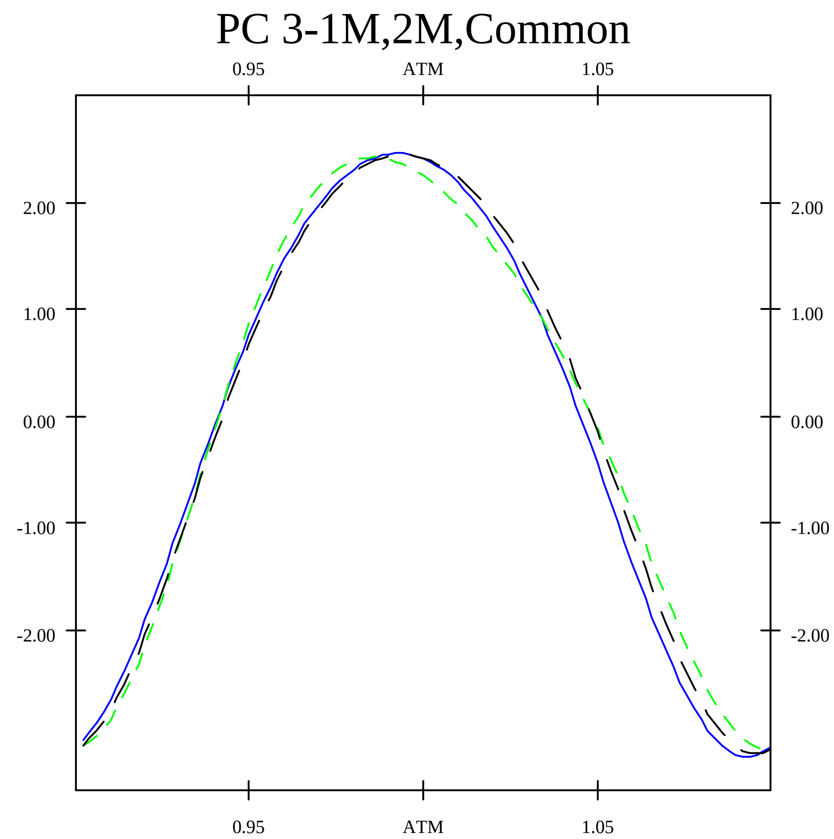
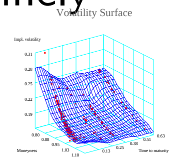


Figure 8: Third eigenfunctions, $\alpha = 10^{-7}$ solid blue – 1M maturity group, finely dashed green 2M maturity group, dashed black – common eigenfunction.

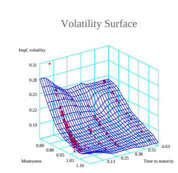


Conclusion

Eigenfunctions from ordinary FPCA are too rough

Smoothed eigenfunctions have level, slope, twist shape

Eigenfunctions for 1M, 2M group are similar
CPC framework is used to estimation



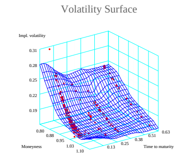
Outlook

estimation of SCPC using simultaneous diagonalization ✓

Other functional Basis (B-Spline)

Alternative FPCA Estimation

Testing the Functional CPC or SPC hypothesis



Other functional Basis

B-Spline functional Basis for IVs

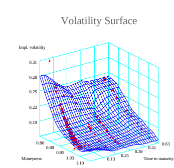
for some datums strong boundary problems

Solution: estimation of basis coefficients using penalized least square regression with roughness penalty $R(\bullet) = ||L \bullet ||$, L linear differential operator

Questions:

Choice of the "best" L, that reflects the financial theory

smoothing functional data contra smoothing eigenfunctions



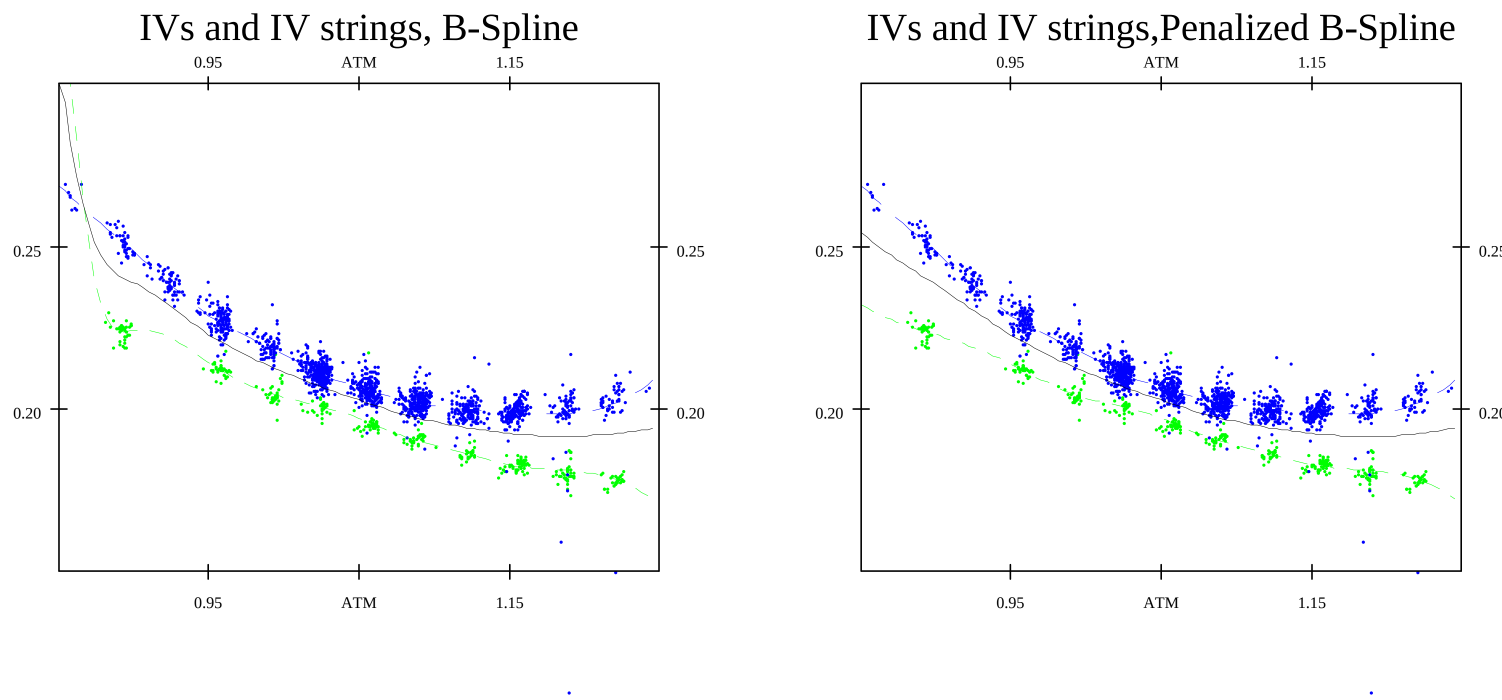
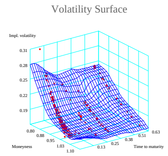


Figure 9: B-Spline basis example,left cubic B-splines, knots: $0.9 + 0.025k, k = 1, ..., 9$, right coefficient estimated by penalized least squares with rougness penalty $R = ||L||, L = D^3$. Day=20010530, as in figure 2.



Alternative FPCA Estimation

Focus on matrix $\mathbf{M}, \mathbf{M}_{l,k} = \langle \sigma_l - \overline{\sigma}, \sigma_k - \overline{\sigma} \rangle$, $l, k = 1, \ldots, n$

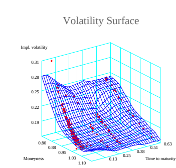
Nonzero eigenvalues of $\mathbf{M}$ and sample covariance operator in 2 are same, eigenfunctions can be recalculated from eigenvectors of $\mathbf{M}$ and σ_i.

Questions:

Appropriate estimator $\hat{\mathbf{M}}$ for noisy data with irregular designs

Testing the FCPC / SCPC

see Kneip, A.,& Utikal, K., (2001)



References

Benko, M.,& Härdle, W., (2005). *Common functional IV analysis*, in Cizek, P., Härdle W., Weron, R., (Eds.): *Statistical Tools for Finance and Insurance*, Springer, Berlin

Fengler, M., Härdle, W., & Villa, C., (2003). *The Dynamics of Implied Volatilities: A CPC Approach*, Review of Derivative Research 6, pp. 179-202.

Flury, B., (1988). *Common Principal Components and Related Models*, Wiley, New York

Kneip, A.,& Utikal, K., (2001). *Inference for Density Families Using Functional Principal Components Analysis*, Journal of the American Statistical Association, 96,p. 519-531

Ramsay, J.,& Silverman, B., (1997). *Functional Data Analysis*, Springer, New York