Variance Swap Dynamics

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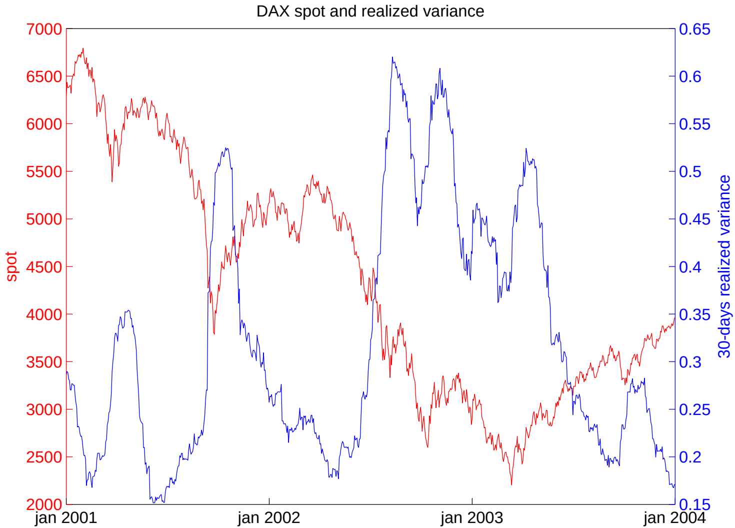

DAX spot and realized variance
spot
30-days realized variance
7000
6500
6000
5500
5000
4500
4000
3500
3000
2500
2000
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
jan 2001
jan 2002
jan 2003
jan 2004

Volatility

Investors on equity markets "trade volatility" for

- speculation (trade implied volatility)
- hedging (vega hedging)

To these ends, European options are often used.

But volatility can also be traded directly...

Realized variance

The realized variance of a stock price process (S_t) over the business days $0 = t_0 < t_1 < \ldots < t_n = T$ is defined as

$$\frac{252}{n} \sum_{i=1}^{n} (\log S_{t_i} - \log S_{t_{i-1}})^2$$

The markets of variance swaps have become very liquid.

Modern option pricing approaches derive models for the stock from models for variance swaps.

Option pricing models

Option pricing models are judged by their fit to observed price surfaces of European calls and puts.

Modern options like e.g. cliquets

$$f \vee \left(\sum_{i=1}^{n} f_i \vee \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right) \wedge c_i\right) \wedge c$$

depend on the **dynamics** of the price surfaces.

Hence, forecasting is an essential model criterion today.

Yield curve forecasting

Thus, we analyze the forecasting of variance swap curves that serve as building blocks for modern option pricing models.

Diebold and Li (2006) obtained good forecasting results in the Nelson-Siegel framework for yield curves.

We analyze similar models motivated by the popular stochastic volatility model of Heston and compare our results.

Outline

1. introduction ✓
2. modeling variance swap curves
3. forecasting variance swap curves
4. conclusion, outlook

Variance swap curves

Models of variance swaps are based on the approximation of the quadratic variation:

$$\sum_{i=1}^{n} (\log S_{t_i} - \log S_{t_{i-1}})^2 \approx < \log S >_T$$

for business days $0 = t_0 < t_1 < \ldots < t_n = T$.

Denoting this quadratic variation by $V(T) \stackrel{\text{def}}{=} < \log S >_T$ we identify the variance swap prices as $V(T)/T$.

Curves

curve	definition
variance swap curve	$T \to V(T)/T$
variance swap curve in “volatility strikes”	$T \to \sqrt{V(T)/T}$
variance curve	$T \to V(T)$
forward variance curve	$T \to v(T) := V'(T)$

As all these curves change daily they should have a time subscript t.

Constructing variance curves

In order to transform the discrete observations into curves we apply on each day a local quadratic regression to the variance prices.

Observing the variance swap prices $V(x_i)/x_i$ for the maturities $x_1, \ldots, x_n$ on day t we construct the variance and forward variance curve by

$$\min_{\beta} \sum_{i=1}^{n} \{V(x_i) - \beta_0 - \beta_1(x_i - x) - \beta_2(x_i - x)^2\} K_h(x_i - x).$$

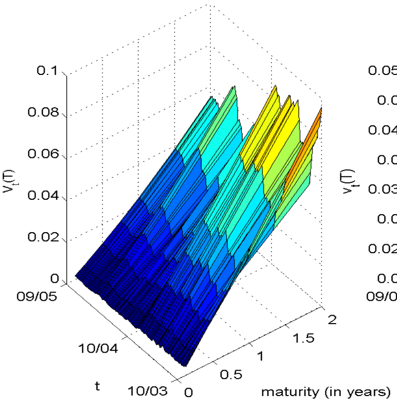
Then we have $V(x) = \hat{\beta}_0(x)$ and $v(x) = \hat{\beta}_1(x)$.

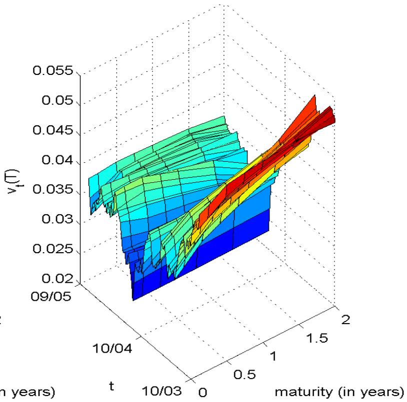
Data grid

Using this approach we construct variance curves at the maturities 1.5, 3, 6, 9, 12, 18 and 24 months on a weekly basis from

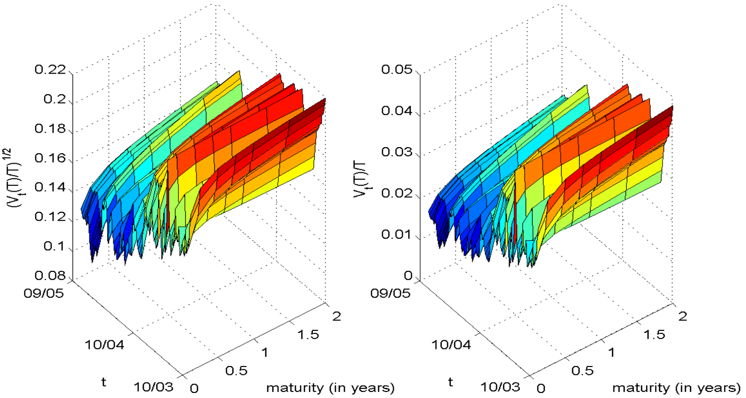
1. prices of variance swaps
2. on S & P 500 index
3. between 1 October 2003 and 30 September 2005

Data grid II

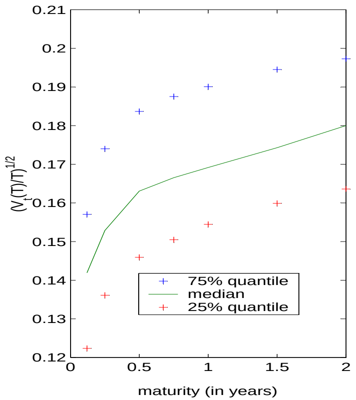

0.1
0.08
0.06
0.04
0.02
0
$v_t(T)$
09/05
10/04
10/03
t
0
0.5
1
1.5
2
maturity (in years)

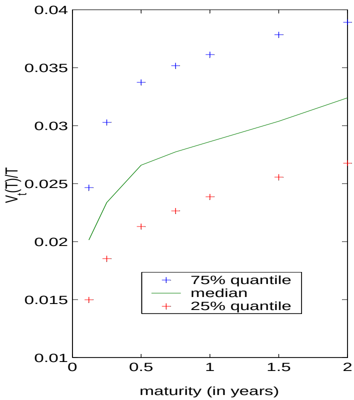

0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
$v_t(T)$
09/05
10/04
10/03
t
0
0.5
1
1.5
2
maturity (in years)

Data grid III



Data grid IV


0.21
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
$(V_t(T)/T)^{1/2}$
0
0.5
1
1.5
2
maturity (in years)
75% quantile
median
25% quantile


0.04
0.035
0.03
0.025
0.02
0.015
0.01
$V_t(T)/T$
0
0.5
1
1.5
2
maturity (in years)
75% quantile
median
25% quantile

Models

We consider

1. the Heston model
2. the Nelson-Siegel approach
3. a semiparametric factor model

Heston model

The Heston model is given by

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^1$$
$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^2$$

where W^1 and W^2 are Wiener processes.

Using the approximation by the quadratic variation the prices of variance swaps are given by

$$\theta + (\zeta_0 - \theta)\frac{1 - \exp(-\kappa T)}{\kappa T}.$$

Heston model II

Reparametrization leads to the forward variance curve model

$$v(T) = z_1 + z_2 \exp(-\kappa T)$$

that implies the variance swap prices

$$z_1 + z_2 \frac{1 - \exp(-\kappa T)}{\kappa T}. \tag{1}$$

Nelson-Siegel approach

The Nelson-Siegel parametrization

$$v(T) = z_1 + z_2 \exp(-\kappa T) + z_3 \kappa T \exp(-\kappa T)$$

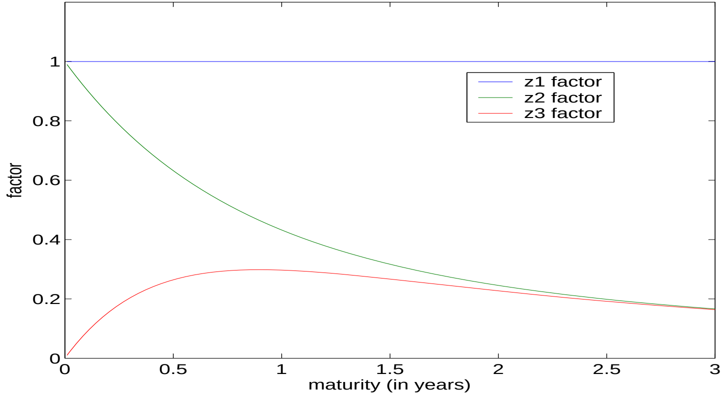
is a generalization of the above forward variance curve model.

The variance swap prices $V(T)/T$ are given in this model by

$$z_1 + z_2 \frac{1 - \exp(-\kappa T)}{\kappa T} + z_3 \{ \frac{1 - \exp(-\kappa T)}{\kappa T} - \exp(-\kappa T) \}. \quad (2)$$

because of $V(T) = \int_0^T v(t) dt$.

Factors


1
0.8
0.6
0.4
0.2
0
factor
0
0.5
1
1.5
2
2.5
3
maturity (in years)
z1 factor
z2 factor
z3 factor

Meaning of factors

1st factor: long term factor; level

2nd factor: short term factor; slope

3rd factor: medium term factor; curvature

Semiparametric factor model

Let $Y_{i,j}$ be an observed price of a variance swap on day i with maturity $T_j \in \{0.12, 0.25, 0.5, 0.75, 1.0, 1.5, 2.0\}$.

Let $X_{i,j}$ be a one-dimensional variable representing the time-to-maturity.

Then the model regresses $Y_{i,j}$ on $X_{i,j}$ by

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}),$$

Estimation of parametric models

Loadings z and parameter κ can be estimated by nonlinear LS.

problems:

1. numerically difficult
2. the factors change with changing κ

solution:

1. common approach in Nelson-Siegel in ir: fix κ
2. common approach in Heston in option pricing: fix κ

Hence, we use $\kappa = 2$ as in Bergomi (2004) and estimate the factor loadings z by OLS.

Estimation of semiparametric model

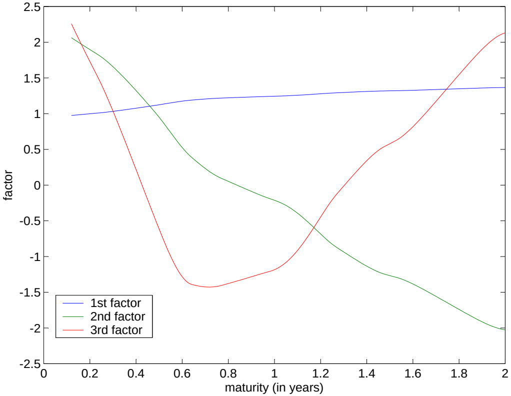
The factors $\widehat{m}_l$ and the loadings $\widehat{\beta}_{i,l}$ are estimated by minimizing:

$$\sum_{i=1}^{I}\sum_{j=1}^{J_i}\int\left\{Y_{i,j}-\sum_{l=0}^{L}\widehat{\beta}_{i,l}\widehat{m}_l(u)\right\}^2 K_h(u-X_{i,j})\,du$$

The minimization procedure searches through all functions $\widehat{m}_l:\mathbb{R}\longrightarrow\mathbb{R}$ and time series $\widehat{\beta}_{i,l}\in\mathbb{R}$ by an iterative procedure.

Afterwards the estimates are orthogonalized and normalized, see Fengler (2005) for details.

Factors of semiparametric model



In-sample fit

Maturity (Months)	Heston	Nelson-Siegel	semiparametric model
1.5	0.17	0.09	0.26
3	0.10	0.11	0.09
6	0.11	0.04	0.10
9	0.06	0.03	0.01
12	0.03	0.06	0.04
18	0.05	0.03	0.00
24	0.07	0.05	0.01

Table 1: MAE of variance swap curves residuals $[E^{-2}]$.

Residuals of in-sample fit

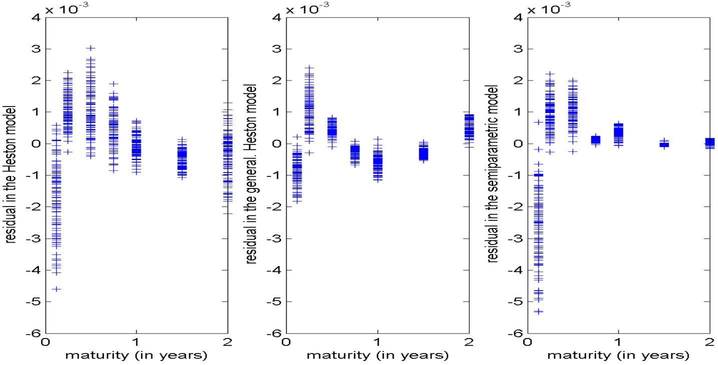


Figure 1: Variance swap curve residuals, 01/10/03 - 30/09/05. left: Heston, middle: generalized Heston, right: semiparametric model.

Factor loadings

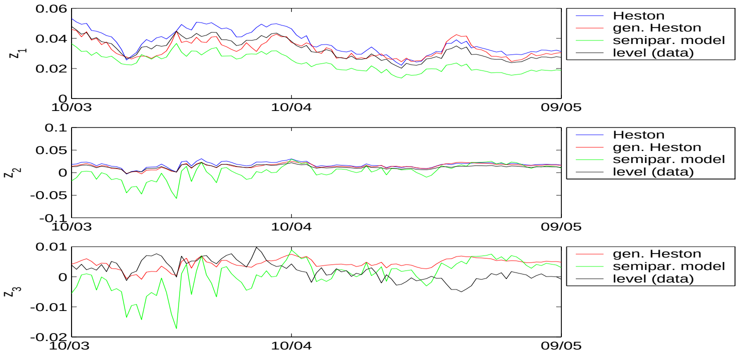


Figure 2: Factor loadings in the models and in the data.

Modeling the loadings series

Diebold and Li (2006) model loadings in the Nelson-Siegel framework for ir by AR(1).

Cont and da Fonseca (2002) model loadings in a principal components framework for ivs by AR(1).

For comparability we follow this accepted approach.

Because of low correlation between the loadings we do not consider VAR. Moreover, the results do not improve for ARIMA.

Autocorrelations

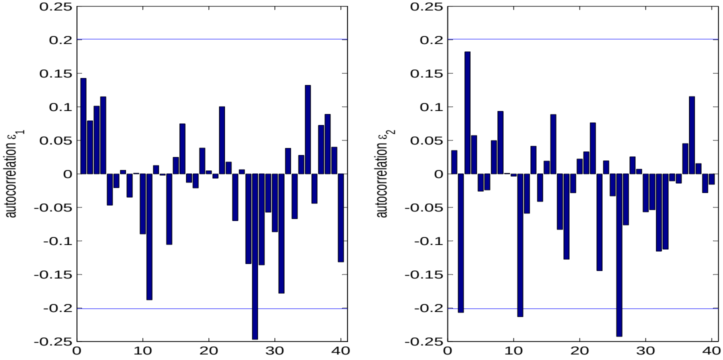


Figure 3: Factor loadings in the Heston model.

Forecasting the loadings series

The forecasts of variance swap curves τ weeks ahead are given by

$$\widehat{V_{t+\tau}(T)}/T = \hat{z}_{1,t/t+\tau} f_1(T) + \hat{z}_{2,t/t+\tau} f_2(T) + \hat{z}_{3,t/t+\tau} f_3(T)$$

where $\hat{z}_{i,t/t+\tau}$ are the forecasts of the i-th factor loading and f_1, f_2, f_3 are the factors.

These loading forecasts can be computed by repeated 1-day forecasts.

Benchmark models

1. The static Heston model
   Heston without forecasting the loadings

$$\widehat{V_{t+\tau}(T)}/T = \frac{V_t(T+\tau) - V_t(\tau)}{T}$$

   where V_t denotes the variance curve at time t.

2. The random walk
   no change forecast:

$$\widehat{V_{t+\tau}(T)}/T = V_t(T)/T$$

Forecasting errors

The forecast errors at time $t+\tau$ are defined as:

$$\widehat{V_{t+\tau}(T)}/T - V_{t+\tau}(T)/T$$

for $T = 1.5, 3, 6, 9, 12, 18$ or 24 months.

We consider the two error measures, the mean absolute error:

$$MAE \stackrel{\text{def}}{=} \frac{1}{n}\sum_t \|\widehat{V_{t+\tau}(T)}/T - V_{t+\tau}(T)/T\|$$

and the mean absolute relative error:

$$MARE \stackrel{\text{def}}{=} \frac{1}{n}\sum_t \|\frac{\widehat{V_{t+\tau}(T)}/T - V_{t+\tau}(T)/T}{V_{t+\tau}(T)/T}\|$$

Forecasting results: MAE, 1-month

Maturity	Heston	Nelson-Siegel	semipara.	static Heston	RW
1.5	0.39	0.45	0.54	0.42	0.32
3	0.31	0.37	0.34	0.30	0.27
6	0.32	0.46	0.35	0.29	0.28
9	0.33	0.53	0.39	0.31	0.29
12	0.35	0.55	0.38	0.33	0.30
18	0.37	0.51	0.38	0.35	0.30
24	0.36	0.44	0.37	0.34	0.32

Table 2: MAE out-of-sample 1-months-ahead forecasting results $[E^{-2}]$.

Forecasting results: MARE, 1-month

Maturity	Heston	Nelson-Siegel	semipara.	static Heston	RW
1.5	27.7	33.4	38.9	30.2	20.9
3	17.2	21.4	19.3	16.8	14.8
6	15.3	23.0	17.3	14.5	13.2
9	15.2	24.8	18.1	14.4	12.7
12	15.2	24.5	16.4	14.4	12.4
18	14.6	21.1	15.3	13.7	12.0
24	13.5	16.7	13.6	12.7	11.6

Table 3: MARE out-of-sample 1-months-ahead forecasting results $[E^{-2}]$.

Forecasting results: MAE, 6-month

Maturity	Heston	Nelson-Siegel	semipara.	static Heston	RW
1.5	0.54	0.82	0.97	1.16	0.33
3	0.44	0.79	0.74	0.90	0.32
6	0.46	0.92	0.77	0.76	0.36
9	0.50	0.98	0.84	0.69	0.38
12	0.53	0.99	0.79	0.64	0.40
18	0.53	0.87	0.74	0.57	0.44
24	0.50	0.71	0.64	0.54	0.48

Table 4: MAE out-of-sample 6-months-ahead forecasting results [E^{-2}].

Forecasting results: MARE, 6-month

Maturity	Heston	Nelson-Siegel	semipara.	static Heston	RW
1.5	39.8	62.3	71.8	84.4	24.0
3	26.0	46.1	43.2	52.5	18.9
6	24.6	46.9	39.7	39.5	18.9
9	24.9	47.0	40.5	34.0	18.9
12	24.8	44.8	36.1	30.3	18.8
18	23.1	36.9	31.3	25.1	19.1
24	20.6	28.6	26.0	22.2	19.2

Table 5: MARE out-of-sample 6-months-ahead forecasting results [E^{-2}].

Conclusion in-sample

- semiparametric better than Nelson-Siegel better than Heston
- but all models similar fits
- all models fit long maturities better
- fit for short maturities not satisfactorily
- residuals show structural problems for short maturities (also for semiparametric)
- Nelson-Siegel gives better fit to yield curves

Conclusion out-of-sample

- random walk better than all models
- confirms Duffie and Kan (1996) and disagrees with Diebold and Li (2006)
- two factor models better than three factor models
- for long forecasts ahead static Heston not good
- for long forecasts ahead random walk only a bit better than (dynamic) Heston

Outlook

- variance swap modeling and forecasting interesting because yield curve approach perform badly
- bad performance of Heston for short maturities well known
- other models (e.g. Bates (1996)) can be analyzed
- other mean reversion speeds (κ) can be considered
- other forecasting techniques can be applied

For Further Reading

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Forecasting the term structure of government bond yields.
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  Springer, Berlin, 2005.

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  A closed-form solution for options with stochastic volatility with applications to bond and currency options.
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For Further Reading

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Parsimonious modeling of the yield curve.
Journal of Business, 60, 1987.