Calibration Risk for Exotic Options in the Heston model

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Motivation

In March 1997 Bank of Tokyo suffered a $ 83 million loss in the derivative markets because of a ‚wrong‘pricing model. (Cont (2005))

In 1999 the derivative losses because of model risk summed up to $ 5 billion in the banking industry. (Williams (1999))

Model risk

The choice of a good pricing model is essential.

Schoutens et al. (2004) calibrate different stochastic volatility models (Heston, Bates, ...) and exponential Levy models (Variance Gamma, CGMY, ...) to an implied volatility surface and price exotic options (barriers, cliquets, ...). They find huge price differences among the models (up to 200%).

The risk of using a wrong model for pricing options is called **model risk**.

Calibration risk

After a model is chosen it has to be calibrated to observed data.

Normally (e.g. Schoutens (2004)) option prices are used as input in the calibration. But other choices like the implied volatility surface are intuitively preferred by traders.

We analyze how the prices of exotic options change with the calibration criterion. We observe significant prices changes and conclude that there is also **calibration risk**.

Aims

We analyze calibration risk in the popular option pricing model of Heston for exotic options.

In particular, we analyze the price differences with respect to

- option type
- time to maturity
- goodness of fit

Outline of the talk

1. motivation ✓
2. introduction
3. models and data
4. calibration
5. exotic options
6. conclusions
7. outlook

Model uncertainty

Different parametric forms for the process of the underlying lead to different prices of exotic options although the plain vanilla prices coincide:

option	lookback	barrier	cliquet
price range	15 %	200 %	40 %

Table 1: Price differences of exotic options for different models (Schoutens et al. (2004)).

Risk measures

Financial risks are quantified by risk measures.

Artzner et al. (1999) have introduced risk measures as monotone, translation invariant, subadditive, positive homogene functions.

Such a risk measure ρ represents the worst case expected payoff for a class $\mathcal{P}$ of probabilistic models:

$$\rho(X) = \sup_{P \in \mathcal{P}} \mathsf{E}_P(-X)$$

Model uncertainty and risk measures

Cont (2005) proposes a quantitative framework for measuring model risk that takes into account special features of model risk like bid-ask spreads or the existence of hedging instruments.

Cont's simplest measure for model risk is given by

$$\mu_{\mathcal{P}}(X) \stackrel{\text{def}}{=} \sup_{P \in \mathcal{P}} \mathsf{E}_P(X) - \inf_{P \in \mathcal{P}} \mathsf{E}_P(X)$$

Risk measures

As this risk measure is given in terms of probability distributions we can use it also for quantifying calibration risk.

Then the class of probabilistic models $\mathcal{P}$ will result from different calibration methods applied to one model.

Thus we identify the price range for several exotic options using a time series of implied volatility surfaces.

Calibration methods

Calibration of option pricing models can be done in two ways:

- by Monte Carlo Markov Chains (see e.g. Eraker (2001))
- by minimization of an error functional (see e.g. Bakshi et al. (1997))

We focus on the minimization of a quadratic error functional because it is most common in practice.

Calibration risk

Calibration risk arises from different specifications of the error functional.

We consider here as error measures differences of prices or implied volatilities in absolute or relative terms.

These measures have been applied before (see e.g. Schoutens et al. (2004), Mikhailov et al. (2003)) but never been analyzed together.

Options

We analyze calibration risk for the following options:

- up and out calls
- down and out puts
- cliquets

Moreover, we look at the relation between calibration risk and model risk (understood as the choice of a parametric model).

Models

We focus on the Heston model because of its popularity in practice.

In addition, we consider the Bates model, an extension of the Heston model. It gives a better fit to data and hence we can analyze the impact of goodness of fit for the same type of model.

Moreover, these two models allow us to consider model risk and analyze its relation to calibration risk.

The Heston model

The price process is given by

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^{(1)}$$

where the volatility process is modelled by a square-root process:

$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^{(2)},$$

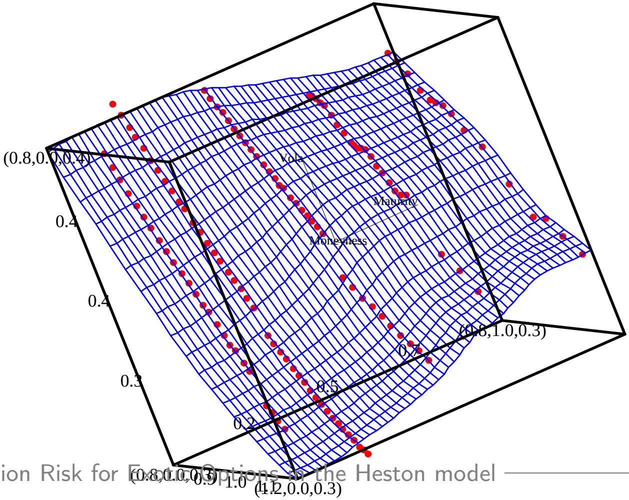
and W^1 and W^2 are Wiener processes with correlation ρ.

The Heston model

The mean-reversion speed ξ, the long vol η and the short vol V_0 control the term structure of the implied volatility surface (i.e. time to maturity direction).

The correlation ρ and the vol of vol θ control the smile/skew (i.e. moneyness direction).

Volatility surface



The Bates model

In this model, the price process is given by

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t} dW_t^{(1)} + dZ_t \\
dV_t &= \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^{(2)}
\end{aligned}
$$

where Z is a compound Poisson process.

This model extends the Heston model and has three more parameters.

The Bates model

The meaning of the parameters ξ, η, θ, ρ and V_0 is the same as in the Heston model.

The parameters of the compound Poisson process control the smile/skew especially for short times to maturity.

Data

- Eurex settlement volatilities of European options
- underlying : dax
- time period: March 2003 - April 2004
- risk free interest rate: Euribor
- no dividends because dax is performance index

Because of computation time we consider only one day each week. Hence, we consider 51 implied volatility surfaces.

Data

Arbitrage:
The implied volatility surfaces have been preprocessed in order to eliminate arbitrage.

Illiquidity:
Only options with moneyness $m \in [0.75, 1.35]$ for small times to maturity $T \leq 1$ have been considered because of illiquidity.

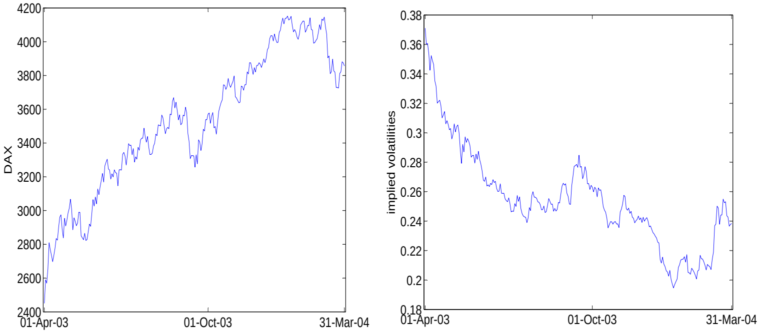


Figure 1: DAX and ATM implied volatility with 1 year to maturity on the trading days from 01 April 2003 to 31 March 2004.

	mean number of maturities	mean number of obervations	mean money-ness range
short maturities ($0.25 \le T < 1.0$)	3.06	64	0.553
long maturities ($1.0 \le T$)	5.98	76	0.699
total	9.04	140	0.649

Table 2: Description of the implied volatility surfaces.

Error functionals I

For the minimization we consider the four objective functions based on the root weighted square error:

$$ap \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i (P_i^{mod} - P_i^{mar})^2}$$

$$rp \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i \left(\frac{P_i^{mod} - P_i^{mar}}{P_i^{mar}}\right)^2}$$

Error functionals II

$$ai \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i (IV_i^{mod} - IV_i^{mar})^2}$$

$$ri \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i \left(\frac{IV_i^{mod} - IV_i^{mar}}{IV_i^{mar}}\right)^2}$$

where *mod* refers to a model quantity and *mar* to a quantity observed on the market, *P* to a price and *IV* to an implied volatility.

Data design

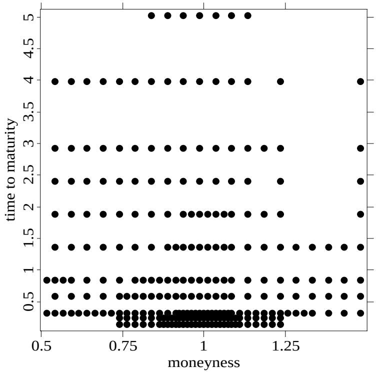


Figure 2: Grid of the DAX implied volatility surface on March 1st, 2004. (only for moneyness between 0.5 and 1.5)

Data design

The observations per day have a special design:

- data come in strings
- strings are not uniformly distributed in time-to-maturity
- strings have different moneyness ranges
- strings are moving in time
- strings disappear
- new strings appear

Data design

We use the RMSE to measure the difference between market and model.

In order to take into account the special data design we use weights such that

- each string get the same weight
- all observations in a string have the same weight

These weights imply an average time to maturity of 2.02. Hence, it is a good weighting to analyze options with 1,2 or 3 years to expiration.

Calibration method

The error functionals are minimized with respect to the model parameters by a global stochastic minimization routine.

Cont et al. (2004) have shown that these error functionals can have local minima. Hence it is essential to use a stochastic/global optimizer. We apply differential evolution (see Storn (1997)).

Other people use gradient descent methods. But we have found them to be significantly inferior to differential evolution.

Calibration method

The plain vanilla prices are calculated by a method of Carr et al. (1999):

$$C(K,T) = \frac{\exp\{-\alpha \ln(K)\}}{\pi} \int_0^{+\infty} \exp\{-\mathbf{i} v \ln(K)\} \psi_T(v) dv$$

for a damping factor $\alpha > 0$. The function ψ_T is given by

$$\psi_T(v) = \frac{\exp(-rT)\phi_T\{v - (\alpha+1)\mathbf{i}\}}{\alpha^2 + \alpha - v^2 + \mathbf{i}(2\alpha+1)v}$$

where ϕ_T is the characteristic function of $\log(S_T)$.

Calibration method

mean objective fct.	AP	RP $[E^{-2}]$	AI $[E^{-2}]$	RI $[E^{-2}]$
AP	7.3	9.7	0.81	3.1
RP	11.	6.1	0.74	2.9
AI	9.4	7.3	0.68	2.6
RI	8.8	7.0	0.70	2.5

Table 3: Calibration errors in the Heston model for 51 days.

Calibration method

mean / objective fct.	AP	RP $[E^{-2}]$	AI $[E^{-2}]$	RI $[E^{-2}]$
AP	7.0	13.	0.76	2.8
RP	12.	5.1	0.67	2.6
AI	8.9	6.4	0.60	2.3
RI	8.7	6.2	0.62	2.2

Table 4: Calibration errors in the Bates model for 51 days.

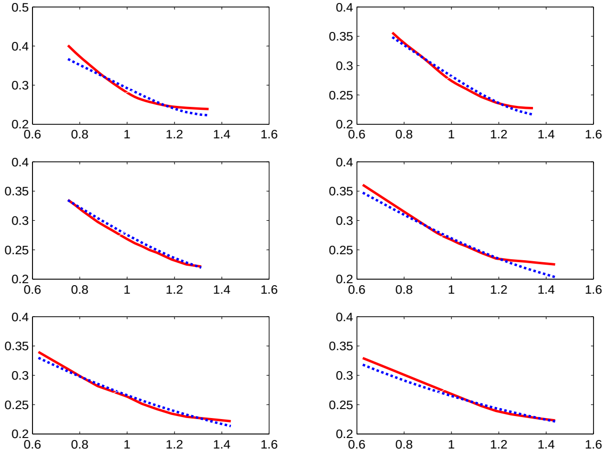


Figure 3: IVS on 25/06/03 for AI in Heston (maturities: 0.26, 0.52, 0.78, 1.04, 1.56, 2.08). (market: blue, model: red; X: moneyness, Y:iv)

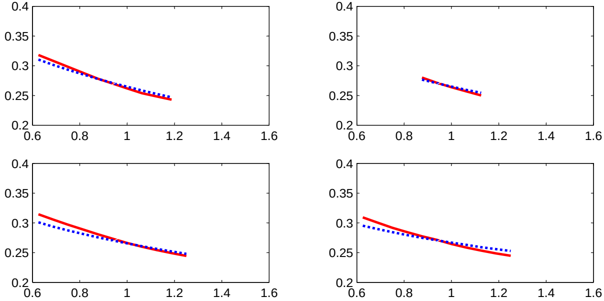


Figure 4: IVS on 25/06/03 for AI in Heston (maturities: 2.60, 3.12, 3.64, 4.70). (market: blue, model: red; X: moneyness, Y:iv)

	ξ	η	θ	ρ	V_0
AP	0.87	0.07	0.34	-0.82	0.07
	(0.48)	(0.02)	(0.08)	(0.08)	(0.02)
RP	1.38	0.07	0.44	-0.74	0.08
	(0.35)	(0.02)	(0.06)	(0.03)	(0.02)
AI	1.32	0.07	0.43	-0.77	0.08
	(0.40)	(0.02)	(0.06)	(0.04)	(0.02)
RI	1.20	0.07	0.41	-0.75	0.08
	(0.35)	(0.02)	(0.06)	(0.05)	(0.02)

Table 5: Mean parameters (std.) in the Heston model for 51 days.

	ξ	η	θ	ρ	V_0	λ	$\bar{k}$	δ
AP	0.92	0.07	0.33	-0.94	0.07	0.33	0.07	0.08
	(0.50)	(0.02)	(0.08)	(0.07)	(0.02)	(0.21)	(0.03)	(0.06)
RP	1.56	0.07	0.45	-0.89	0.08	0.54	0.05	0.08
	(0.47)	(0.02)	(0.07)	(0.07)	(0.02)	(0.23)	(0.03)	(0.06)
AI	1.43	0.07	0.43	-0.95	0.07	0.50	0.06	0.09
	(0.44)	(0.02)	(0.06)	(0.06)	(0.02)	(0.22)	(0.03)	(0.04)
RI	1.36	0.07	0.41	-0.93	0.07	0.52	0.05	0.08
	(0.44)	(0.02)	(0.07)	(0.09)	(0.02)	(0.26)	(0.04)	(0.08)

Table 6: Mean parameters (std.) in the Bates model for 51 days.

Monte Carlo simulation

We compue the prices of the options by Euler discretization using 1000000 paths.

	Heston			Bates		
	$T = 1$	$T = 2$	$T = 3$	$T = 1$	$T = 2$	$T = 3$
up and out calls	0.17	0.10	0.08	0.17	0.11	0.09
down and out puts	0.18	0.11	0.08	0.19	0.12	0.10
cliquet options	0.06	0.05	0.05	0.07	0.06	0.05

Table 7: Maximal relative standard error of Monte Carlo simulations. $[E^{-2}]$

Monte Carlo simulation

The payoffs of the barrier options are based on a maximum/minimum in continuous time.

In the simulations these extrema are replaced by maxima/minima over 250 days a year.

Although there is some discretization bias it can regarded as payoff of an approximate instrument.

Barrier options

The prices of up and out calls are given by

$$\exp(-rT)\,\mathsf{E}[(S_T - K)^+ \mathbf{1}_{\{M_T < B\}}]$$

where

$$M_T \stackrel{\text{def}}{=} \max_{0 \le t \le T} S_t .$$

Barrier options II

We consider for the barrier B and the strike K

$$B = 1 + T * 0.2$$
$$K = 1 - T * 0.1$$

where $T = 1, 2, 3$ denotes time to maturity.

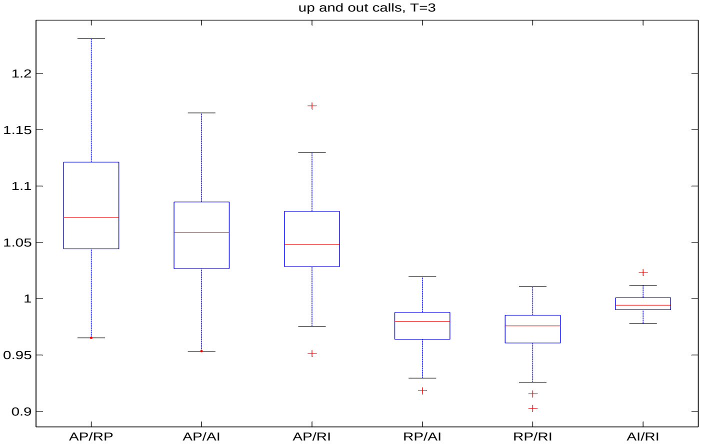


Figure 5: Relative prices of the up and out calls in the Heston model for 3 years to maturity.

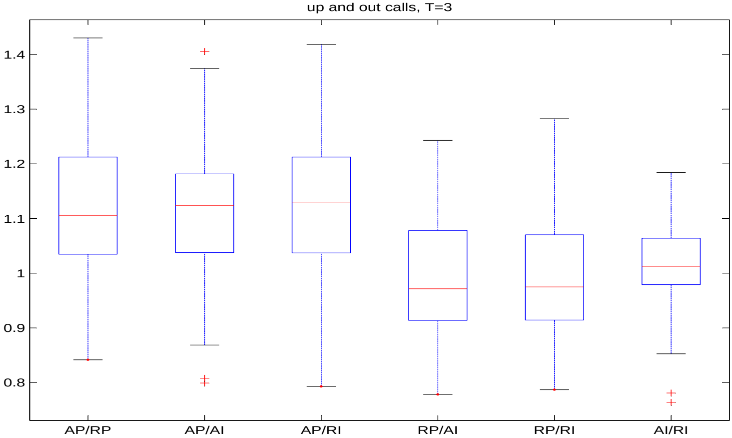


Figure 6: Relative prices of the up and out calls in the Bates model for 3 years to maturity.

		AP/RP	AP/AI	AP/RI	RP/AI	RP/RI	AI/RI
Heston	$T = 1$	0.986	0.968	0.967	0.984	0.984	0.999
	$T = 2$	1.051	1.024	1.022	0.979	0.978	0.998
	$T = 3$	1.072	1.059	1.048	0.980	0.976	0.994
Bates	$T = 1$	0.988	0.985	1.002	1.002	1.006	1.012
	$T = 2$	1.070	1.083	1.104	0.970	0.986	1.018
	$T = 3$	1.106	1.123	1.129	0.972	0.975	1.013

Table 8: Median of price quotients of up and out calls.

Cliquet options

We consider cliquet options with prices

$$\exp(-rT)\,\mathsf{E}[H]$$

where the payoff H is given by

$$H \stackrel{\text{def}}{=} \min(c_g, \max(f_g, \sum_{i=1}^{N} \min(c_l^i, \max(f_l^i, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})))).$$

Here c_g (f_g) is a global cap (floor) and c_g^i (f_g^i) is a local cap (floor) for the period $[t_{i-1}, t_i]$.

Here we consider three periods with $t_i = i\frac{T}{3}$ $(i = 0, \ldots, 3)$ and the caps and floors are given by

$$
\begin{aligned}
c_g &= \infty \\
f_g &= 0 \\
c_l^i &= 0.08,\ \ i = 1, 2, 3 \\
f_l^i &= -0.08,\ \ i = 1, 2, 3
\end{aligned}
$$

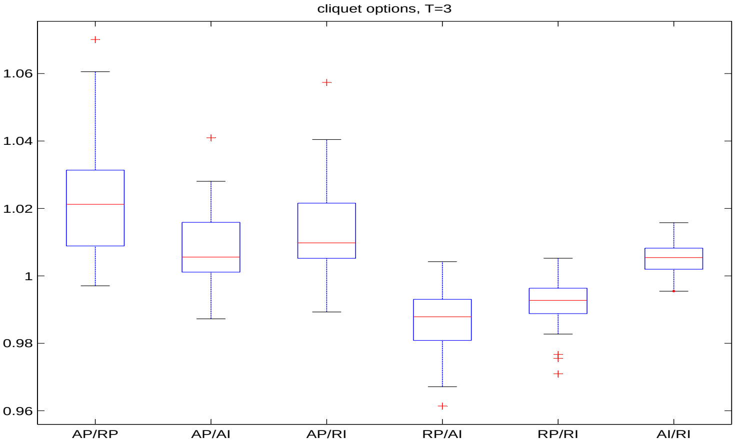


Figure 7: Relative prices of the cliquet options in the Heston model for 3 years to maturity.

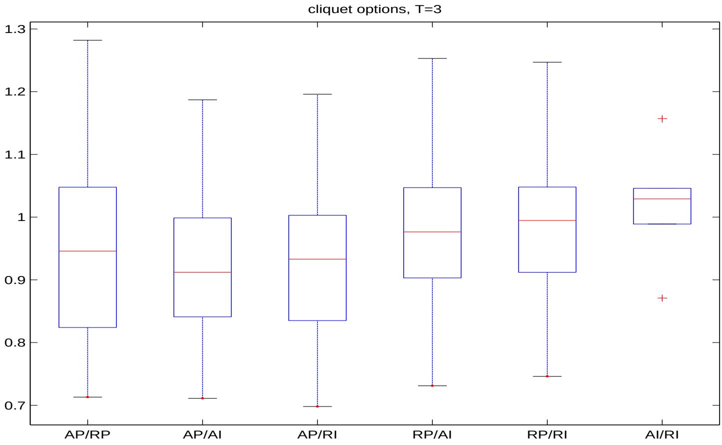


Figure 8: Relative prices of the cliquet options in the Bates model for 3 years to maturity.

		AP/RP	AP/AI	AP/RI	RP/AI	RP/RI	AI/RI
Heston	$T = 1$	0.983	0.976	0.989	0.993	1.006	1.013
	$T = 2$	1.002	0.991	1.000	0.989	0.998	1.010
	$T = 3$	1.022	1.008	1.014	0.987	0.992	1.005
Bates	$T = 1$	0.917	0.899	0.917	0.987	1.005	1.024
	$T = 2$	0.931	0.903	0.923	0.980	0.999	1.029
	$T = 3$	0.946	0.912	0.933	0.976	0.995	1.029

Table 9: Median of price quotients of cliquet options.

Model and calibration risk

Calibration risk leads to significant price differences.

What is bigger/more important model or calibration risk?
Are these risks independent?

Model and calibration risk

risk	barrier	cliquet
model	200 %	40 %
calibration	13 %	6 %

Table 10: Maximal price differences of exotic options.

(Model risk from Schoutens et al (2004) for a wide range of models)

Model risk is bigger than calibration risk.

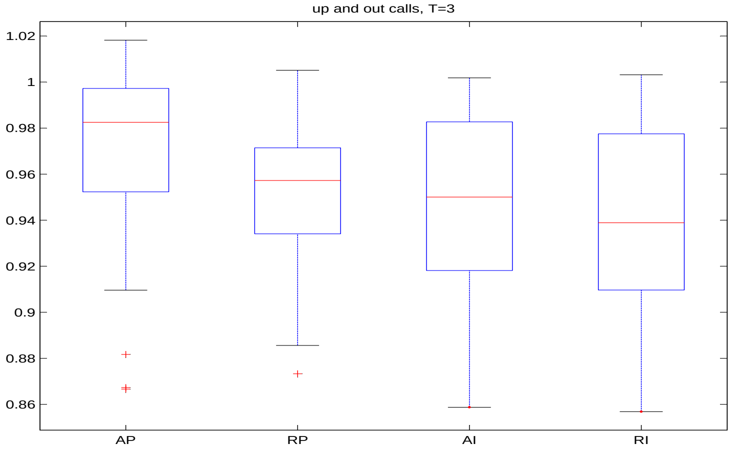


Figure 9: Bates prices over Heston prices for up and out calls with 3 years to maturity on 51 days.

		AP	RP	AI	RI
up and out calls	$T = 1$	0.973	0.953	0.944	0.941
	$T = 2$	0.980	0.954	0.953	0.940
	$T = 3$	0.983	0.957	0.950	0.939
down and out puts	$T = 1$	0.933	0.892	0.877	0.878
	$T = 2$	0.918	0.883	0.872	0.860
	$T = 3$	0.916	0.881	0.873	0.860
cliquets	$T = 1$	1.057	1.100	1.109	1.119
	$T = 2$	1.076	1.128	1.130	1.144
	$T = 3$	1.086	1.138	1.140	1.162

Table 11: Median of Bates prices over Heston prices.

Calibration risk

- exotic prices from AP differ from exotic prices from RP,AI,RI
- price differences grow for longer times to maturity
- price differences bigger for barrier options than for cliquets
- models with better fits do not have less calibration risk

Calibration and model risk

- model risk and calibration risk not independent
- model risk (between Heston and Bates) smallest for AP and biggest for RI

Choice of the error functional

If the choice of the parametric model is not clear for the considered option then calibration w.r.t AP minimizes risk.

For a given model calibration w.r.t AI gives (relatively) stable parameters, good fits and exotics prices with small variances that lie between the prices from AP and RP calibrations.

Heston model

The existence of this calibration risk raises the question if the Heston model is appropriate at all.

Tests on other models should be conducted to clarify if calibration risk is a general phenomenon in option pricing.

Outlook

Traders in banks are not only interested in good calibrations.
It is vital to have stable results, i.e. stable prices and greeks.

Possible regularizations:

- parameters
- (finite dimensional) distributions of the stock process
- other liquid markets (e.g. variance swap market)

How does regularization influence calibration risk?

For Further Reading

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