COPULAE IN TEMPORE VARIENTES
(Inhomogeneous Dependence Modelling with Time Varying Copulae)

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Value-at-Risk

1. *value of linear portfolio* $w = (w_1, \ldots, w_d)^\top$ of assets $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$:

$$V_t = \sum_{j=1}^{d} w_j S_{j,t}$$

2. *profit and loss* (P&L) function:

$$\begin{aligned} L_{t+1} &= V_{t+1} - V_t = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1) \\ X_{t+1} &= \log S_{t+1} - \log S_t \end{aligned}$$

3. *Value-at-Risk* at level α:

$$VaR(\alpha) = F_L^{-1}(\alpha)$$

Log returns DCX & VW at 20030408

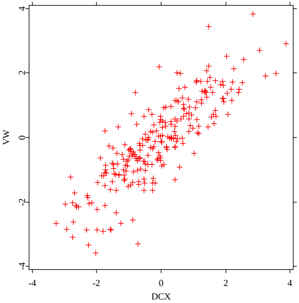


Figure 1: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20020415-20030408. maxmindep.xpl

Log returns DCX & VW at 20041027

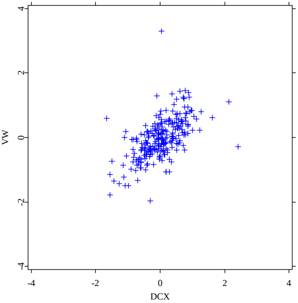


Figure 2: Standardized log returns, DaimlerChrysler (DCX) and Volkswagen (VW), 20031103-20041027. maxmindep.xpl

VaR depends on the distribution F_X of log returns $X = (X_1, \ldots, X_d)^\top$.

1. How to model the dependency among $X_1, \ldots, X_d$?
2. How does F_X and the dependency among $X_1, \ldots, X_d$ vary over time ?

Traditional approach (*RiskMetrics*)

1. log returns conditionally normal

$$X_t \sim N(0, \Sigma_t)$$

2. covariance matrix Σ_t estimated with $\lambda = 0.05$ by

$$\widehat{\Sigma}_t = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X_s X_s^T$$

Drawbacks from multivariate normal distribution:

1. no heavy-tails
2. joint extreme values relatively infrequent
3. symmetry (elliptical distribution)

Copula based approach

Log returns conditionally distributed with copula C:

$$X_t \sim C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d); \theta_t\}$$

where $F_{X_1}, \dots, F_{X_d}$ are marginal distributions and θ_t is the copula (dependence) parameter.

(Embrechts, 1999)

Adaptive Copulae

Global parameter $\theta_t = \theta$, too optimistic.

1. local parametric assumption: θ_t nearly constant on *homogeneity intervals*
2. find largest interval where homogeneity is acceptable ($\omega\rho\alpha\kappa\lambda\epsilon$)
3. for each t, *adaptively* find homogeneity interval

Estimate dependence parameter θ_t in a time varying interval

Local Parametric Assumption

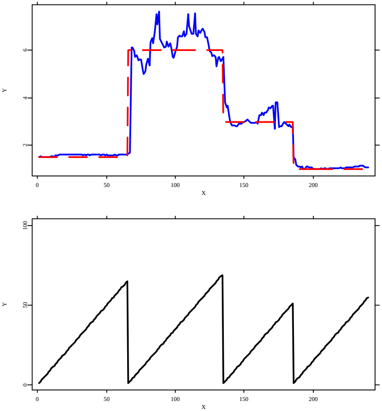


Figure 3: Parameter θ_t (blue), size of homogeneity interval at t (black).

Modelling Dependence over Time

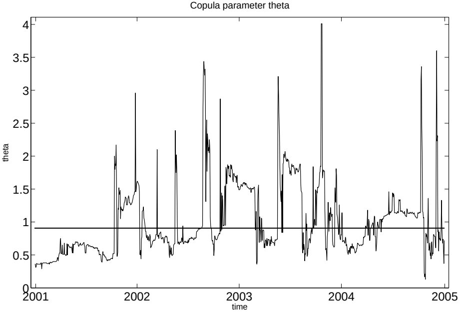


Figure 4: Dependence over time for DaimlerChrysler, Volkswagen, Bayer, BASF, Allianz and Münchener Rückversicherung, 20000101-20041231

Outline

1. Motivation ✓
2. Copulae and Value-at-Risk
3. Copula Estimation
4. Adaptive Copulae
5. Applications
6. References
7. Appendix

Copulae

Theorem (Sklar's theorem)

For a distribution function F with marginals $F_{X_1}, \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$ with

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

If $F_{X_1}, \ldots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \ldots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \ldots, F_{X_d}$.

With copula density

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}$$

the density function of $F(x_1, \ldots, x_d)$ is

$$f(x_1, \ldots, x_d) = c\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \prod_{j=1}^{d} f_j(x_j)$$

where $u_j = F_{X_j}(x_j)$ and $f_j(x_j) = F'_{X_j}(x_j)$, $j = 1 \ldots d$

1. Gaussian Copula

$$C_{\Psi}^{Ga}(u_1, \ldots, u_d) = \Phi_{\Psi}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}$$

Φ univariate standard normal cdf

Φ_{Ψ} d-dimensional standard normal cdf with correlation matrix Ψ

- Gaussian copula contains *the dependence structure*
- *normal* marginal distributions + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distributions + Gaussian copula = *meta-Gaussian* distributions

Explicit expression for the Gaussian copula

$$
\begin{aligned}
C_{\Psi}^{Ga}(u_1, \ldots, u_d) &= \Phi_{\Psi}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\} \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} e^{(-\frac{1}{2} r^{\top} \Psi^{-1} r)} dr_1 \ldots dr_d
\end{aligned}
$$

where

$r = (r_1, \ldots, r_d)^{\top}$, $u_j = \Phi(x_j)$

- $C_{\Psi}^{Ga}(u_1, \ldots, u_d)$ allows to generate joint symmetric dependence, but no tail dependence (i.e., there are no joint extreme events)

2. Frank Copula, $0 < \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = -\frac{1}{\theta} \log \left[1 + \frac{\prod_{j=1}^{d} \left\{ \exp(-\theta u_j) - 1 \right\}}{\{\exp(-\theta) - 1\}^{d-1}} \right]$$

- dependence becomes maximal when $\theta \longrightarrow \infty$
- independence is achieved when $\theta \longrightarrow 0$

3. Gumbel-Hougaard copula, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = \exp\left[- \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{\theta^{-1}} \right]$$

- for $\theta > 1$ allows to generate dependence in the upper tail (Schmidt, 2005)
- For $\theta = 1$ reduces to the product copula, i.e. $C_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$.
- for $\theta \longrightarrow \infty$, we obtain the Fréchet-Hoeffding upper bound:

$$C_\theta(u_1, \ldots, u_d) \stackrel{\theta \to \infty}{\longrightarrow} \min(u_1, \ldots, u_d).$$

4. Ali-Mikhail-Haq copula, $-1 \leq \theta < 1$

$$C_\theta(u_1, \dots, u_d) = \frac{\prod_{j=1}^{d} u_j}{1 - \theta \left\{ \prod_{j=1}^{d} \left(1 - u_j\right) \right\}}$$

- independence is achieved when $\theta = 0$
- the Fréchet-Hoeffding bounds are not achieved

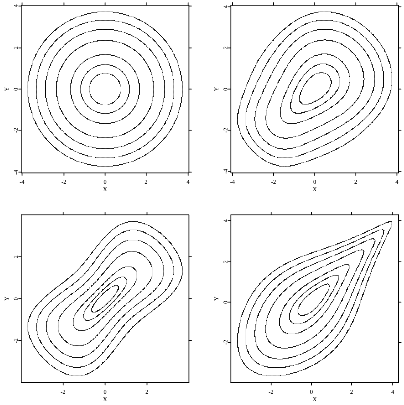
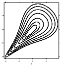


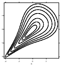
Figure 5: Pdf contour plots, $F(x_1, x_2) = C\{\Phi(x_1), \Phi(x_2)\}$ with Gaussian ($\rho = 0$), AMH ($\theta = 0.9$), Frank ($\theta = 8$), Gumbel ($\theta = 2$) copulae. cont4.xpl



5. Clayton copula, $\theta > 0$

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left(\sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-\theta^{-1}}$$

- dependence becomes maximal when $\theta \longrightarrow \infty$
- independence is achieved when $\theta \longrightarrow 0$
- the distribution tends to the lower Fréchet-Hoeffding bound when $\theta \longrightarrow 1$
- allows to generate asymmetric dependence and lower tail dependence, but no upper tail dependence



Value-at-Risk with Copulae

The process $\{X_t\}_{t=1}^{T}$ of log-returns can be modelled as

$$X_{j,t} = \sigma_{j,t}\varepsilon_{j,t}$$

with

$$\sigma_{j,t}^2 = E[X_{j,t}^2 \mid \mathcal{F}_{t-1}]$$

where $\mathcal{F}_t$ is the available information at time t.

The standardized innovations

$$\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{d,t})^\top$$

are independent with distribution function

$$F(\varepsilon_t) = C\{F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d); \theta_t\}$$

where

1. C is copula with parameter θ
2. ε_j have continuous marginal distributions F_j, $j = 1, \ldots, d$

VaR Estimation with Copulae

For log-returns $\{x_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, d$, estimation of VaR at level α:

1. determine innovations $\hat{\varepsilon}_t$ (e.g. by deGARCHing)
2. specificy and estimate marginal distributions $F_j(\hat{\varepsilon}_j)$
3. specificy a copula C and estimate dependence parameter θ
4. simulate innovations ε and losses L
5. determine $\widehat{VaR}(\alpha)$, the empirical α-quantile of F_L.

Copula estimation

The distribution of $X = (X_1, ..., X_d)^\top$ with marginals $F_{X_j}(x_j, \delta_j)$, $j = 1, ..., d$ is given by:

$$F_X(x_1, \dots, x_d) = C\{F_{X_1}(x_1;\delta_1), \dots, F_{X_d}(x_d;\delta_d);\theta\}$$

and its density is given by

$$f(x_1, \dots, x_d; \delta_1, \dots, \delta_d, \theta)$$

$$= c\{F_{X_1}(x_1;\delta_1), \dots, F_{X_d}(x_d;\delta_d);\theta\} \prod_{j=1}^{d} f_j(x_j;\delta_j)$$

where c is a copula density.

For a sample of observations $\{x_t\}_{t=1}^{T}$ and $\vartheta = (\delta_1, \ldots, \delta_d, \theta)^\top \in \mathbb{R}^{d+1}$ the likelihood function is

$$L(\vartheta; x_1, \ldots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \ldots, x_{d,t}; \delta_1, \ldots, \delta_d, \theta)$$

and the corresponding log-likelihood function

$$\ell(\vartheta; x_1, \ldots, x_T) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \delta_1), \ldots, F_{X_d}(x_{d,t}; \delta_d); \theta\}$$

$$+ \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{j,t}; \delta_j)$$

Full Maximum Likelihood (FML)

- FML estimates vector of parameters ϑ in one step through

$$\tilde{\vartheta}_{FML} = \arg\max_{\vartheta} \ell(\vartheta).$$

- the estimates $\tilde{\vartheta}_{FML} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve

$$(\partial\ell/\partial\delta_1, \ldots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

- Drawback: with an increasing dimension the algorithm becomes too burdensome computationally.

Inference for Margins (IFM)

1. estimate parameters δ_j from the marginal distributions:

$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j) = \arg\max_{\delta} \left\{ \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j) \right\}$$

2. estimate the dependence parameter θ by maximizing the *pseudo log-likelihood* function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}$$

- The estimates $\hat{\vartheta}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0.$$

- Advantage: numerically stable.

Canonical Maximum Likelihood (CML)

- CML maximizes the *pseudo log-likelihood* function with *empirical* marginal distributions

$$\ell(\theta) = \sum_{t=1}^{T} \log c\{\widehat{F}_{X_1}(x_{1,t}), \ldots, \widehat{F}_{X_d}(x_{d,t}); \theta\}$$

$$\widehat{\vartheta}_{CML} = \arg\max_{\theta} \ell(\theta)$$

where

$$\widehat{F}_{X_j}(x) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbf{1}\{X_j, t \leq x\}$$

- Advantage: no assumptions about the parametric form of marginal distributions.

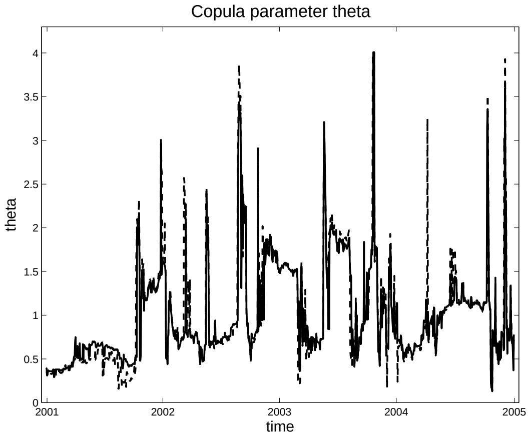


Figure 6: Copula dependence parameter θ for DaimlerChrysler, Volkswagen, Bayer, BASF, Allianz and Münchener Rückversicherung, 20000101-20041231. Estimated IFM approach (blue line) and CML approach (green line). plotrealtheta.xpl

Moving Window

- use static windows of size $w = 250$ scrolling in time t for VaR estimation:

$$\{x_t\}_{t=s-w+1}^{s}$$

  for $s = w, \dots, T$
- the VaR estimation procedure generates a time series $\{\widehat{VaR}_t\}_{t=w}^{T}$ and $\{\hat{\theta}_t\}_{t=w}^{T}$ of dependence parameters estimates.

Adaptive Copula estimation

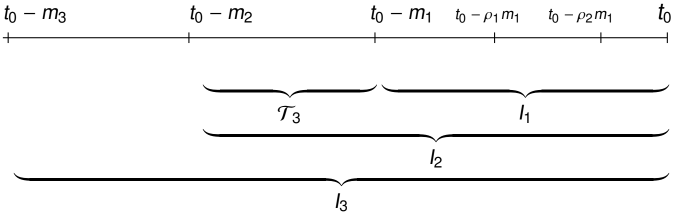
1. adaptively estimate largest interval where homogeneity hypothesis is accepted
2. *Local Change Point detection (LCP)*(Mercurio, Spokoiny, 2004): sequentially test θ_t is constant (i.e. $\theta_t = \theta$) within some interval I (local parametric assumption).
3. *"Oracle" choice*: largest interval $I = [t_0 - m_{k^*}, t_0]$ where small modelling bias condition (SMB)

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta$$

holds, m_{k^*} is the ideal scale, θ is ideally estimated from $I = [t_0 - m_{k^*}, t_0]$

Choice of homogeneity interval

1. Define set of intervals $\mathcal{I} = \{I_k, k = -1, 0, 1, ..., K\}$ such that $I_k = [t_0 - m_k, t_0]$
2. m_k: $m_{-1} < m_0 < ... \leq t_0$ and $m_{-1} = \rho_2 m_1$, $m_0 = \rho_1 m_1$ for $\rho_1 > \rho_2 \in (0, 1)$
3. sets $\mathcal{T}_k \subset I_k$ for $k = 1, \dots, K$ as $\mathcal{T}_k = [t_0 - m_{k-1}, t_0 - m_{k-2}]$.



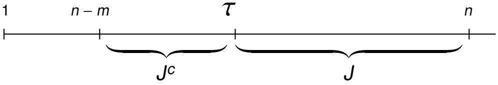
LCP procedure

1. $k = 1$
2. repeat until rejection or largest interval is reached
   - test homogeneity hypothesis $H_{0,k}$ within I_k on $\mathcal{T}_k$ against change-point alternative
   - $k = k + 1$
3. if $H_{0,k}$ is rejected, estimated homogeneity interval is last accepted interval $\widehat{I} = I_{\hat{k}}$, $\hat{k} = k - 2$
4. if largest possible interval is reached, $\widehat{I} = I_{K-1}$
5. estimate θ from observations in $\widehat{I}$, i.e., define $\hat{\theta}_{t_0} = \hat{\theta}_{\hat{I}}$

Test of homogeneity against a change point alternative

Interval $I = [n-m, n[$, $\mathcal{T}_I$ a set of points $\tau \in I$, $J = [\tau, n[$ and $J^c = [n-m, \tau[$

$$
\begin{aligned}
H_0 &: \quad \forall \tau \in \mathcal{T}_I, \quad \theta_t = \theta \;\; \forall t \in J, \quad \forall t \in J^c \\
H_1 &: \quad \exists \tau \in \mathcal{T}_I, \quad \theta_t = \theta_1 \;\forall t \in J, \quad \theta_t = \theta_2 \neq \theta_1 \;\forall t \in J^c
\end{aligned}
$$



Likelihood ratio test for the fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2} \{\ell_J(\theta_1) + \ell_{J^c}(\theta_2)\} - \max_{\theta} \ell_I(\theta) \\ &= \ell_J(\hat{\theta}_J) + \ell_{J^c}(\hat{\theta}_{J^c}) - \ell_I(\hat{\theta}_I) \\ &= \hat{\ell}_J + \hat{\ell}_{J^c} - \hat{\ell}_I \end{aligned}$$

where $\ell_I(\theta)$ is log-likelihood corresponding to H_0 and $\ell_J(\theta_1) + \ell_{J^c}(\theta_2)$ to H_1. Test statistics for unknown change point location:

$$T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}$$

Reject H_0 if $T_I > \lambda_I$

Choice of critical values

Event "accept homogeneity in I_{k-1}, reject in I_k" may be represented by the set

$$\mathcal{B}_k = \bigcap_{j=1}^{k-1} \{T_{I_j} \le \lambda_{I_j}\} \cap \{T_{I_k} > \lambda_{I_k}\}$$

and it holds $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots$. Thus, defining $\beta_{I_k} = P(\mathcal{B}_k)$ and $\alpha_{I_k} = P\left(\bigcup_{j=1}^{k} \mathcal{B}_j\right)$ we verify

$$\alpha_{I_k} = \sum_{j=1}^{k} \beta_{I_j} \tag{1}$$

Select by Monte Carlo critical values λ_{I_k} for every interval I_k to provide a prescribed probability of false alarm α:

$$\alpha = \sum_{j=1}^{K} \beta_{I_j}$$

where

$$P_{H_0}\left(\bigcap_{j=1}^{k-1} \{T_{I_j} \leq \lambda_{I_j}\} \cap \{T_{I_k} > \lambda_{I_k}\} \right) = \beta_{I_k}$$

Simulated Examples
1. Clayton copula: sudden jump in dependence

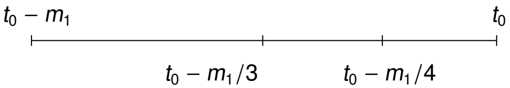
Simulated sets of 300 observations from d-dimensional Clayton copula with parameter

$$\theta_t = \begin{cases} 0.1 & \text{if} \quad 1 \le t \le 100 \\ \vartheta & \text{if} \quad 101 \le t \le 200, \\ 0.1 & \text{if} \quad 201 \le t \le 300 \end{cases}$$

for $\vartheta = 1.5, 3, 6$ and $d = 2, 6, 10$.

Implementation

1. Interval candidates $\mathcal{I}$: $\mathcal{I} = \{I_k : I_k = [t_0 - m_k, t_0]\}$ with $m_k = \left[m_1 c^k\right]$, $k = 1, ..., K$, $m_{-1} = \rho_2 m_1$, $m_0 = \rho_1 m_1$
2. $\rho_1 = 1/3$, $\rho_2 = 1/4$, $m1 = 20$, $c = 1.25$



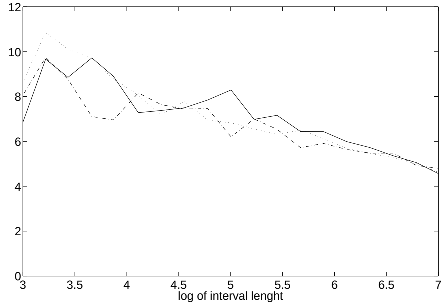


Figure 7: Critical values λ_{I_k} for $\alpha = 0.05$, $m_1 = 20$, $c = 1.25$, $d = 2$ (dotted), 6 (dashed) and 10 (full)

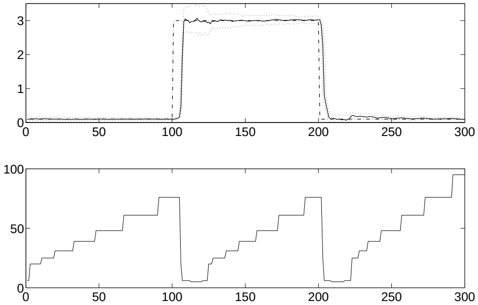


Figure 8: θ_t (dashed), median (full), 0.25 and 0.75 quantiles (dotted) from $\hat{\theta}_t$ (top), median from $|\hat{I}_t|$ (bottom), based on 200 simulations, Clayton copula, $d = 6$, $\vartheta = 3$

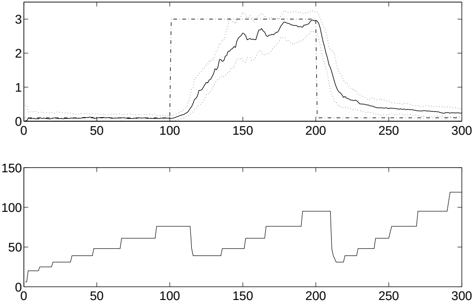


Figure 9: θ_t (dashed), median (full), 0.25 and 0.75 quantiles (dotted) from $\hat{\theta}_t$ (top), median from $|\hat{I}_t|$ (bottom), based on 200 simulations, Clayton copula, $d = 2$, $\vartheta = 3$

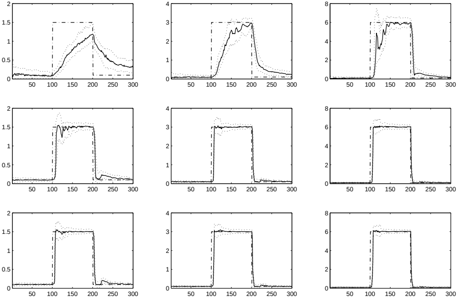


Figure 10: From left to right $\vartheta = 1.5, 3, 6$, from top to bottom $d = 2, 6, 10$, 200 simulations from Clayton copula, $m_1 = 20$ and $c = 1.25$

Kullback-Leibler Divergence

Kullback-Leibler divergence is defined as:

$$\mathcal{K}(P_{\vartheta}, P_{\vartheta'}) = E_{\vartheta} \log \frac{p(y, \vartheta)}{p(y, \vartheta')}$$

ϑ	$\mathcal{K}_2(0.1, \vartheta)$	$\mathcal{K}_2(\vartheta, 0.1)$	$\mathcal{K}_6(0.1, \vartheta)$	$\mathcal{K}_6(\vartheta, 0.1)$	$\mathcal{K}_{10}(0.1, \vartheta)$	$\mathcal{K}_{10}(\vartheta, 0.1)$
1.5	0.41	0.26	3.52	1.57	7.30	2.89
3.0	1.28	0.56	11.49	3.25	24.69	5.89
6.0	3.51	1.01	31.52	5.56	68.35	10.00

Table 1: Kullback-Leibler divergence between d-dimensional Clayton copulae with parameters 0.1 and ϑ

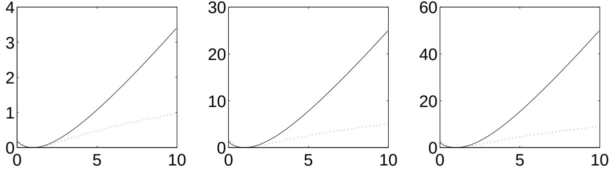


Figure 11: $\mathcal{K}_d(0.1, \vartheta)$ (dashed), $\mathcal{K}_d(\vartheta, 0.1)$ (full), corresponding to upward and downward jumps, d-dimensional Clayton copula, $d = 2$ (left), 6 (middle) and 10 (right)

Detection Delay Statistics

rule		mean	std.dev.	max	min
40%	$\mathcal{K}_2(1.5, 0.1)$	35.64	16.49	93.00	1.00
50%		41.70	19.23	100.00	2.00
60%		50.04	21.93	100.00	7.00
40%	$\mathcal{K}_6(1.5, 0.1)$	8.84	3.53	31.00	2.00
50%		9.35	4.39	34.00	2.00
60%		10.00	5.57	34.00	3.00
40%	$\mathcal{K}_2(0.1, 1.5)$	9.27	10.39	62.00	00
50%		14.70	13.38	74.00	0.00
60%		25.78	21.04	100.00	1.00
40%	$\mathcal{K}_6(0.1, 1.5)$	5.33	2.69	14.00	1.00
50%		5.87	3.13	15.00	1.00
60%		6.31	3.57	20.00	1.00

Table 2: Statistics for detection delay to downward jump at $t = 100$, $\mathcal{K}_d(1.5, 0.1)$ and upward jump at $t = 200$, $\mathcal{K}_d(0.1, 1.5)$ for 200 simulations from d-dimensional Clayton copula, $m_1 = 20$, $c = 1.25$

rule		mean	std.dev.	max	min
40%	$\mathcal{K}_2(3, 0.1)$	23.05	9.41	62.00	6.00
50%		26.34	11.79	67.00	6.00
60%		29.31	13.82	84.00	6.00
40%	$\mathcal{K}_6(3, 0.1)$	5.82	1.07	9.00	2.00
50%		6.07	0.97	9.00	3.00
60%		6.32	0.92	9.00	4.00
40%	$\mathcal{K}_2(0.1, 3)$	8.27	5.79	30.00	1.00
50%		10.62	6.20	32.00	6.00
60%		12.87	7.30	45.00	1.00
40%	$\mathcal{K}_6(0.1, 3)$	2.89	1.39	7.00	1.00
50%		3.01	1.45	7.00	1.00
60%		3.07	1.46	7.00	1.00

Table 3: Statistics for detection delay to downward jump at $t = 100$, $\mathcal{K}_d(3, 0.1)$ and upward jump at $t = 200$, $\mathcal{K}_d(0.1, 3)$ for 200 simulations d-dimensional Clayton copula, $m_1 = 20$, $c = 1.25$

rule		mean	std.dev.	max	min
40%	$\mathcal{K}_2(6, 0.1)$	14.31	6.39	38.00	4.00
50%		15.24	7.60	41.00	4.00
60%		16.00	8.62	41.00	4.00
40%	$\mathcal{K}_6(6, 0.1)$	6.22	0.80	7.00	4.00
50%		6.43	0.68	7.00	4.00
60%		6.62	0.61	9.00	5.00
40%	$\mathcal{K}_2(0.1, 6)$	5.55	2.43	16.00	1.00
50%		6.07	2.73	17.00	1.00
60%		6.66	3.21	22.00	1.00
40%	$\mathcal{K}_6(0.1, 6)$	1.61	0.74	4.00	1.00
50%		1.62	0.75	4.00	1.00
60%		1.74	0.81	4.00	1.00

Table 4: Statistics for detection delay to downward jump at $t = 100$, $\mathcal{K}_d(6, 0.1)$ and upward jump at $t = 200$, $\mathcal{K}_d(0.1, 6)$ for 200 simulations from d-dimensional Clayton copula, $m_1 = 20$, $c = 1.25$

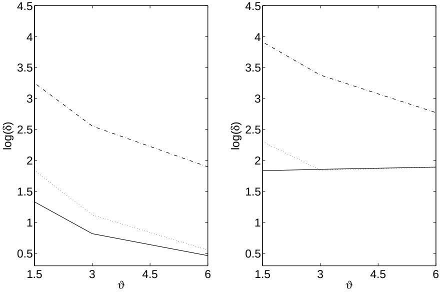
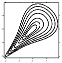


Figure 12: Logarithm of mean detection delays at $r = 0.6$ for different upward (left) and downward (right) jump sizes, d-dimensional Clayton Copula, $d = 2$ (dashed), 6 (dotted) and 10 (full)



2. Clayton copula: linear change in dependence

Simulated sets of 300 observations from d-dimensional Clayton copula with parameter

$$\theta_t = \begin{cases} 0.1 & \text{if} \quad 1 \leq t \leq 100 \\ 0.1 + \frac{1}{50}\Delta(t - 100) & \text{if} \quad 101 \leq t \leq 150 \\ \vartheta & \text{if} \quad 151 \leq t \leq 250 \\ \vartheta - \frac{1}{50}\Delta(t - 250) & \text{if} \quad 251 \leq t \leq 300 \\ 0.1 & \text{if} \quad 301 \leq t \leq 400 \end{cases}$$

for $\vartheta = 1.5, 3, 6$ and $d = 2, 6, 10$.

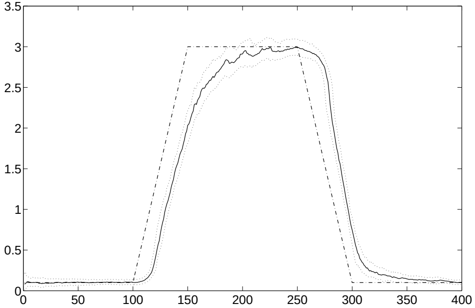
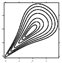


Figure 13: Pointwise median (full), 0.25, 0.75 quantiles (dotted) from estimated parameter $\hat{\theta}_t$ and true parameter θ_t (dashed), $\vartheta = 3$, $d = 6$. Based on 200 simulations from Clayton copula, $m_1 = 20$ and $c = 1.25$



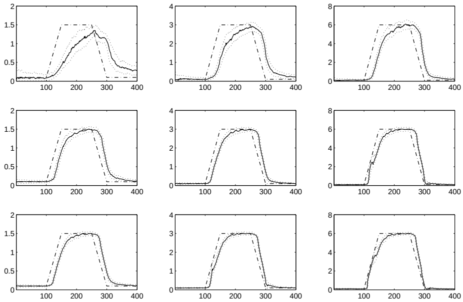
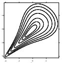


Figure 14: From left to right $\vartheta = 1.5, 3, 6$, from top to bottom $d = 2, 6, 10$. Based on 200 simulations from Clayton copula, $m_1 = 20$ and $c = 1.25$



3. Gaussian copula: sudden jump in correlation

Simulations from 2-dimensional Gaussian copula with parameter given by

$$\rho_t = \begin{cases} 0 & \text{if} \quad 1 \le t \le 100 \\ \varrho & \text{if} \quad 101 \le t \le 200 \\ 0 & \text{if} \quad 201 \le t \le 300 \end{cases}$$

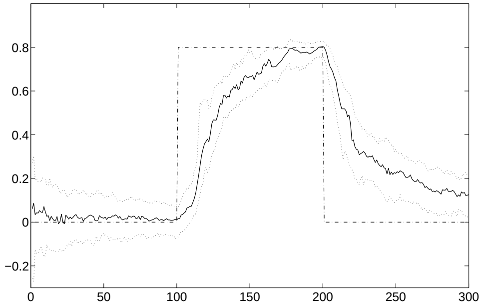


Figure 15: Pointwise median (full), 0.25, 0.75 quantiles (dotted) from estimated parameter $\hat{\rho}_t$ and true parameter ρ_t (dashed), $\varrho = 0.8$. Based on 100 simulations from Gaussian copula, $m_1 = 20$ and $c = 1.25$

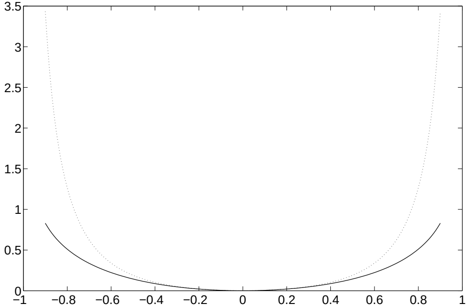


Figure 16: $\mathcal{K}_2(0,\varrho)$ (dashed), $\mathcal{K}_2(\varrho,0)$ (full), corresponding to upward and downward jumps, 2-dimensional Gaussian copula

Detection Delay Statistics

t	r	mean	std dev.	max	min
100	40%	18.11	7.15	43	6
	50%	19.69	7.74	43	6
	60%	22.24	9.42	46	8
200	40%	16.02	9.08	45	2
	50%	20.42	13.19	63	2
	60%	25.21	18.16	100	2

Table 5: Statistics for detection delay δ to downward ($t = 101$) and upward ($t = 201$) jump of size 0.8 at rule r, based on 100 simulations from Gaussian copula, $m_1 = 20$, $c = 1.25$

Simulations from a 3-dimensional Gaussian copula with correlation given by

$$\Psi_t = \begin{cases} \mathcal{I}_3 & \text{if} \quad 1 \leq t \leq 100 \\ \mathcal{R} & \text{if} \quad 101 \leq t \leq 200 \\ \mathcal{I}_3 & \text{if} \quad 201 \leq t \leq 300 \end{cases}$$

where $\mathcal{I}_3$ is the identity matrix of size 3 and

$$\mathcal{R} = \begin{pmatrix} 1 & 0.8 & 0 \\ 0.8 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix}$$

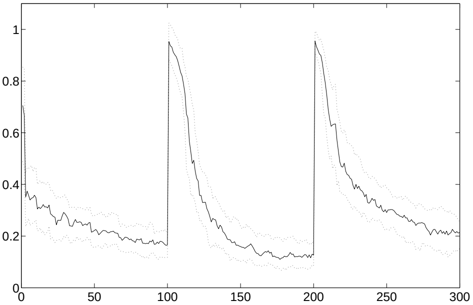


Figure 17: Pointwise median (full), 0.25, 0.75 quantiles (dotted) of distance $d(\hat{\Psi}_t, \Psi_t)$ between estimated and true correlation matrices. Based on 200 simulations from Gaussian copula, $d = 3$, $m_1 = 20$ and $c = 1.25$

Applications

Portfolio of 6 DAX stocks: Siemens, Thyssen, Schering, E.on, Henkel and Lufthansa

$\sigma^2_{t,j}$, $j = 1, .., 6$ estimated at time t by exponential smoothing

$$\hat{\sigma}^2_{t,j} = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X^2_{s,j}$$

Clayton copula

$$C_\theta(u_1, ..., u_d) = (u_1^{-\theta} + ... + u_d^{-\theta} - d + 1)^{-1/\theta}$$

with density

$$c_\theta(u_1, ..., u_d) = \prod_{j=1}^{d} \{1 + (j-1)\theta\} \prod_{j=1}^{d} u_j^{-(\theta+1)} \left\{ \sum_{j=1}^{d} u_j^{-\theta} - d + 1 \right\}^{-(1/\theta+d)}$$

Recall: $\theta = 0$ indicates independence

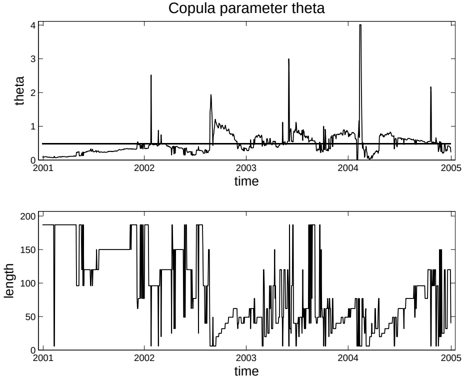
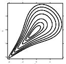


Figure 18: Upper panel: estimated copula dependence parameter θ for SIE, THY, SCH, EOA, HEN, LHA and global parameter. Lower panel: estimated intervals of time homogeneity, $m_1 = 20$, $c = 1.25$ and $\alpha = 0.05$



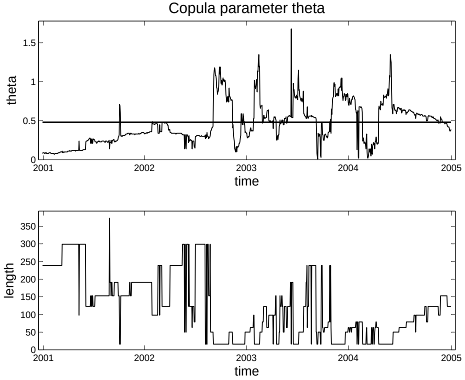


Figure 19: Upper panel: estimated copula dependence parameter θ for SIE, THY, SCH, EOA, HEN, LHA and global parameter. Lower panel: estimated intervals of time homogeneity, $m_1 = 50$, $c = 1.25$ and $\alpha = 0.05$

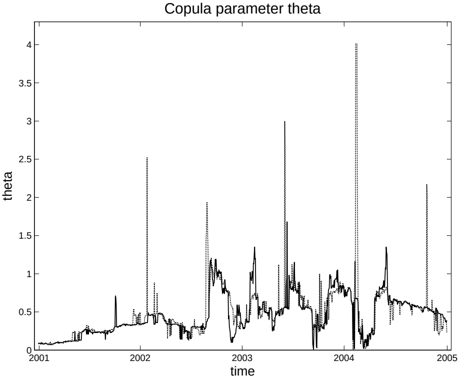
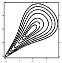


Figure 20: Adaptively estimated copula parameter θ for SIE, THY, SCH, EOA, HEN, LHA, $m_1 = 20$ (dashed line) and $m_1 = 50$ (solid line)



Backtesting

Compare the estimated values for the VaR with the true realizations $\{l_t\}$ of the P&L function

The *exceedances ratio* is given by

$$\hat{\alpha} = \frac{1}{T-w} \sum_{t=w}^{T} \mathbf{1}\{l_t < \widehat{VaR}_t(\alpha)\}$$

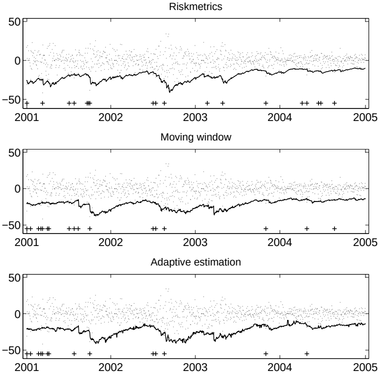


Figure 21: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.01$; $w = (3, 2, 3, 2, 3, -1)^\top$, estimated with *RiskMetrics* (top), moving window and LCP procedure (bottom)

Portfolio	Exceedances ratio $\alpha(\times 10^2)$ 5	4	3	2	1
$(1,1,1,1,1,1)$	6.2807	5.3974	4.6124	3.7291	1.8646
$(1,2,3,2,1,3)$	6.4769	5.6919	4.4161	3.3366	1.7664
$(2,1,2,3,1,3)$	6.2807	5.6919	4.8086	3.3366	2.1590
$(3,2,3,2,3,1)$	6.4769	4.9068	4.4161	3.7291	1.8646
$(3,1,2,1,3,2)$	6.1825	5.4956	4.4161	3.1403	1.5702
$(1,3,1,2,3,1)$	6.1825	5.4956	4.7105	3.0422	1.9627
$(2,1,3,2,1,3)$	6.4769	5.5937	4.3180	3.3366	1.8646
$(2,3,3,2,1,1)$	6.3788	5.6919	4.2198	3.4347	2.0608
$(3,1,2,2,2,3)$	6.1825	5.4956	4.6124	3.2385	1.8646
....	......	......	......	......	
$(2,3,3,2,1,-1)$	6.2807	5.2993	4.3180	3.2385	1.8646
$(3,1,2,2,2,-3)$	5.6919	4.9068	4.3180	3.7291	1.7664
$(2,3,1,1,2,-3)$	5.9863	5.1030	4.3180	3.5329	1.9627
$(2,3,2,3,2,-3)$	5.8881	5.4956	4.1217	3.8273	1.7664
$(3,2,3,2,3,-3)$	6.2807	5.2012	4.5142	3.4347	1.4720
avg.	6.1825	5.3688	4.4570	3.4184	1.8237
std.dev.	0.2333	0.2342	0.1918	0.3231	0.1685
$\sum_{w \in W}(\hat{\alpha}-\alpha)^2$	0.3481	0.4623	0.5179	0.5068	0.1694
$\sum_{w \in W}(\hat{\alpha}-\alpha)^2/\alpha$	6.9626	11.557	17.264	25.342	16.936

Table 6: Exceedances ratio $\hat{\alpha}$ for different portfolios, estimated using *RiskMetrics*

Portfolio	Exceedances ratio $\alpha(\times 10^2)$ 5	4	3	2	1
$(1, 1, 1, 1, 1, 1)$	5.9863	5.3974	4.6124	3.7291	1.6683
$(1, 2, 3, 2, 1, 3)$	5.8881	5.1030	3.6310	2.3553	1.3739
$(2, 1, 2, 3, 1, 3)$	7.2620	6.4769	5.4956	3.9254	2.2571
$(3, 2, 3, 2, 3, 1)$	5.5937	5.0049	4.3180	3.3366	1.6683
$(3, 1, 2, 1, 3, 2)$	6.0844	5.2012	4.3180	3.3366	1.4720
$(1, 3, 1, 2, 3, 1)$	5.1030	4.0236	3.1403	2.8459	1.5702
$(2, 1, 3, 2, 1, 3)$	6.9676	5.5937	4.5142	3.3366	2.0608
$(2, 3, 3, 2, 1, 1)$	6.5751	5.2993	4.6124	3.6310	1.9627
$(3, 1, 2, 2, 2, 3)$	7.1639	6.1825	5.3974	4.4161	2.1590
....	......	......	......	......	
$(2, 3, 3, 2, 1, -1)$	6.1825	5.2993	4.4161	3.2385	1.9627
$(3, 1, 2, 2, 2, -3)$	6.2807	5.2993	5.1030	3.4347	2.1590
$(2, 3, 1, 1, 2, -3)$	5.6919	5.0049	4.3180	2.7478	1.6683
$(2, 3, 2, 3, 2, -3)$	5.5937	4.8086	4.0236	3.1403	1.6683
$(3, 2, 3, 2, 3, -3)$	5.5937	4.9068	3.6310	2.6497	1.6683
avg.	5.9985	5.1398	4.2812	3.1935	1.7623
std.dev.	0.7515	0.6483	0.6528	0.5257	0.2644
$\sum_{w \in W} (\hat{\alpha} - \alpha)^2$	0.3692	0.4085	0.4919	0.4054	0.1556
$\sum_{w \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	7.3835	10.212	16.398	20.271	15.556

Table 7: Exceedances ratio $\hat{\alpha}$ for different portfolios, estimated with Clayton copula and moving window

Portfolio	Exceedances ratio $\alpha(\times 10^2)$				
	5	4	3	2	1
$(1,1,1,1,1,1)$	5.4956	4.6124	3.8273	2.8459	1.3739
$(1,2,3,2,1,3)$	5.1030	3.9254	3.2385	2.5515	1.3739
$(2,1,2,3,1,3)$	6.7713	5.4956	4.6124	3.2385	2.2571
$(3,2,3,2,3,1)$	5.2012	4.2198	3.4347	2.4534	1.3739
$(3,1,2,1,3,2)$	5.2993	4.6124	3.4347	2.2571	1.3739
$(1,3,1,2,3,1)$	4.0236	3.2385	2.6497	2.4534	1.1776
$(2,1,3,2,1,3)$	5.9863	5.1030	4.1217	2.9441	1.5702
$(2,3,3,2,1,1)$	5.8881	5.0049	4.1217	3.2385	1.6683
$(3,1,2,2,2,3)$	6.7713	5.4956	4.7105	3.1403	1.6683
....	......	......	......	......	
$(2,3,3,2,1,-1)$	5.2993	4.9068	4.0236	3.2385	1.5702
$(3,1,2,2,2,-3)$	5.8881	4.9068	4.3180	2.7478	1.6683
$(2,3,1,1,2,-3)$	5.3974	4.8086	3.6310	2.4534	1.5702
$(2,3,2,3,2,-3)$	5.2993	4.1217	3.4347	2.3553	1.6683
$(3,2,3,2,3,-3)$	5.0049	4.1217	2.9441	2.1590	1.3739
avg.	5.4383	4.5101	3.6310	2.6619	1.5088
std.dev	0.7529	0.6621	0.5902	0.3864	0.2293
$\sum_{w\in W}(\hat{\alpha}-\alpha)^2$	0.1765	0.1633	0.1757	0.1395	0.0742
$\sum_{w\in W}(\hat{\alpha}-\alpha)^2/\alpha$	3.5299	4.0822	5.8563	6.9750	7.4237

Table 8: Exceedances ratio $\hat{\alpha}$ for different portfolios, estimated with Clayton copula and LCP

	Exceedances ratio $\alpha(\times 10^2)$				
Method	5	4	3	2	1
Riskmetrics	6.9626	11.557	17.264	25.342	16.936
Moving Window	7.3835	10.212	16.398	20.271	15.556
LCP	3.5299	4.0822	5.8563	6.9750	7.4237

Table 9: Relative squared deviation $\sum_{w \in W}(\hat{\alpha}-\alpha)^2/\alpha$ for Riskmetrics, Moving Window and LCP

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Appendix I

For all $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, every copula C satisfies

$$W(u_1, \ldots, u_d) \le C(u_1, \ldots, u_d) \le M(u_1, \ldots, u_d)$$

where

$$M(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d)$$

and

$$W(u_1, \ldots, u_d) = \max\left(\sum_{i=1}^{d} u_i - d + 1, 0\right)$$

- $M(u_1, \ldots, u_d)$ is called *Fréchet-Hoeffding upper bound*
- $W(u_1, \ldots, u_d)$ is called *Fréchet-Hoeffding lower bound*

(Hoeffding, 1940)

Appendix II

For a random vector $X = (X_1, X_2)^\top$

- *upper tail dependence coefficient* is defined as

$$\delta = \lim_{u \to 1} P\left\{X_1 > F_{X_1}^{-1}(u) | X_2 > F_{X_2}^{-1}(u)\right\}$$

- *lower tail dependence coefficient* is defined as

$$\gamma = \lim_{u \to 0} P\left\{X_1 \leq F_{X_1}^{-1}(u) | X_2 \leq F_{X_2}^{-1}(u)\right\}$$

X is *upper/lower* tail dependent if $\delta > 0$ resp. $\gamma > 0$.

- *upper tail dependence coefficient for Copula C:*

$$\delta = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u}$$

- *lower tail dependence coefficient for Copula C:*

$$\gamma = \lim_{u \to 0} \frac{C(u, u)}{u}$$

  - Gaussian copula: $\delta = \gamma = 0$
  - Clayton copula: $\delta = 0$, $\gamma = 2^{1/\theta}$
  - Gumbel copula: $\delta = 2 - 2^{1/\theta}$, $\gamma = 0$

Appendix III

Some definitions

1. underlying measure $\mathbb{P}$,

$$X_t \sim C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d), \theta_t\} = \mathbb{P}$$

2. parametric measure $\mathbb{P}_\theta$

1. test statistics for the change point location at τ

$$
\begin{aligned}
T_{I,\tau} &= \max_{\theta_1,\theta_2} \{\ell_J(\theta_1) + \ell_{J^c}(\theta_2)\} - \max_{\theta} \ell_I(\theta) \\
&= \hat{\ell}_J + \hat{\ell}_{J^c} - \hat{\ell}_I
\end{aligned}
$$

2. fitted log-likelihod for change point alternative with unknown location

$$
T_{I_k} = \max_{\tau \in \mathcal{T}_{I_k}} T_{I,\tau}
$$

3.

$$
\tilde{\tau}_{I_k} = \operatorname*{argmax}_{\tau \in \mathcal{T}_{I_k}} T_{I_k,\tau}
$$

Random sets, for $1 \leq k \leq K$

1.
$$\mathcal{C}_k = \{T_{I_k} \leq \lambda_k\}$$

2. I_k accepted
$$\mathcal{A}_k = \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k$$

3. I_k accepted, I_{k+1} rejected
$$\mathcal{B}_k = \mathcal{A}_k \backslash \mathcal{C}_{k+1}$$

4. for rejected I_k,
$$\hat{I} = [t_0 - m_{k-2}, t_0]$$

1. MLE estimator at I_k

$$\widetilde{\theta}_{I_k}$$

2. adaptive estimator $\hat{\theta}_{I_k}$ at step k

$$\hat{\theta}_{I_k} = \widetilde{\theta}_{I_k} \mathbf{1}(\mathcal{A}_k) + \sum_{l=1}^{k-1} \widetilde{\theta}_{\hat{l}} \mathbf{1}(\mathcal{B}_l)$$

3. adaptive estimator $\hat{\theta}$ at step K

$$\hat{\theta} = \hat{\theta}_{I_K}$$

1. *Small modelling bias (SMB)* condition for an interval I,

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta,$$

where θ is constant and *Kullback-Leibler divergence*

$$\mathcal{K}(P_\vartheta, P_{\vartheta'}) = E_\vartheta \log \frac{p(y, \vartheta)}{p(y, \vartheta')}$$

2. For $\Delta > 0$ and set of intervals $\mathcal{I}$:

$$k^* = \{s \in \{1, \ldots, K\} : k \leq s, \Delta_{I_k}(\theta) \leq \Delta\}$$

Theorem 1 (parametric case):

Under the local parametric assumption θ can be estimated by $\widetilde{\theta}_{I_k}$ in an interval I_k such that it holds:

$$E_{\mathbb{P}_\theta} \left| \ell \left(\widetilde{\theta}_{I_k}, \theta \right) \right|^r \leq \mathcal{R}_{2r}$$

- estimation risk is bounded by constant $\mathcal{R}_{2r}$
- in the Gaussian regression case $\mathcal{R}_{2r} = E|\xi|^{2r}$ where $\xi \sim N(0, 1)$

Theorem 2 (nonparametric case):

Under the *SMB* condition (i.e. $\Delta_{I_k}(\theta) \le \Delta$) it holds

$$E_{\mathbb{P}} \left| \ell\left(\widetilde{\theta}_{I_k}, \theta\right) \right|^{r/2} \le \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta),$$

i.e. $\widetilde{\theta}_{I_k}$ is a “good” estimator of θ in an interval I_k.

- $\exp(\Delta)$ is payment for approximation

Sequential choice of critical values

Choose λ_{I_k} such that for $k = 1, \ldots, K$

$$E_{\mathbb{P}_\theta} \left| \ell \left(\widetilde{\theta}_{I_k}, \hat{\theta}_{I_k} \right) \right|^r \leq \rho \mathcal{R}_{2r}$$

i.e. in the parametric case the risk of adaptive estimate is of same order as risk of non-adaptive "oracle" estimate $\widetilde{\theta}_{I_K}$:

$$E_{\mathbb{P}_\theta} \left| \ell \left(\widetilde{\theta}_{I_K}, \hat{\theta} \right) \right|^r \leq \rho \mathcal{R}_{2r}$$

Theorem 3 (critical values):

For λ_{I_k}, $k = 1, \ldots, K$ such that

$$E_{\mathbb{P}_\theta} \left| \ell \left(\widetilde{\theta}_{I_k}, \hat{\theta}_{I_k} \right) \right|^r \leq \rho \mathcal{R}_{2r} \qquad (2)$$

there exist ι_0, ι such that

$$\lambda_{I_k} \leq \iota_0 \log K + \iota(K - k) \qquad (3)$$

- simplified procedure: select ι_0, ι such that (2) holds

Propagation to nonparametric:

Under the *SMB* condition, i.e. $\Delta_{I_k}(\theta) \le \Delta$ for $k \le k^*$, it follows from theorem 2 for $\rho > 0$

$$
\begin{aligned}
E_{\mathbb{P}} \left| \ell\left(\widetilde{\theta}_{I_{k^*}}, \theta\right)\right|^{r/2} &\le \mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta) \\
E_{\mathbb{P}} \left| \ell\left(\widetilde{\theta}_{I_{k^*}}, \hat{\theta}_{I_{k^*}}\right)\right|^{r/2} &\le (\rho \mathcal{R}_{2r})^{1/2} \cdot \exp(\Delta)
\end{aligned}
$$

- as long as SMB holds, local parametric approach is justified and adaptive estimate behaves similarly to ”oracle” estimate

Theorem 4 (stability after propagation):

Adaptive estimator $\hat{\theta}$ provides same quality of estimation as non-adaptive oracle estimator $\widetilde{\theta}_{I_{k^*}}$:

$$E_{\mathbb{P}} \left| L\left(\widetilde{\theta}_{I_{k^*}}, \hat{\theta}\right)\right|^{r/2} \leq const \cdot \left\{\mathcal{R}_{2r}^{1/2} \cdot \exp(\Delta) + \lambda_{I_{k^*}}^{r/2}\right\}$$

- $\exp(\Delta)$ is payment for approximation
- $\lambda_{I_{k^*}}$ is payment for adaptation