Variance swaps

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Why investors may wish to trade volatility?

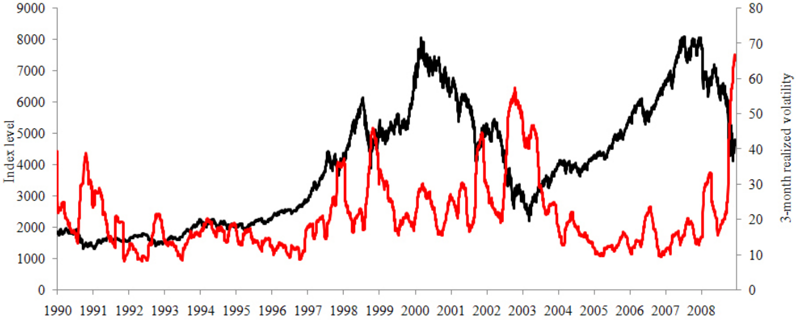


Figure 1: DAX level vs. DAX 3M realized vola(2000-2005)

Why investors may wish to trade volatility?

Volatility:

- reverts to it's long-term mean
- jumps when markets crash
- experiences different regimes
- negatively correlated with the underlying

Volatility trading

- Taking directional bets on volatility (variance)
- Trading spreads on indices
- Hedging volatility exposure
- Dispersion trading

Outline

1. Motivation ✓
2. Definition
3. Replication and hedging
4. Dispersion trading strategy

Variance swap

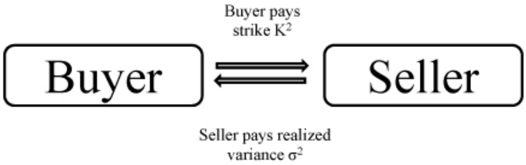


Figure 2: Cash flow of a Variance swap at expiry

Definition

Variance swap is a forward contract that at maturity pays the difference between realized variance σ_R^2 and predefined strike K_{var}^2 multiplied by notional N_{var}.

$$(\sigma_R^2 - K_{var}^2) \cdot N_{var}$$

$$\sigma_R = \sqrt{\frac{252}{T} \sum_{t=1}^{T} \left(\log \frac{S_t}{S_{t-1}} \right)^2} \cdot 100$$

Notional can be expressed in Vega and variance terms:

$$N_{vega} = N_{var} \cdot 2K_{var}$$

Construction

Example: 3-month variance swap long
Trade size is 2500 variance notional (represents a payoff of 2500 per point difference between realized and implied variance).
If K_{var} is 20% ($K_{var}^2 = 400$) and the subsequent variance realized over the course of the year is $(15\%)^2$ (quoted as $\sigma_R^2 = 225$), the investor will make a loss because realized variance is below the level bought.
Overall loss: $437500 = 2500 \cdot (400 - 225)$.

Marking-to-market

- Variance is **additive**.
- At time t we price a variance swap initiated at 0 and maturing at T:

$$(\sigma^2_{R,(0,T)} - K^2_{var,(0,T)}) \cdot N_{var}$$

At time t the part of volatility is already realized!

- Let $\sigma^2_{R,(0,t)}$ - realized variance between 0 and t, $K^2_{var,(t,T)}$ - strike of the variance swap initiated at t and maturing at T. Then

$$\frac{1}{T}\left\{t\sigma^2_{R,(0,t)} - (T-t)K^2_{var,(t,T)}\right\} - K^2_{var,(0,T)}$$

is a swap payoff at time t (per unit of variance notional).

Replication and hedging - intuitive approach

- A call option with BS price: $C_{BS}(S, K, \sigma\sqrt{\tau})$.
- Variance Vega:

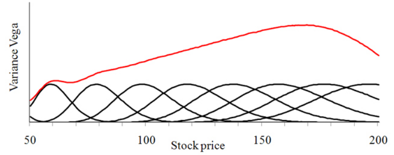
$$V = \frac{\partial C_{BC}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma} \frac{\exp(-d_1^2/2)}{\sqrt{2}\pi}$$

where

$$d_1 = \frac{\log(S/K) + (\sigma^2 \tau)/2}{\sigma\sqrt{\tau}}$$

Replication and hedging - intuitive approach

If we need a position in future realized variance $C_{BS}(S, K, \sigma\sqrt{\tau})$ is imperfect vehicle since V is sensitive to stock price moves. Solution - construct an portfolio of options with $V = const$.



Variance Vega
50
100
150
200
Stock price

Replication - more rigorous approach

Assumptions:

- existence of futures market with delivery dates $T' \geq T$
- futures contract F_t(underlying) follows a diffusion process with no jumps
- existence of European futures options market, for these options all strikes are available (market is complete)
- continuous trading is possible

Log contract

Let us consider the following function:

$$f(F_t) = \frac{2}{T}\left\{\log\frac{F_0}{F_t} + \frac{F_t}{F_0} - 1\right\}$$

This function is twice differentiable with derivatives:

$$f'(F_t) = \frac{2}{T}\left(\frac{1}{F_0} - \frac{1}{F_t}\right)$$

and

$$f''(F_t) = \frac{2}{TF_t^2}$$

at time $t = 0$ the function $f(F_t)$ has a value of zero.

To find the dynamic of $f(F_t)$ use Itô's lemma. In general for every smooth twice differentiable function $f(F_t)$ Itô's lemma gives:

$$f(F_t) = f(F_0) + \int_0^T f'(F_t)dF_t + \frac{1}{2}\int_0^T F_t^2 f''(F_t)\sigma_t^2 dt$$

Substituting the above introduced function gives obtain expression for the realized variance:

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\left(\log\frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right) - \\ -\frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right) dF_t \qquad (1)$$

- Equation 1 shows that the value of a realized variance for the time interval from 0 to T equals to:

$$\frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right) dF_t$$

- continuously rebalanced futures position. This position costs nothing to initiate and easy to replicate;

$$\frac{2}{T}\left(\log \frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right)$$

- **log contract**, static position of a contract that pays $f(F_T)$ at expiry and has to be replicated.

Carr and Madan (2002) argue that the market structure assumed above allows to represent any twice differentiable payoff function $f(F_T)$:

$$
\begin{aligned}
f(F_T) = f(k) + f'(k)\left([(F_T - k)^+ - (k - F_T)^+\right] \\
+ \int_0^k f''(K)(K - F_T)^+ dK \\
+ \int_k^\infty f''(K)(F_T - K)^+ dK
\end{aligned}
$$

For $f(F_T)$ expansion around F_0 we gives:

$$\log\left(\frac{F_0}{F_T}\right) + \frac{F_T}{F_0} - 1 =$$

$$= \int_0^{F_0} \frac{1}{K^2}(K - F_T)^+ dK + \int_{F_0}^{\infty} \frac{1}{K^2}(F_T - K)^+ dK$$

To obtain the strike K_{var}^2 of a variance swap take a risk-neutral expectation:

$$K_{var}^2 = \frac{2}{T}e^{rT}\int_0^{F_0}\frac{1}{K^2}P_0(K)dK + \frac{2}{T}e^{rT}\int_{F_0}^{\infty}\frac{1}{K^2}C_0(K)dK$$

But it is impossible to find vanilla options with a complete strike range (from 0 to ∞) traded on the market. How to replicate a fair future realized variance in reality?

Discrete approximation

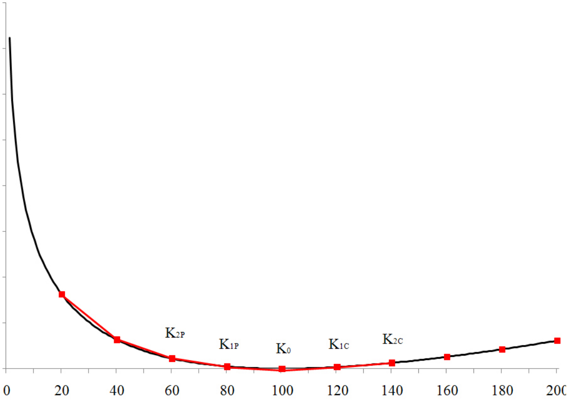


Figure 4: Discrete approximation of a log payoff

Discrete approximation

Dermann et al. (1998) proposed approach for finding weights for replicating portfolio of options:

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0}$$

The second segment - combination of call with strikes K_0 and K_{1c}:

$$w_c(K_{1c}) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0)$$

Setting up a replicating portfolio

Example: Constructing a variance swap

Suppose we are replicating a 3-month maturity DAX variance swap starting 07.09.2001. Index value is 4730, 3-month risk free rate (FIBOR/EURIBOR) 4.2%. The replicating portfolio consists of 13 OTM vanilla options: 7 puts and 6 calls with strikes 85 ip apart. Option prices are calculated using data from implied volatility surfaces of DAX vanilla options.

The prices, and numbers of options required for each strike are given in table below.

The total cost of replicating variance 534.67, K_{var}**=23.12**.

Payoff of the variance swaps

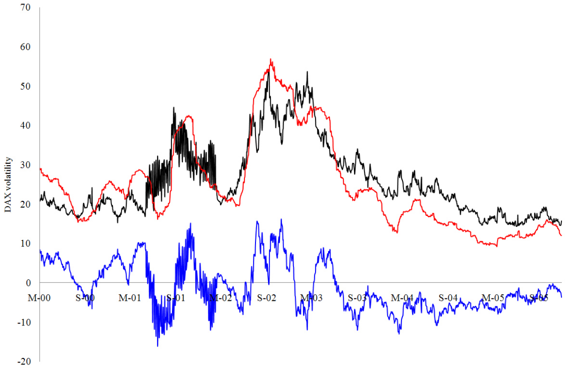


Figure 5: 3-month DAX variance swap strike, realized volatilty, payoff of a long swap position

Basket volatility

For a basket of any assets the variance is defined:

$$\sigma^2_{Basket} = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where σ_i - standard deviation of the return of an i-th constituent (also called volatility), w_i - weight of an i-th constituent in the basket, ρ_{ij} - correlation coefficient between the i-th and the j-th constituent.

Average index correlation

Assume $\rho_{ij} = \mathit{const}$ for any pair of i, j and call this parameter $\overline{\rho}$ - **average index correlation, or dispersion**:

$$\overline{\rho} = \frac{\sigma^2_{index} - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_i \sigma_j}$$

Dispersion trading strategy

From the previous definition follows the idea of dispersion trading. Long dispersion means:

- **short** index variance
- **long** variance of index constituents

Corresponding positions in variances can be taken by buying (selling) variance swaps.

- **short** swap on index realized variance
- **long** swaps on each of n index constituents

Payoff of dispersion strategy

The payoff of the direct dispersion strategy is a sum of variance swap payoffs of each of i-th constituent

$$(\sigma^2_{R,i} - K^2_{var,i}) \cdot N_i$$

and of short position in index swap

$$(K^2_{var,index} - \sigma^2_{R,index}) \cdot N_{index}$$

where

$$N_i = N_{index} \cdot w_i$$

Payoff of dispersion strategy

The payoff of the overall strategy is:

$$N_{index} \cdot \left(\sum_{i=1}^{n} w_i \sigma_{R,i}^2 - \sigma_{R,Index}^2 \right) - ResidualStrike$$

$$ResidualStrike = N_{index} \cdot \left(\sum_{i=1}^{n} w_i K_{var,i}^2 - K_{var,Index}^2 \right)$$

Implementing dispersion strategy

the success of the volatility dispersion strategy lies in determining:

- the direction of the strategy
- constituents for the offsetting variance basket

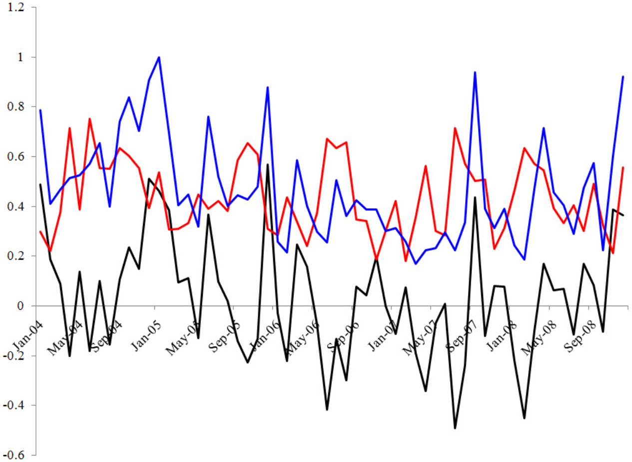


Figure 6: Average implied correlation (blue), average realized correlation (red), payoff of the direct dispersion strategy (black)

Basic vs improved dispersion strategies on DAX Index

Strategy	Mean	Std. Dev.	Skewness	Kurtosis	J-B	Prob.
Basic	0.032	0.242	0.157	2.694	0.480	0.786
Improved	0.077	0.232	-0.188	3.012	0.354	0.838

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