Time Varying Hierarchical Archimedean Copulae

Wolfgang Härdle
Ostap Okhrin
Yarema Okhrin

Institut für Statistik and Ökonometrie
CASE – Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin
Universität Bern

Correlation

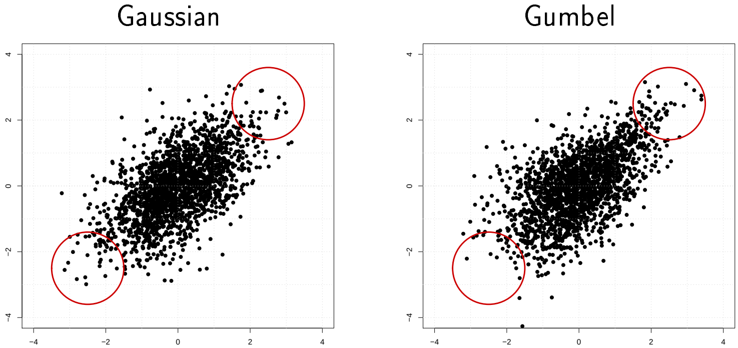


Figure 1: Scatterplots for two distributions with $\rho = 0.4$

- same marginal distributions
- same linear correlation coefficient

Simple AC over time

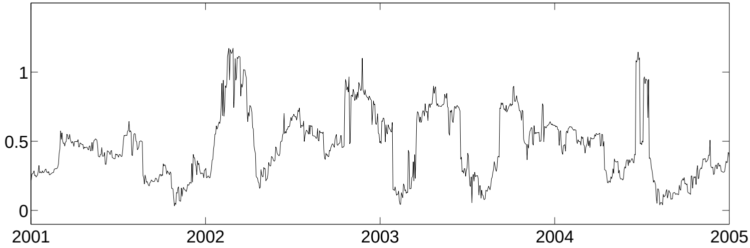


Figure 2: Estimated copula dependence parameter $\widehat{\theta}_t$ with the Local Change Point method for 6-dimensional data: DC, VW, Bayer, BASF, Allianz and Münchener Rückversicherung. Clayton Copula.
Giacomini et. al (2008)

Grid-type Copula

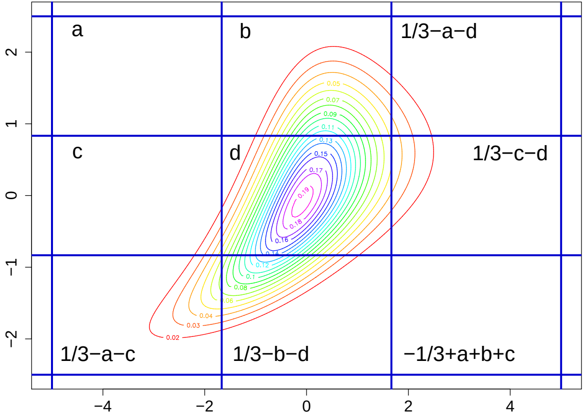


Figure 3: Grid-Type Copula approximation with $\mathfrak{a}_{i_1,\ldots,i_d}(n)$-matrix

Grid-type Copula

Grid-type copula with 9 subsquares: $dim = 2$ and $n = 3$.

$$\begin{bmatrix} a & b & 1/3 - a - b \\ c & d & 1/3 - c - d \\ 1/3 - a - c & 1/3 - b - d & -1/3 + a + b + c \end{bmatrix}$$

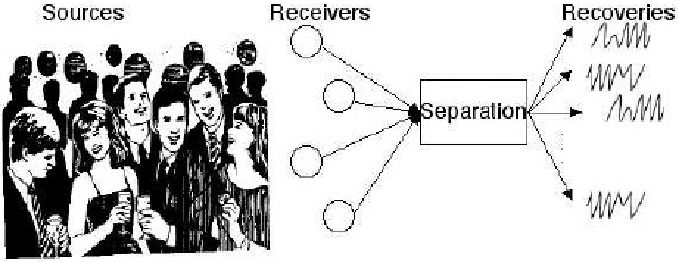
with suitable real numbers $a, b, c \in [0, 1/3]$ and

$$d = 1 - 4a - 2b - 2c.$$

For this choice $corr(X_1, X_2) = 0$!

COPICA

- Coctail-party problem



Blind Source Separation (BSS)

- Recover the original sources from their mixtures without knowing the mixing process
- Model: $\mathbf{X}(t) = A\mathbf{S}(t)$.
  - $\mathbf{X}(t) = \{x_1(t), \ldots, x_m(t)\}^\top$: observation at time t
  - $\mathbf{S}(t) = \{s_1(t), \ldots, s_M(t)\}^\top$: independent unknown sources, $s_1(t), \ldots, s_M(t)$, at time t
  - A: unknown mixing matrix
- Goal of BSS: Given $\mathbf{X}(1), \ldots, \mathbf{X}(T)$
  - Recover the original "independent" sources, $s_i(t), i = 1, \ldots, M, \; t = 1, \ldots, T$.
  - Infer the unknown mixing matrix A.

CDO Dynamics

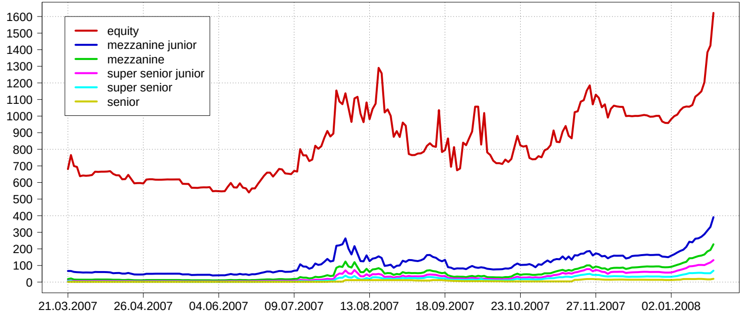


Figure 4: Time series of iTraxx spreads, Series 7, Maturity: 5 years, 21.03.2007-22.01.2008

Dependence Matters

The normal world is not enough.

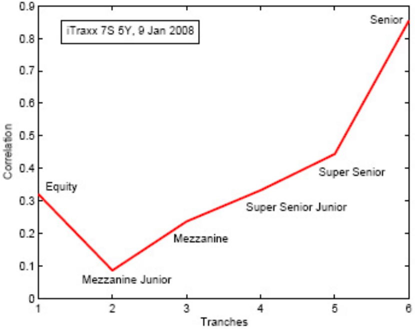


Figure 5: Gaussian one factor model with constant correlation.

Base Correlation Over Time

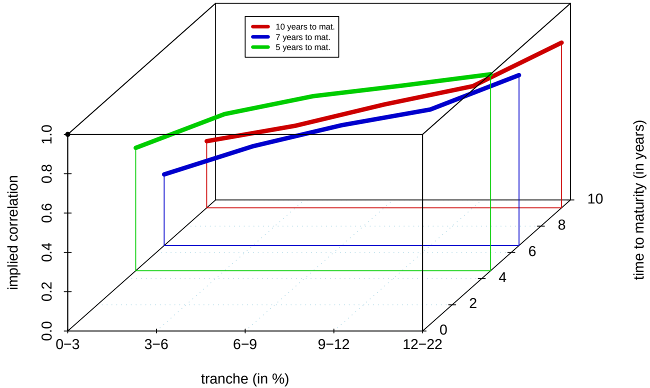


Figure 6: Film of base correlation over time.

Why are copulae important?

- interpretability
- margins
- flexible range of dependence
- closed-form representation of cdf and pdf
- fat tails
- dimension reduction

Main Idea

- combine interpretability with flexibility without loosing statistical precision
- determine the optimal structure of HAC for a given data set
- find the intervals of the homogeneity of the dependency

Outline

1. Motivation ✓
2. Archimedean copulae
3. Quality of the Fit
4. Copulae in Tempore Varintes
5. LCP for the HAC
6. References

Archimedean Copulae

Multivariate Archimedean copula $C : [0,1]^d \rightarrow [0,1]$ defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \dots + \phi^{-1}(u_d)\}, \qquad (1)$$

where $\phi : [0, \infty) \rightarrow [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.

Example

$$\phi_{Gumbel}(u, \theta) = \exp\{-u^{1/\theta}\}, \text{ where } 1 \le \theta < \infty$$

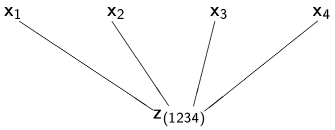
$$\phi_{Clayton}(u, \theta) = (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1, \infty) \backslash \{0\}$$

Disadvantages: too restrictive, single parameter, exchangeable

Hierarchical Archimedean Copulae

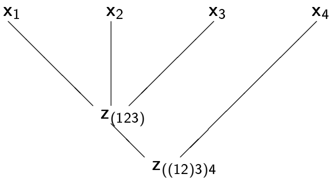
Simple AC with s=(1234)

$$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$$



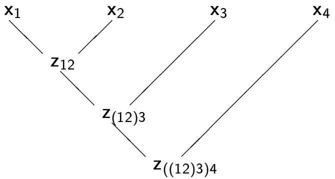
AC with s=((123)4)

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$$



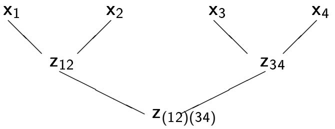
Fully nested AC with s=(((12)3)4)

$$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$$



Partially Nested AC with s=((12)(34))

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$$



Hierarchical Archimedean Copulae

Advantages of HAC:

- flexibility and wide range of dependencies:
  for $d = 10$ more than $2.8 \cdot 10^8$ structures
- dimension reduction:
  $d - 1$ parameters to be estimated
- subcopulae are also HAC

Hierarchical Archimedean Copulae

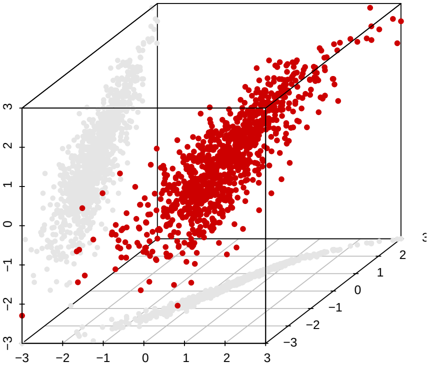


Figure 7: Scatterplot of the
$C_{Gumbel}[C_{Gumbel}\{\Phi(x_1), t_2(x_2); \theta_1 = 2\}, \Phi(x_3); \theta_2 = 10]$

Hierarchical Archimedean Copulae

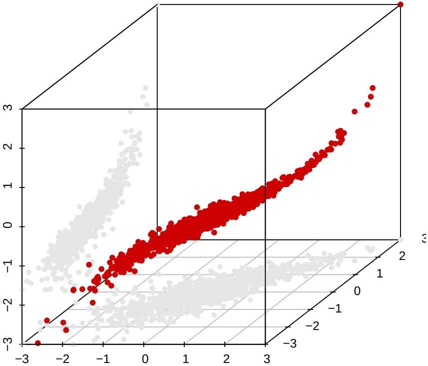


Figure 8: Scatterplot of the
$C_{Gumbel}[\Phi(x_2), C_{Gumbel}\{t_2(x_1), \Phi(x_3); \theta_1 = 2\}; \theta_2 = 10]$

Determining Structure

$$
\begin{array}{rcl|}
(12) & \rightsquigarrow & \widehat{\theta}_{12} \\
(13) & \rightsquigarrow & \widehat{\theta}_{13} \\
(14) & \rightsquigarrow & \widehat{\theta}_{14} \\
(23) & \rightsquigarrow & \widehat{\theta}_{23} \\
(24) & \rightsquigarrow & \widehat{\theta}_{24} \\
(34) & \rightsquigarrow & \widehat{\theta}_{34} \\
\hline
(123) & \rightsquigarrow & \widehat{\theta}_{123} \\
(124) & \rightsquigarrow & \widehat{\theta}_{124} \\
(234) & \rightsquigarrow & \widehat{\theta}_{234} \\
(134) & \rightsquigarrow & \widehat{\theta}_{134} \\
(1234) & \rightsquigarrow & \widehat{\theta}_{1234}
\end{array}
$$

Determining Structure

$$
\left.
\begin{array}{rcl}
(12) & \rightsquigarrow & \widehat{\theta}_{12} \\
(13) & \rightsquigarrow & \widehat{\theta}_{13} \\
(14) & \rightsquigarrow & \widehat{\theta}_{14} \\
(23) & \rightsquigarrow & \widehat{\theta}_{23} \\
(24) & \rightsquigarrow & \widehat{\theta}_{24} \\
(34) & \rightsquigarrow & \widehat{\theta}_{34} \\
\hline
(123) & \rightsquigarrow & \widehat{\theta}_{123} \\
(124) & \rightsquigarrow & \widehat{\theta}_{124} \\
(234) & \rightsquigarrow & \widehat{\theta}_{234} \\
(134) & \rightsquigarrow & \widehat{\theta}_{134} \\
(1234) & \rightsquigarrow & \widehat{\theta}_{1234}
\end{array}
\right| \text{ best fit (13) } \rightsquigarrow
$$

$$
z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}
$$

Determining Structure

$$
\left.\begin{array}{rcl}
(12) & \rightsquigarrow & \widehat{\theta}_{12} \\
\color{red}{(13)} & \rightsquigarrow & \widehat{\theta}_{13} \\
(14) & \rightsquigarrow & \widehat{\theta}_{14} \\
(23) & \rightsquigarrow & \widehat{\theta}_{23} \\
(24) & \rightsquigarrow & \widehat{\theta}_{24} \\
(34) & \rightsquigarrow & \widehat{\theta}_{34} \\
\hline
(123) & \rightsquigarrow & \widehat{\theta}_{123} \\
(124) & \rightsquigarrow & \widehat{\theta}_{124} \\
(234) & \rightsquigarrow & \widehat{\theta}_{234} \\
(134) & \rightsquigarrow & \widehat{\theta}_{134} \\
(1234) & \rightsquigarrow & \widehat{\theta}_{1234}
\end{array}\right|
\text{ best fit (13) } \rightsquigarrow
\left.\begin{array}{c}
\boxed{z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}} \\
\\
(13)2 \rightsquigarrow \widehat{\theta}_{(13)2} \\
(13)4 \rightsquigarrow \widehat{\theta}_{(13)4} \\
24 \rightsquigarrow \widehat{\theta}_{24} \\
\hline
(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}
\end{array}\right|
$$

Determining Structure

$(12) \rightsquigarrow \widehat{\theta}_{12}$
$(13) \rightsquigarrow \widehat{\theta}_{13}$
$(14) \rightsquigarrow \widehat{\theta}_{14}$
$(23) \rightsquigarrow \widehat{\theta}_{23}$
$(24) \rightsquigarrow \widehat{\theta}_{24}$
$(34) \rightsquigarrow \widehat{\theta}_{34}$

$(123) \rightsquigarrow \widehat{\theta}_{123}$
$(124) \rightsquigarrow \widehat{\theta}_{124}$
$(234) \rightsquigarrow \widehat{\theta}_{234}$
$(134) \rightsquigarrow \widehat{\theta}_{134}$
$(1234) \rightsquigarrow \widehat{\theta}_{1234}$

best fit (13) $\rightsquigarrow$

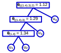
$$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$$

$(13)2 \rightsquigarrow \widehat{\theta}_{(13)2}$
$(13)4 \rightsquigarrow \widehat{\theta}_{(13)4}$
$24 \rightsquigarrow \widehat{\theta}_{24}$

$(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}$

best fit ((13)4) $\rightsquigarrow$

$$z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}$$



Determining Structure

$$
\begin{array}{l}
(12) \rightsquigarrow \widehat{\theta}_{12} \\
(13) \rightsquigarrow \widehat{\theta}_{13} \\
(14) \rightsquigarrow \widehat{\theta}_{14} \\
(23) \rightsquigarrow \widehat{\theta}_{23} \\
(24) \rightsquigarrow \widehat{\theta}_{24} \\
(34) \rightsquigarrow \widehat{\theta}_{34} \\
\hline
(123) \rightsquigarrow \widehat{\theta}_{123} \\
(124) \rightsquigarrow \widehat{\theta}_{124} \\
(234) \rightsquigarrow \widehat{\theta}_{234} \\
(134) \rightsquigarrow \widehat{\theta}_{134} \\
(1234) \rightsquigarrow \widehat{\theta}_{1234}
\end{array}
$$

best fit (13) $\rightsquigarrow$

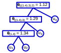
$$
\boxed{z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}}
$$

$$
\begin{array}{c}
(13)2 \rightsquigarrow \widehat{\theta}_{(13)2} \\
(13)4 \rightsquigarrow \widehat{\theta}_{(13)4} \\
24 \rightsquigarrow \widehat{\theta}_{24} \\
\hline
(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}
\end{array}
$$

best fit ((13)4) $\rightsquigarrow$

$$
\boxed{z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}}
$$

$$
((13)4)2 \rightsquigarrow \widehat{\theta}_{((13)4)2}
$$



Determining Structure

$$
\left.\begin{array}{rcl}
(12) & \rightsquigarrow & \widehat{\theta}_{12} \\
\color{red}{(13)} & \rightsquigarrow & \widehat{\theta}_{13} \\
(14) & \rightsquigarrow & \widehat{\theta}_{14} \\
(23) & \rightsquigarrow & \widehat{\theta}_{23} \\
(24) & \rightsquigarrow & \widehat{\theta}_{24} \\
(34) & \rightsquigarrow & \widehat{\theta}_{34} \\
\hline
(123) & \rightsquigarrow & \widehat{\theta}_{123} \\
(124) & \rightsquigarrow & \widehat{\theta}_{124} \\
(234) & \rightsquigarrow & \widehat{\theta}_{234} \\
(134) & \rightsquigarrow & \widehat{\theta}_{134} \\
(1234) & \rightsquigarrow & \widehat{\theta}_{1234}
\end{array}\right| \text{best fit (13)} \rightsquigarrow
\begin{array}{c}
\boxed{z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}} \\[1em]
(13)2 \rightsquigarrow \widehat{\theta}_{(13)2} \\
\color{red}{(13)4} \rightsquigarrow \widehat{\theta}_{(13)4} \\
24 \rightsquigarrow \widehat{\theta}_{24} \\
\hline
(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}
\end{array}
\Bigg| \text{best fit ((13)4)} \rightsquigarrow
\begin{array}{c}
\boxed{z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}} \\[1em]
\color{red}{((13)4)2} \rightsquigarrow \widehat{\theta}_{((13)4)2}
\end{array}
$$

Criteria for grouping: goodness-of-fit tests, parameter-based method, etc.

Estimation: multistage MLE with nonparametric and parametric margins

Data and Copula

daily returns of four companies listed in DAX index

company: Commerzbank (CBK), Merck (MRK), ThyssenKrupp (TKA) and Volkswagen (VOW)

timespan = [13.11.1998 - 18.10.2007] ($n = 2400$)

$\mathcal{M} = \{\phi = \exp(-u^{1/\theta})\}$ - Gumbel generator

Data and Copula

- GARCH-residuals are conditionally distributed with estimated copula

$$\varepsilon \sim C\{F_1(x_1), \dots, F_d(x_d); \theta_t\}$$

  where $F_1, \dots, F_d$ are marginal distributions and θ_t are the copula parameters.
- margins are $t_{3.163}$, $t_{3.420}$, $t_{3.023}$ and $t_{2.879}$ distributed

Changes of the Quality of the Fit over Time

$$ML = \sum_{i=1}^{n} \log\{f(u_{i1}, \ldots, u_{id}, \widehat{\theta})\},$$

where f denotes the joint multivariate density function.

$$AIC = -2ML + 2m, \qquad BIC = -2ML + 2\log(m),$$

where m is the number of parameters to be estimated.

Changes of the Quality of the Fit over Time

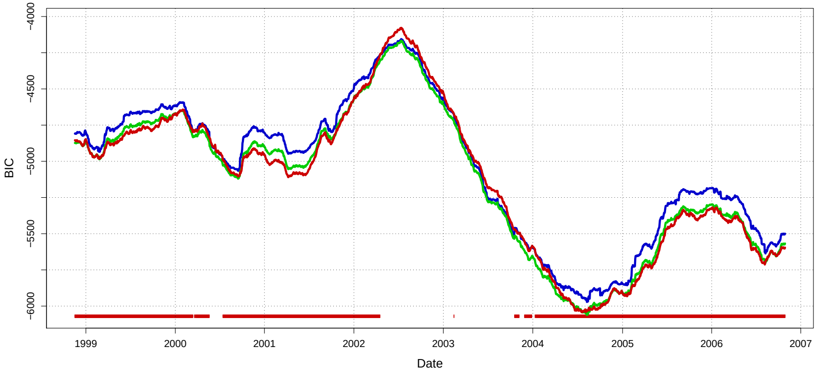


Figure 9: Time-varying HAC: BIC for the multivariate t distribution, multivariate N distribution and estimated HAC. Horizontal red line represents intervals where HAC-based distribution outperforms N

Changes of the Quality of the Fit over Time

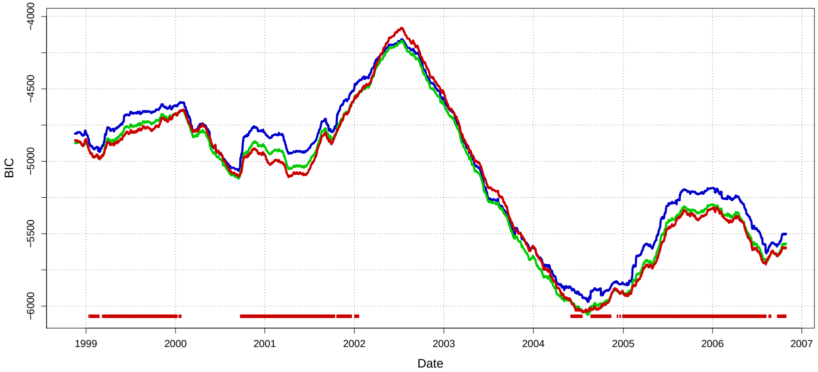


Figure 10: Time-varying HAC: BIC for the multivariate t distribution, multivariate $\mathbb{N}$ distribution and estimated HAC. Horizontal red line represents intervals where HAC-based distribution outperforms t

Changes of the Quality of the Fit over Time

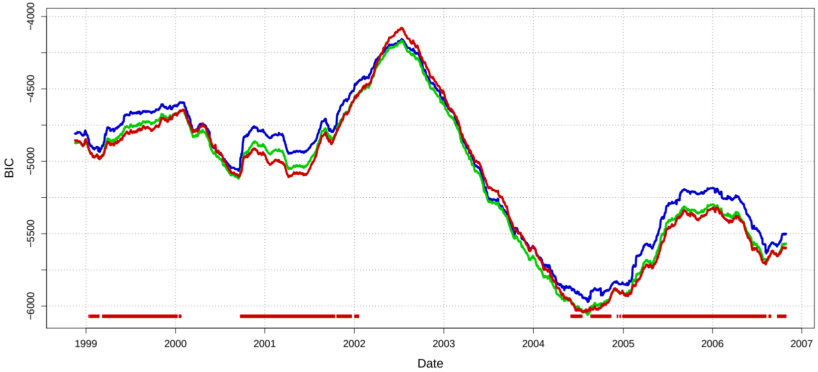


Figure 11: Time-varying HAC: BIC for the multivariate t distribution, multivariate $\mathbb{N}$ distribution and estimated HAC. Horizontal red line represents intervals where HAC-based distribution outperforms t and N

Copulae in tempore variantes

window for 250 days

$\mathbf{\Theta}_t(d \times d)$ - matrix of the pairwise θ based on the 250 days before t

$$
\begin{aligned}
||\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1}||_1 &= \max_{1 \le i \le d} \sum_{j=1}^{d} |\widehat{\theta}_{ij,t} - \widehat{\theta}_{ij,t-1}|, \\
||\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1}||_2 &= \sqrt{\lambda_{\max}\{(\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1})(\widehat{\mathbf{\Theta}}_t - \widehat{\mathbf{\Theta}}_{t-1})^\top\}}
\end{aligned}
$$

Copulae in tempore variantes

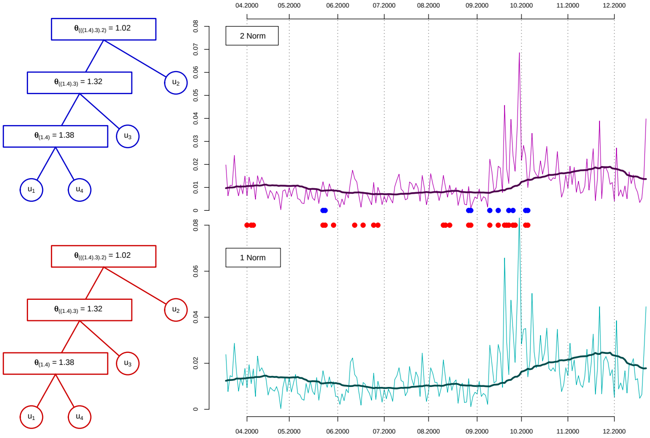
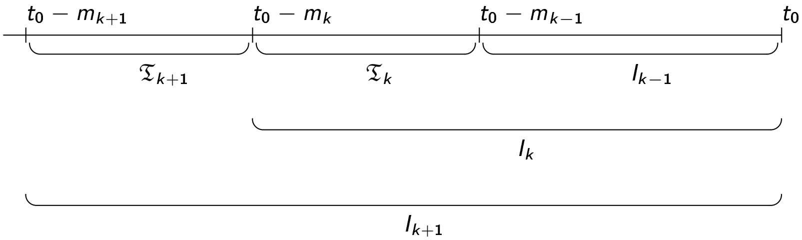


Figure 12: Film of time-varying HAC

Local Change Point Detection

1. define family of nested intervals
   $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ with length m_k as
   $I_k = [t_0 - m_k, t_0]$
2. define $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$



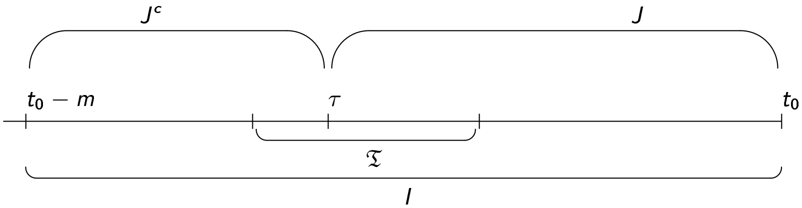
Local Change Point Detection

1. test homogeneity $H_{0,k}$ against the change point alternative in $\mathfrak{T}_k$ using I_{k+1}
2. if no change points in $\mathfrak{T}_k$, accept I_k. Take $\mathfrak{T}_{k+1}$ and repeat previous step until $H_{0,k}$ is rejected or largest possible interval I_K is accepted
3. if $H_{0,k}$ is rejected in $\mathfrak{T}_k$, homogeneity interval is the last accepted $\widehat{I} = I_{k-1}$
4. if largest possible interval I_K is accepted $\widehat{I} = I_K$

Test of homogeneity

Interval $I = [t_0 - m, t_0], \mathfrak{T} \subset I$

$$\begin{aligned} H_0 &: \forall \tau \in \mathfrak{T},\ \theta_t = \theta,\ s_t = s, \\ &\quad \forall t \in J = [\tau, t_0], \forall t \in J^c = I \setminus J \\ H_1 &: \exists \tau \in \mathfrak{T},\ \theta_t = \theta_1,\ s_t = s_1;\ \forall t \in J, \\ &\quad \theta_t = \theta_2 \neq \theta_1;\ s_t = s_2 \neq s_1, \forall J^c \end{aligned}$$



Test of homogeneity

Likelihood ratio test statistic for fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2}\{L_J(\theta_1) + L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= ML_J + ML_{J^c} - ML_I \end{aligned}$$

Test statistic for unknown change point location:

$$T_I = \max_{\tau \in \mathfrak{T}_I} T_{I,\tau}$$

Reject H_0 if for a critical value ζ_I

$$T_I > \zeta_I$$

Selection of I_k and ζ_k

- set of numbers m_k defining the length of I_k and $\mathfrak{T}_k$ are in the form of a geometric grid
- $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$, $m_0 = 20$ and $c = 1.25$, where $[x]$ means the integer part of x
- $I_k = [t_0 - m_k, t_0]$ and $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$
- estimated from the whole data sample structure
  $s^* = ((1.4)_{1.40}.3)_{1.36}.2)_{1.11}$ is set to be true
- ζ_l is selected by a simulation from the true structure s^*

LCP for HAC

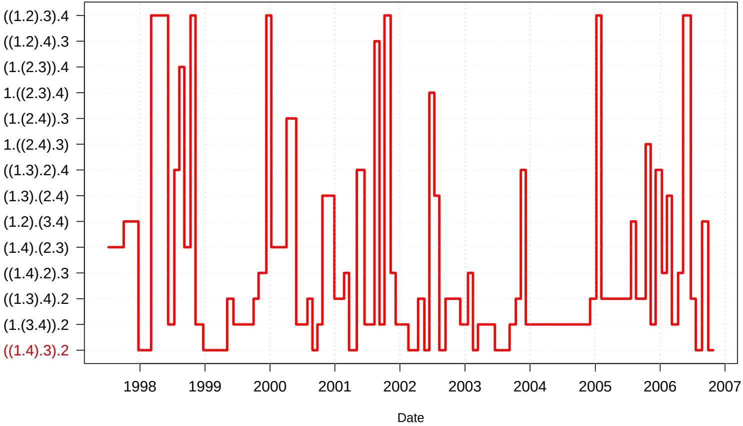


Figure 13: Structure of the estimated HAC on the intervals of homogeneity

LCP for HAC

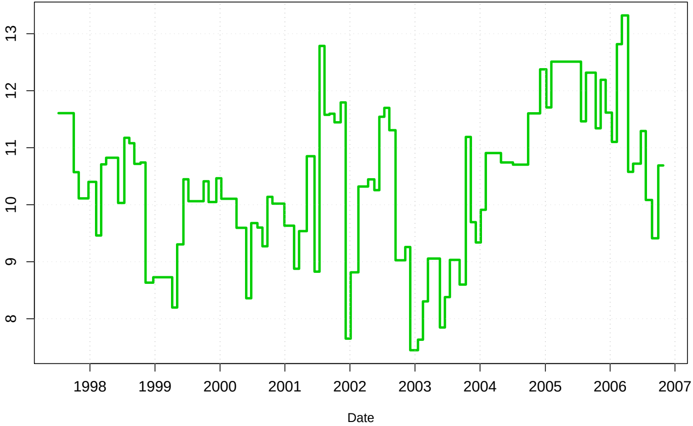


Figure 14: ML for the estimated HAC on the intervals of homogeneity

E. Giacomini, W. Härdle, V. Spokoiny
Inhomogeneous Dependency Modelling with Time Varying Copulae
J. Bus. Econ. Statist., 2008

H. Joe
Multivariate Models and Dependence Concepts
London: Chapman&Hall, 1997.

D. Mercurio, and V. Spokoiny
Estimation of Time Dependent Volatility via Local Change Point Analysis
Ann. Statist., **32**: 577-602, 2004

R. Nelsen
An intoduction to copulas
Springer, 1999

O. Okhrin, Y. Okhrin, W. Schmid
On the structure and estimation of hierarchical Archimedean copulas
submitted to J. Econometrics

D. Straßburger and D. Pfeifer
Dependence Matters!

C. Bluhm, L. Overbeck and C. Wagner
An Introduction to Credit Risk Modeling
CRC Press, 2002

M. Feld
Implied Correlation Smile
MSc Thesis, http://edoc.hu-berlin.de/