Tying the straps tighter for generalized linear models

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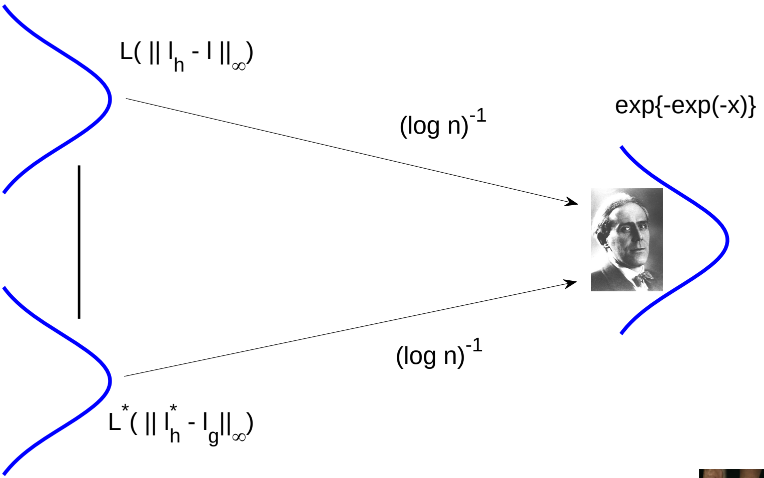
Motivation

- Bootstrap improves finite sample performance
- Regression Problem: resample from residuals
- For nonlinear functionals (e.g. sup), little improvement
- $\|l_n(x) - l(x)\|_\infty \stackrel{\text{def}}{=} \sup_x |l_n(x) - l(x)|$, where $l(x)$ is a mean or quantile curve, and $l_n(x)$ is a quantile or mean smoother

Challenges



Song et. al. (2012)

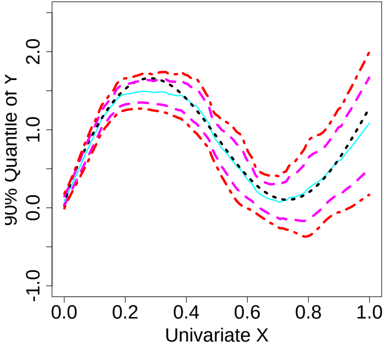


Figure 1: The real 0.9 quantile curve, 0.9 quantile estimate with corresponding 95% uniform confidence band from asymptotic theory and confidence band from bootstrapping.

Theorem (Song et. al. (2010))

An approximate $(1-\alpha)\times 100\%$ *confidence band over* $[0,1]$ *is*

$$\begin{aligned} l_h(t) \quad \pm \quad & (nh)^{-1/2}\{p(1-p)/\hat{f}_X(t)\}^{1/2}\hat{f}^{-1}\{l_h(t)|t\} \\ & \times\{d_n + c(\alpha)(2\delta\log n)^{-1/2}\}\cdot\{\lambda(K)\}^{1/2}, \qquad (1) \end{aligned}$$

where $c(\alpha) = \log 2 - \log|\log(1-\alpha)|$ *and* $\hat{f}_X(t)$, $\hat{f}\{l_h(t)|t\}$ *are consistent estimates for* $f_X(t)$, $f\{l(t)|t\}$.

n	Cov. Prob.
50	0.144 (0.642)
100	0.178 (0.742)
200	0.244 (0.862)

Table 1: Simulated coverage probabilities based on asymptotics (bootstrap methods).

Bootstrap Improvement

- Bootstrap improvements for the quantile smoother have been shown
- Quantile estimator is of Bounded Influence
- M- smoothers or a general QMLE may be expected to have same performance?
- How to handle high dimensional objects?

Opportunities

Extend this to $x \in \mathbb{R}^d$ and improve band precision?

- Hall (1991): bootstrap improvement
- Hahn (1995): consistency of bootstraping cdf
- Horowitz (1998): bootstrap (pointwise) for median
- Additive Models: Horowitz (2001), Horowitz and Lee (2005) and Stone(1985)

How to tie the straps tighter?

Outline

1. Motivation $\checkmark$
2. Bootstrap Confidence Bands
3. Bootstrap Confidence Bands for Additive Models
4. Monte Carlo Study
5. Applications

Regression with a general loss function

- $\{(X_i, Y_i)\}_{i=1}^n$ i.i.d. rv's, $x \in J^* = (a, b)$
- Suppose $Y_i = l(X_i) + \varepsilon_i$, $\varepsilon_i \sim F_{\varepsilon|X_i}(\cdot)$. Both l & F are smooth.
- Suppose

$$l(x) = \arg\min_\theta \mathsf{E}_{(Y|X=x)} \rho(Y - \theta), \tag{2}$$

where $\rho(.)$ is a loss function of Hampel/Huber type or more generally a negative (quasi) log likelihood.

Robust Statistics

For $\rho(.)$ is a trimmed mean (Tukey's biweight),

$$\rho(x) = \begin{cases} x^2, & |x| \leq k, \\ k^2, & |x| > k \end{cases} \tag{3}$$

or a form of Winsorized mean (Huber):

$$\rho(x) = \left\{ \begin{array}{rl} x^2/2, & |x| \leq k, \\ -k^2/2 + k|x|, & |x| > k. \end{array} \right\}. \tag{4}$$

The Bootstrap Couple

- $\{U_i\}_{i=1}^n$: i.i.d. uniform $U[0,1]$ rv's
- Bootstrap sample

$$Y_i^* = l_g(X_i) + \hat{F}^{-1}_{(\varepsilon_i|X_i)}(U_i), \quad i = 1, \ldots, n$$

- Couple with the true conditional distribution:

$$Y_i^{\#} = l(X_i) + F^{-1}_{(\varepsilon_i|X_i)}(U_i), \quad i = 1, \ldots, n.$$

Given $X_1, \ldots, X_n$: $Y_1, \ldots, Y_n$ and $Y_1^{\#}, \ldots, Y_n^{\#}$ are equally distributed.

Bootstrapping Approximation Rate

Theorem

If assumptions (A1)–(A3) hold, then

$$\sup_{x\in J^*} |l_h^*(x) - l_g(x) - l_h(x) + l(x)| = \mathcal{O}_p(\delta_n) = \mathcal{O}_p(h^2\Gamma_n),$$

with Γ_n a slowly varying sequence (a sequence a_n is slowly varying if $n^{-\alpha}a_n \to 0$ for any $\alpha > 0$).

Bootstrap improves the rate of convergence

Sketch of the Proof

- The basic elements: smoothness of $F_{Y|X}(\cdot)$ and bounded influence of $\rho(.)$
-
$$
\begin{aligned}
\max_i |\varepsilon_i^\sharp - \varepsilon_i^*| &= \mathcal{O}_p(h^2\Gamma_n)\\
\max_i |\psi(\varepsilon^*) - \psi(\varepsilon^\sharp)| &= \mathcal{O}_p(h^2\Gamma_n)\\
\max_{||X_i - X_j|| \le ch} |l_g(X_i) - l_g(X_j) - \{l(X_i) - l(X_j)\}| &= \mathcal{O}_p(h^2\Gamma_n)
\end{aligned}
$$
-
$$
|\hat{F}^{-1}(u|X_i) - F^{-1}(u|X_i)| \le h^2\Gamma_n, \forall u \in B.
$$
-
$$
\mathsf{E}_{\hat{F}_{\varepsilon|x=X_i}} \psi(\varepsilon_i^*) = 0 = \mathsf{E}_{F_{\varepsilon|x=X_i}} \psi(\varepsilon^\sharp) \tag{5}
$$

$$
\begin{aligned}
G_n^*(\theta, X_i) &\stackrel{\text{def}}{=} n^{-1} \sum_{j=1}^{n} W_{h,j}(X_i)[\psi\{\varepsilon_j^* - \theta + \hat{l}_g(X_j)\}] \\
&= n^{-1} \sum_{j=1}^{n} W_{h,j}(X_i)\{\psi(Y_j^* - \theta)\} \\
G_n^\sharp(\theta, X_i) &\stackrel{\text{def}}{=} n^{-1} \sum_{j=1}^{n} W_{h,j}(X_i)[\psi\{\varepsilon_j^\sharp - \theta + l(X_j)\}] \\
&= n^{-1} \sum_{j=1}^{n} W_{h,j}(X_i)\{\psi(Y_j^\sharp - \theta)\} \\
T_{n,1}(X_i) &\stackrel{\text{def}}{=} G_n^*\{\hat{l}_g(.), X_i\} - G_n^\sharp\{l(.), X_i\}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{E}_{F(\varepsilon|X_i)} T_{n,1}(X_i) &= 0 \\
\mathrm{Var}_{(\varepsilon|X_i)} T_{n,1}(X_i) &= n^{-2} \sum_{j=1}^{n} W_{h,j}^2(X_i) \, \mathrm{Var}\{\psi(\varepsilon_j^*) - \psi(\varepsilon_i^\sharp)\} \\
&= \mathcal{O}_p(n^{-1} h^3)
\end{aligned}
$$

$$
\begin{aligned}
\hat{l}_{h,g}^*(X_i) - \hat{l}_g(X_i) &= -\frac{G_n^*\{\hat{l}_g(X_i), X_i\}}{G_n'^*\{\hat{l}_{h,g}^*(X_i), X_i\}} + o_p(h^2), \qquad (6) \\
\hat{l}_h(X_i) - l(X_i) &= -\frac{G_n^\sharp\{l(X_i), X_i\}}{G_n'^\sharp\{\hat{l}_h(X_i), X_i\}} + o_p(h^2). \qquad (7)
\end{aligned}
$$

Why Oversmoothing?

- Take care of bias with tuning parameter: g
- Härdle and Marron (1991), let

$$
\begin{aligned}
b_h(x) &\stackrel{\mathrm{def}}{=} \mathsf{E}\, l_h^{\#}(x) - l(x) \\
\hat{b}_{h,g}(x) &\stackrel{\mathrm{def}}{=} \mathsf{E}^*\, l_h^*(x) - l_g(x)
\end{aligned}
$$

- Investigate $\mathsf{E}\left[\left\{\hat{b}_{h,g}(x) - b_h(x)\right\}^2 | X_1, \ldots, X_n\right]$.
  How fast it converges to 0?

Oversmoothing

Theorem

Under some assumptions, for any $x \in J^*$

$$\mathrm{E}\left[\left\{\hat{b}_{h,g}(x) - b_h(x)\right\}^2 | X_1, \ldots, X_n\right] \sim h^4\{\mathcal{O}_p(g^4) + \mathcal{O}_p(n^{-1}g^{-5})\}$$

To minimize RHS, $g = \mathcal{O}(n^{-1/9})$, $g \gg h$, where $h = \mathcal{O}(n^{-1/5})$

The Multivariate Case

$$m(X_i) = \sum_{j=1}^{d} m_j(x_{i,j}), \tag{8}$$

Estimate the additive model via a basis function approach:

$$m_j(x_{i,j}) \approx \sum_{l=1}^{L_j+1} a_{l,j} g_l(x_{i,j}),$$

$\psi(x_{i,j})$s are B-splines, e.g. linear B-splines.

$$\psi_l(x) = \begin{cases} Hx - l + 1 & (l-1)H^{-1} \le x \le lH^{-1} \\ l + 1 - Hx & lH^{-1} \le x \le (l+1)H^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Bootstrap Couple

Quantile

$$Z_i = \begin{cases} 1 & \text{with prob } \tau \\ -1 & \text{with prob } 1-\tau \end{cases},$$

where $\tau = 1/2$ is for symmetric error distribution. The bootstrap couple ε^* (the bootstrap residuals) and $\varepsilon^\sharp$ (the theoretical couple) are:

$$\varepsilon_i^* = Z_i |\hat{\varepsilon}_i| \tag{9}$$

$$\varepsilon_i^\sharp = Z_i \eta_i \tag{10}$$

Bootstrap

$$F_{i,s}(t) \stackrel{\text{def}}{=} \mathrm{P}(|\varepsilon_i| \le t | s\varepsilon_i > 0), \quad i = 1, \ldots, n, \ s \in \{1, -1\}, \tag{11}$$

$$\eta_i \stackrel{\text{def}}{=} F^{-1}_{i,Z_i}\{F_{i,sgn(\varepsilon_i)}(|\varepsilon_i|)\}, \quad i = 1, \ldots, n. \tag{12}$$

Recall $F_{Y|X=x_i}\{l(X_i)\} = \tau$ and $F_{\varepsilon|X=x_i}(0) = \tau$. $F_{i,\mathrm{sgn}(\varepsilon_i)}(|\varepsilon_i|)$ standard uniform

$$F_{i,+1}(t) = \frac{F_i(t) - 1 + \tau}{\tau}, F_{i,-1}(t) = \frac{1 - \tau - F_i(-t)}{1 - \tau}.$$

$$\mathcal{L}(\varepsilon_i^\sharp) = \mathcal{L}(\varepsilon_i). \tag{13}$$

Go to details

Uniform Consistency

Theorem

Let assumptions A.1- A.9 be fulfilled, then

$$\sup_x \left|(\hat{m}_j - m_j)(x) - \{(\hat{m}_j^* - \hat{m}_j)(x)\}\right| = o_p(h^2\Gamma_n),$$

Go to conditions

How to Bootstrap?

- Simulate $\{X_i, Y_i\}_{i=1}^n$,

$$f(x, y) = g\{y - \sin(\pi x)\}\mathbf{1}(x \in [0, 1]) \tag{14}$$

$$g(u) = 9\varphi(u)/10 + \varphi(u/9)/90 \tag{15}$$

- Compute the smoother (Tukey biweight) $\hat{l}_h(x)$, $\hat{\varepsilon}_i = Y_i - \hat{l}_h(X_i)$
- Compute the conditional edf:

$$\hat{F}_{\varepsilon|X_i=x}(t) = \frac{\sum_{i=1}^n K_h(x - X_i)\mathbf{1}(\hat{\varepsilon}_i \le t)}{\sum_{i=1}^n K_h(x - X_i)}$$

with Gaussian kernel

$$K_h(u) = (\sqrt{2\pi})^{-1} \exp\{-u^2/2h\}/h,$$

$h = 0.06$.

- For each $i = 1, \ldots, n$, generate random variable $\varepsilon^* \sim \hat{F}(t|x)$, $i^* = 1, \ldots, n^*$, where n^* is the bootstrap sample size:

$$Y_{i,i^*} = \hat{l}_g(X_i) + \varepsilon^*_{i,i^*},$$

with $g = 0.2$.

- For each sample X_i, Y_i^*, compute $\hat{l}_h^*(.)$ and the random variable

$$d_{i^*} \stackrel{\text{def}}{=} \sup_{x \in J^*} \{\sqrt{\hat{f}_X(x)}\, \mathsf{E}_{y|x}\{\psi'(\varepsilon_i^*)\} / \sqrt{\mathsf{E}_{y|x}\{\psi^2(\varepsilon_i^*)\}} |\hat{l}_h^*(x) - \hat{l}_g(x)|\}$$

- Calculate the $1-\alpha$ quantile d_α^* of $d_1, \ldots, d_{n^*}$.
- Construct the bootstrap uniform band centered around $m_h(x)$

$$\hat{l}_h(x) \pm [\sqrt{\hat{f}_X(x)}\, \mathsf{E}_{y|x}\{\psi'(\varepsilon_i^*)\} / \sqrt{\mathsf{E}_{y|x}\{\psi^2(\varepsilon_i^*)\}}]^{-1} d_\alpha^*$$

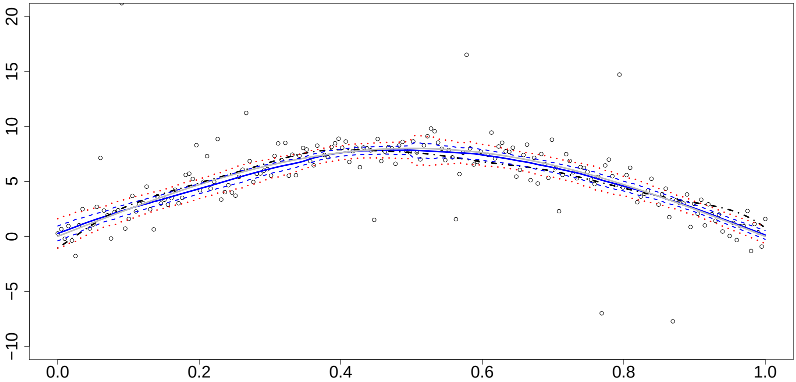


Figure 2: Plot of true curve (grey), BI estimation and bands (blue), local polynomial estimation (black), bootstrap band (red)

n	Cov. Prob.	Area
100	0.88(0.98)	1.23(2.51)
200	0.89(0.98)	0.89(1.95)
400	0.90(0.96)	0.78(1.32)

Table 2: Simulated coverage probabilities areas of nominal asymptotic (bootstrap) 95% confidence bands with 100 repetition.

Bootstrap for Additive Models

- Simulate $\{X_i, Y_i\}_{i=1}^{n}$. The variable $X_i = (x_{1i}, x_{2i}, x_{3i}, x_{4i}) \sim U(-2.5, 2.5)$,

  $m_1(x_1) = \sin(\pi x_1), m_2(x_2) = \Phi(3x_2), m_3(x_3) = x_3^3, m_4(x_4) = x_4^4,$

  and ε_i is as in (15).
- Compute $\hat{m}_1(x_1), \hat{m}_2(x_2), \hat{m}_3(x_3), \hat{m}_4(x_4)$ and $\hat{\varepsilon}_i = Y_i - \sum_{j=1}^{4} \hat{m}_l(x_{i,j})$.
- For each $i = 1, \ldots, n$, generate random variable ε_i^*, $i^* = 1, \ldots, n^*$ as in (9), where n^* is the bootstrap sample size:

$$Y_{i,i^*} = \sum_{j=1}^{4} \hat{m}_l(x_{i,j}) + \varepsilon_{i,i^*}^*.$$

- For each sample $(x_{1i}, x_{2i}, x_{3i}, x_{4i}, Y_i^*)$, compute $m_j^*(.)$ and the random variable

$$d_{i^*} \stackrel{\mathrm{def}}{=} \sup_{x \in J^*} \{\sqrt{\hat{f}_{x_j}(x_j)}\, \mathsf{E}_{\,y|x}\{\psi'(\varepsilon_i^*)\}/\sqrt{\mathsf{E}_{\,y|x}\{\psi^2(\varepsilon_i^*)\}}|\hat{m}_j^*(x) - \hat{m}_j(x)|\}$$

- Calculate the $1-\alpha$ quantile d_α^* of $d_1, \ldots, d_{n^*}$.
- Construct the bootstrap uniform band centered around $\hat{m}_j(x_j)$

$$\hat{m}_j(x_j) \pm [\sqrt{\hat{f}_{x_j}(x_j)}\, \mathsf{E}_{\,y|x_j}\{\psi'(\varepsilon_i^*)\}/\sqrt{\mathsf{E}_{\,y|x_j}\{\psi^2(\varepsilon_i^*)\}}]^{-1} d_\alpha^*$$

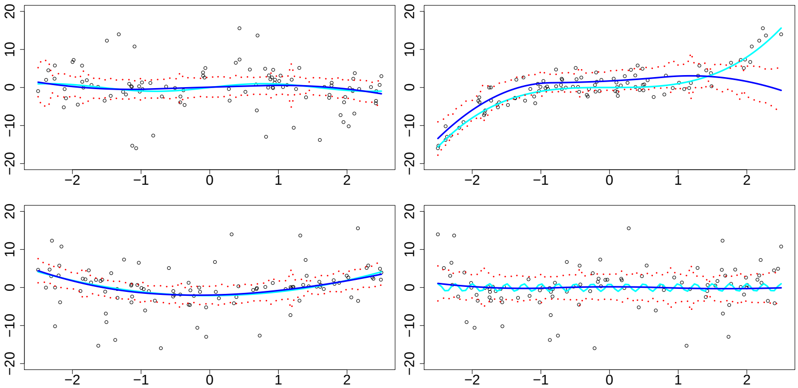


Figure 3: Plot of true curve (dark blue), robust estimation and bands (cyran), bootstrap band (red dotted)

n	Cov. Prob.	Area
100	0.95, 0.98, 0.83, 0.95	6.06, 5.37, 5.44, 5.21
200	0.88, 0.95, 0.93, 0.88	5.50, 4.74, 4.54, 4.65
400	0.84, 0.95, 0.96, 0.84	4.83, 3.63, 3.76, 3.70

Table 3: Simulated coverage probabilities areas of nominal asymptotic (bootstrap) 95% confidence bands with 100 repetition, 100 bootstrap sample.

n	Cov. Prob.	Area
100	0.89, 0.94, 0.85, 0.92	5.88, 5.07, 5.04, 5.30
200	0.90, 0.95, 0.86, 0.88	4.84, 3.84, 3.85, 4.00
400	0.85, 0.90, 0.92, 0.84	4.02, 3.25, 3.11, 3.03

Table 4: Simulated coverage probabilities areas of nominal asymptotic (bootstrap) 90% confidence bands with 100 repetition, 100 bootstrap sample.

Firm expenses analysis

- Joel L. Horowitz and Sokbae Lee (2005)
- Whether a concentrated shareholding is associated with lower expenditure on activities
- The dependent variable Y: general sales and administrative expenses deflated by sales (denoted by MH5)
- The covariates: ownership concentration (denoted by TOPTEN, cumulative shareholding by the largest ten shareholders), firm characteristics: the log of assets, firm age, and leverage (the ratio of debt to debt plus equity)
- $n = 185$

$$
\begin{aligned}
MH5 &= \beta_0 + m_1(TOPTEN) + m_2\{log(Assets)\} \\
&\quad + m_3(Age) + m_4(Leverage) + error
\end{aligned}
$$

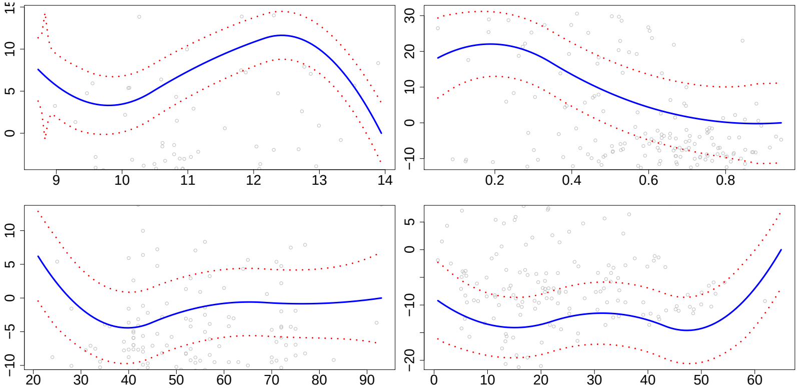


Figure 4: Robust estimation (blue), bootstrap band (red dotted), left up: Log(Asset), right up: Leverage, left below: Age, right below: Leverage.

The impact on stock market

- (Oil, currency, bond, real estate) affect the stock market.
- (http://www.lnstatistical.com /Main.jsp;jsessionid=009E36E74DFA15C80B74EE0BDAEB5746)
- The X variables are: the crude oil price, EUR- USD exchange rate, the 10 year treasury constant maturity inflation index %, and the y variable is S&P 500 index returns.
- $20080903 - 20111128$, $n = 170$

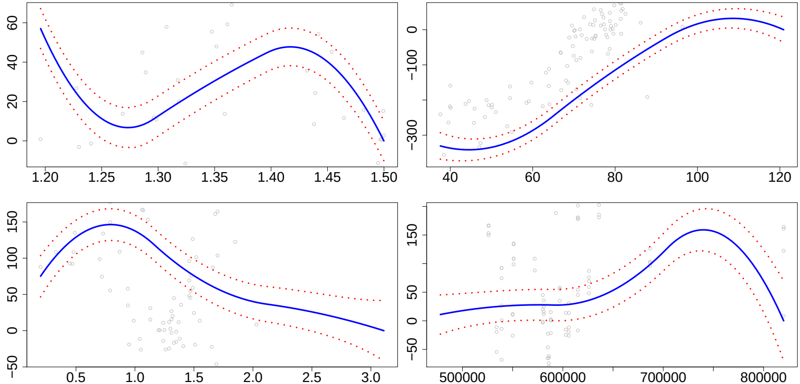


Figure 5: Robust estimation (blue), bootstrap band (red dotted), y S&P index, left up: exchange rates EUR-USD, right up: crude oil price, left below: inflation index, right below: real estate price.

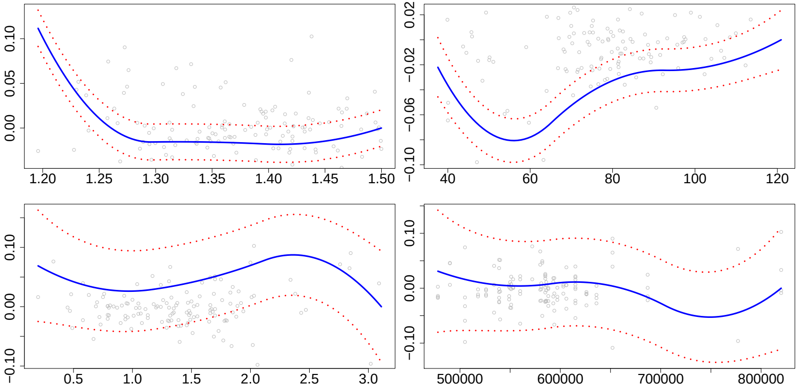


Figure 6: Robust estimation (blue), bootstrap band (red dotted), y S&P index return, left up: exchange rates EUR-USD, right up: crude oil price, left below: inflation index, right below: real estate price.

- Exchange rate (EUR-USD): (< 1.27) negatively correlated with the stock indices, (> 1.43) a positive correlation follows
- Oil prices: negative impact
- Inflation index: the inflation rate high, interest rates typically high; A negative correlation >0.7
- The stock indices raise when the real estate prices gets higher
- ...

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Appendix - Assumptions

A.1 $\psi(.) = \rho'(.)$ being a.s. differentiable and Lipschitz continuous: $\forall \mu_1, \mu_2 \in B$ $|\psi(\mu_1) - \psi(\mu_2)| < C|\mu_1 - \mu_2|$, and we assume that $\exists M > 0$ s.t.$\psi(\mu) \leq M$.
A.2 The support of X is $[0,1]^d$. The conditional density $f_{(y|X=x)(.)}$ is bounded from below $\infty > C_1 > \inf_t f_{(y|X=x)}(t) = c_1 > 0$.
A.3 The kernel function $K(.)$ is a product kernel composed from one dimension kernel with bandwidth $h = h_n$:

$$K_h(s) = \Pi_{j=1}^d K(s_j/h)/h, s = (s_1, \ldots, s_d)^\top \in \mathbb{R}^d. \qquad (16)$$

Appendix - Assumptions

A.4 The bandwidth satisfies $h \sim n^{-1/(4+d)}$. Let g be another bandwidth sequence $g >> h$. (Work our later the speed). Let Γ_n be a slowly increasing sequence in the sense that $n^{-\alpha} a_n \to 0$ for any $\alpha > 0$.

A.5 Assume $\sup_{x \in B} |\hat{l}''_g(x) - l''(x)| = \mathcal{O}_p(1)$, and $\sup_{x \in B} |\hat{l}'(x) - l'(x)| = \mathcal{O}_p(h^2 \Gamma_n)$.

A.6 There is an $\alpha > 0$ such that

$$\begin{aligned} \mathrm{E}_X \{\sum_{i=1}^{d} m_j(X_j)\}^2 &\geq \alpha \max_j \mathrm{E}_{X_j}\{m_j^2(X_j)\} \\ \mathrm{E}_{X_j}\{m_j(X_j)\} &= 0, m_j(\cdots) \in L_2(X_j). \end{aligned}$$

A.7 $\mathsf{E}_{X_j}\{\psi_l^2(x_{i,j})\} = 1$ for any $i \in 1, \ldots, n$ and $j \in 1, \ldots, d$.
$||\Phi_l(X_j)||_\infty \leq C_3/L, a.s.$, where $\Phi_l(X_j)), \ldots, \phi_l^2(x_{n,j})\}^\top$.

A.8 The inverse link function b' satisfies the following: $b' \in C(\mathbb{R})$, $b''(\theta) > 0, \theta \in \mathbb{R}$ while for a compact interval Θ whose interior contains $m([0,1]^d)$, $C_b > \max_{\theta \in \Theta} b''(\theta) > \min_{\theta \in \Theta} b''(\theta) > c_b$ for $C_b > c_b > 0$

A.9 The number of regressors $p = dL + 1$ (more precisely $L = L_j$) with $L \sim n^{1/5}$.

Go back

$$
\begin{aligned}
\mathrm{P}(\varepsilon_i^\sharp < t) &= \tau\,\mathrm{P}[F_{i,+1}^{-1}\{F_{i,\sim\eth\ltimes(\varepsilon_i)}(|\varepsilon_i|)\} < t] + 1 - \tau \\
&= \tau\,\mathrm{P}\{F_{i,\sim\eth\ltimes(\varepsilon_i)}(|\varepsilon_i|) < F_{i,+1}(t)\} + 1 - \tau \\
&= \tau\,\mathrm{P}\{\varepsilon_i < 0, F_{i,-1}(-\varepsilon_i) < F_{i,+1}(t)\} \\
&\qquad + \tau\,\mathrm{P}\{\varepsilon_i > 0, F_{i,+1}(\varepsilon_i) < F_{i,+1}(t)\} + 1 - \tau \\
&= \tau\,\mathrm{P}\{\varepsilon_i < 0, \frac{1 - \tau - F_i(\varepsilon_i)}{1 - \tau} < \frac{F_i(t) - 1 + \tau}{\tau}\} \\
&\qquad + \tau\,\mathrm{P}(0 < \varepsilon_i < t) + 1 - \tau \\
&= \tau\,\mathrm{P}[1 - \tau > F_i(\varepsilon_i) > \frac{1-\tau}{\tau}\{1 - F_i(t)\}] \\
&\qquad + \tau\,\mathrm{P}(0 < \varepsilon_i < t) + 1 - \tau \\
&= \tau[1 - \frac{1-\tau}{\tau}\{1 - F_i(t)\} - \tau] + \tau\{F_i(t) - 1 + \tau\} + 1 - \tau \\
&= F_i(t).
\end{aligned}
$$