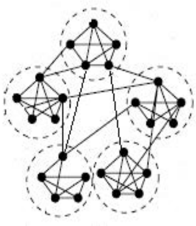
Quantile Regression in Risk Calibration

Shih-Kang Chao
Wolfgang Karl Härdle
Weining Wang

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. - Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de
http://www.case.hu-berlin.de

Dependence Risk



Risk Calibration and Quantile Regression

- Quantification via value-at-risk (VaR)/expected shortfall (ES)
- Quantile VaR: dependence risk?
- Parametric VaR: Chernozhukov and Umantsev (2001), Engle and Manganelli (2004)
- Nonparametric VaR: Cai and Wang (2008), Taylor (2008) and Schaumburg (2010)
- Parametric CoVaR: Adrian and Brunnermeier (2010)(AB)

Risk Calibration

- Marginal Expected Shortfall (MES): Acharya et al. (2010)
- Distressed Insurance Premium (DIP): Huang et al. (2010)

▸ Go to details

- AB: X_j and X_i are two asset returns,

$$\mathsf{P}\left\{ X_j \leq \mathrm{CoVaR}^{\tau}_{j|i} \middle| X_i = \mathrm{VaR}^{\tau}(X_i), M_{t-1} \right\} = \tau.$$

- Advantages:
  - Cloning property
  - Conservative property
  - Adaptiveness

▸ Go to details

CoVaR Construction (AB)

$X_{j,t}$ and $X_{i,t}$ are two asset returns. Two linear quantile regressions:

$$X_{i,t} = \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t}, \tag{1}$$

$$X_{j,t} = \alpha_{j|i} + \beta_{j|i} X_{i,t} + \gamma_{j|i}^\top M_{t-1} + \varepsilon_{j,t}. \tag{2}$$

M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau|M_{t-1}, X_{i,t}) = 0$.

$$\widehat{VaR}_{i,t} = \hat{\alpha}_i + \hat{\gamma}_i^\top M_{t-1},$$

$$\widehat{CoVaR}_{j|i,t} = \hat{\alpha}_{j|i} + \hat{\beta}_{j|i} \widehat{VaR}_{i,t} + \hat{\gamma}_{j|i}^\top M_{t-1}.$$

CoVaR Construction Linear?

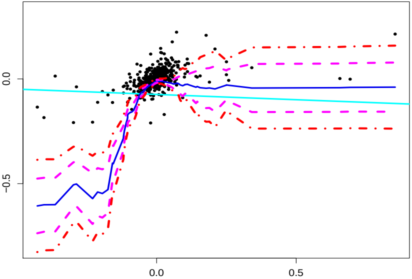


Figure 1: Goldman Sachs (GS, y-axis) and Citigroup (C, x-axis) quantile functions ($\tau = 5\%$). $X_{GS,t} = f(X_{C,t}) + \varepsilon_{GS,t}$. LLQR curve. Linear quantile regression line. 95% asymptotic confidence band and 95% bootstrap confidence band. Data weekly returns 20050131-20100131 (n=546).

Nonlinear Dependence

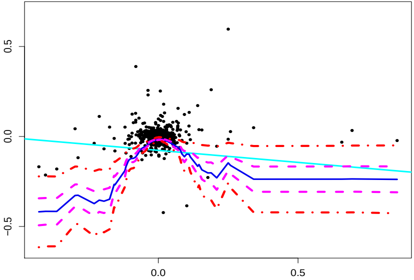


Figure 2: Bank of America (y-axis) and C (x-axis) quantile functions ($\tau = 5\%$). $X_{BOA,t} = f(X_{C,t}) + \varepsilon_{BOA,t}$. LLQR curve. Linear quantile regression line. 95% asymptotic confidence band and 95% bootstrap confidence band. Data weekly returns 20050131-20100131 (n=546).

Nonlinear Dependence

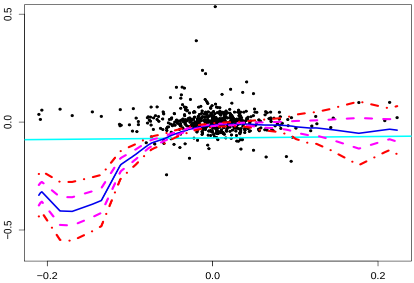


Figure 3: J.P.Morgan (y-axis) and GS (x-axis) quantile functions ($\tau = 5\%$). $X_{JPM,t} = f(X_{GS,t}) + \varepsilon_{JPM,t}$. LLQR curve. Linear quantile regression line. 95% asymptotic confidence band and 95% bootstrap confidence band. Data weekly returns 20050131-20100131 (n=546).

General Specification

- Nonparametric quantile regression:

$$X_{i,t} = f(M_{t-1}) + \varepsilon_{i,t}; \quad (3)$$
$$X_{j,t} = g(X_{i,t}, M_{t-1}) + \varepsilon_{j,t}. \quad (4)$$

M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau|M_{t-1}, X_{i,t}) = 0$.

- Challenges
  - The curse of dimensionality for f, g
  - Numerical Calibration of (3) and (4)

Research Questions

- Measure CoVaR in a nonparametric (semiparametric) way
- Test the performance of the CoVaR
- What can one learn from the semiparametric specification?
- Consequences for econometrical modelling?

Outline

1. Motivation ✓
2. Locally Linear Quantile Regression
3. A Semiparametric Model
4. Empirical CoVaR
5. Backtesting
6. Conclusions and Outlook

Locally Linear Quantile Estimation (LLQR)

- $\{(X_i, Y_i)\}_{i=1}^n \subset \mathbb{R}^2$ i.i.d. bivariate random variables, locally linear kernel quantile estimator estimated as $\hat{l}(x_0) = \hat{a}_{0,0}$:

$$\operatorname*{argmin}_{\{a_{0,0},a_{0,1}\}} \sum_{i=1}^{N} K\left(\frac{x_i - x_0}{h}\right) \rho_\tau \left\{y_i - a_{0,0} - a_{0,1}(x_i - x_0)\right\}. \quad (5)$$

▸ Check Functions

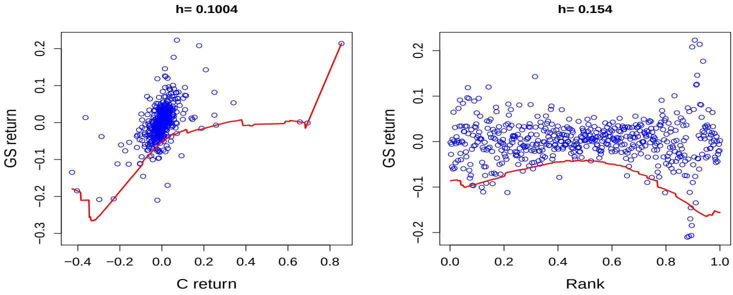
- Choice of Bandwidth: Yu and Jones (1998):

$$h_\tau = h_{mean} \left[\tau(1-\tau)\varphi\{\Phi^{-1}(\tau)\}^{-2}\right]^{1/5},$$

where h_{mean}: local mean regression bandwidth.

Stabilized Estimator

- Calculate $X_{(i:n)}$ (order statistics), then perform LLQR on $\{i/n\}_{i=1}^{n}$ and corresponding $Y_{(i:n)}$ ($\tau = 5\%$)
- $\hat{l}(x)\hat{f}_X^{-1}(x)$ is a consistent estimator for the conditional quantile in the original X space



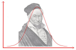
Uniform Confidence Band

Theorem (Härdle and Song (2010))

Under regularity conditions,

$$P\left[(2\delta\log n)^{1/2}\left\{\sup_{x\in J} r(x)|\hat{l}(x)-l(x)|/\lambda(K)^{1/2}-d_n\right\}<z\right] \to \exp\{-2\exp(-z)\},$$

as $n \to \infty$, where $\hat{l}(\cdot)$ is the solution of (5) and d_n is a scaling constant.

Emil Julius Gumbel on BBI: 

Bootstrap Confidence Band

Macroeconomic Drivers

Components of M_t:

1. VIX
2. Short term liquidity spread
3. Change in the 3M T-bill rate
4. Change in the slope of the yield curve
5. Change in the credit spread between 10 years BAA-rated bonds and the T-bond rate
6. S&P500 returns
7. Dow Jones U.S. Real Estate index returns

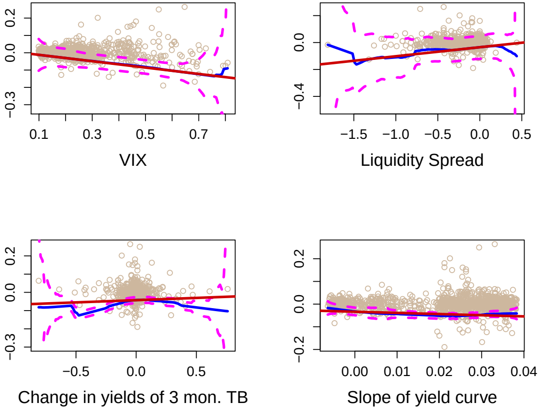


Figure 4: GS daily returns given 7 market variables and LLQR curves. Data 20060804-20110804. $n = 1260$. $\tau = 0.05$.

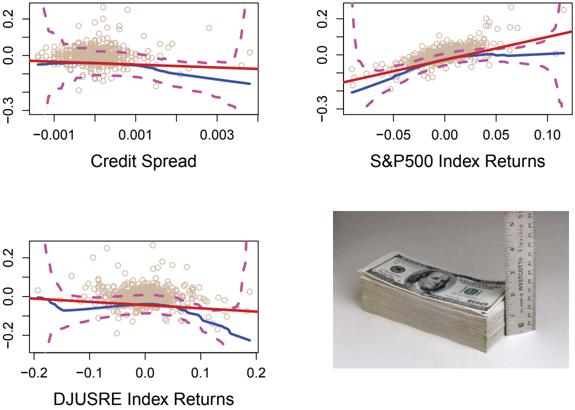


Figure 5: GS daily returns given 7 market variables and LLQR curves. Data 20060804-20110804. $n = 1260$. $\tau = 0.05$.

Partial Linear Model (PLM)

- The linearity observation (Figure 4, 5) implies:

$$X_{i,t} = \alpha_i + \gamma_i^\top M_{t-1} + \varepsilon_{i,t};$$
$$X_{j,t} = \tilde{\beta}_{j|i}^\top M_{t-1} + l_{j|i}(X_{i,t}) + \varepsilon_{j,t}. \qquad (6)$$

l: a general function. M_t: state variables. $F^{-1}_{\varepsilon_{i,t}}(\tau|M_{t-1}) = 0$ and $F^{-1}_{\varepsilon_{j,t}}(\tau|M_{t-1}, X_{i,t}) = 0$.

- Advantages
  - Capturing nonlinear asset dependence
  - Avoid curse of dimensionality

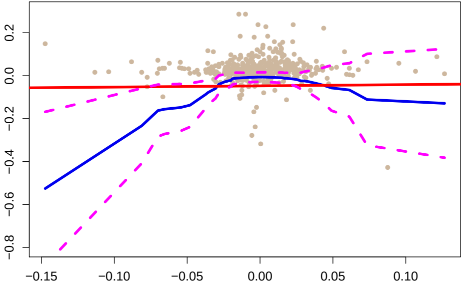


Figure 6: The nonparametric element of the PLM. y-axis=GS daily returns after filtering M_t's effect. x-axis=C daily returns. The LLQR quantile curve. Linear parametric quantile line. 95% Confidence band. Data 20080625-20081223. $n = 126$ (window size). $h =$0.2003. $\tau = 0.05$.

Estimation of Partial Linear Model

- PLM model: Liang, Härdle and Carroll (1999) and Härdle, Ritov and Song (2012)

$$Y_t = \beta^\top M_{t-1} + l(X_t) + \varepsilon_t.$$

- Consider $[0,1]$ (standard rank space). Dividing $[0,1]$ into a_n equally divided subintervals I_{nt}, $a_n \uparrow \infty$. On each subinterval, $l(\cdot)$ is roughly constant.

Estimation of PLM QR

1. Linear element β:

$$\hat{\beta} =$$

$$\operatorname*{argmin}_{\beta} \min_{l_1,\ldots,l_{a_n}} \sum_{t=1}^{n} \rho_\tau \left\{ Y_t - \beta^\top M_{t-1} - \sum_{m=1}^{a_n} l_m \mathbf{1}(X_t \in I_{nt}) \right\}$$

2. Nonlinear element $l(\cdot)$: With data $\{(X_t, Y_t - \hat{\beta}^\top M_{t-1})\}_{t=1}^{n}$, applying LLQR.

Empirical CoVaR

- j: GS daily returns,
  i: C daily returns
  Window Size: 126 days (half a year)
  Data 20060804-20110804
- Three types of VaR (CoVaR):
  - VaR from (1)
  - CoVaR^{AB} from (2)
  - CoVaR^{PLM} from (6)

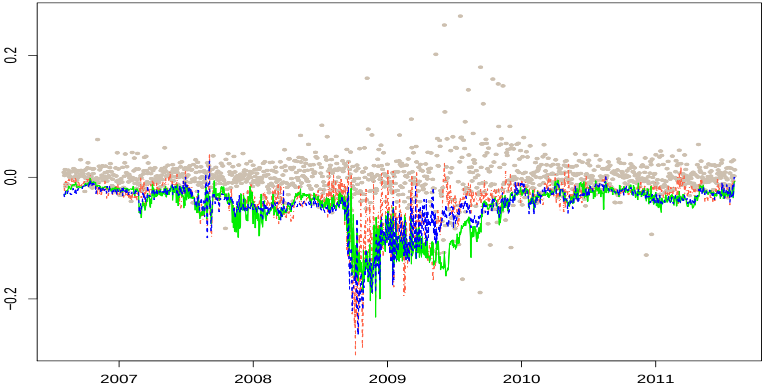


Figure 7: CoVaR of GS given the VaR of C. The x-axis is time. The y-axis is the GS daily returns. PLM CoVaR . AB CoVaR. The linear QR VaR of GS.

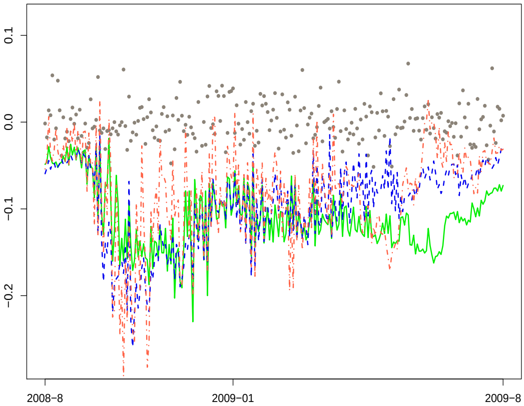


Figure 8: CoVaR of GS given the VaR of C during 20080804-20090804. The x-axis is time. The y-axis is the GS daily returns. PLM CoVaR . AB CoVaR . The VaR of GS.

Backtesting Procedure

- Berkowitz, Christoffersen and Pelletier (2011): If the VaR calibration is correct, violations

$$I_t = \begin{cases} 1, & \text{if } X_{i,t} < \widehat{(Co)VaR}^{\tau}_{t-1}(X_{i,t}) \\ 0, & \text{otherwise.} \end{cases}$$

should form a sequence of martingale difference

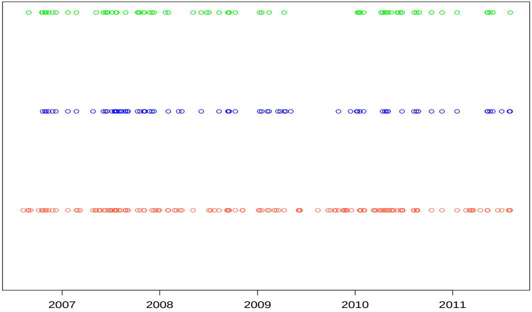


Figure 9: The timings of violations $\{t : I_t = 1\}$. The circles are the violations of the $\widehat{CoVaR}^{PLM}_{GS|C,t}$, totally 68 violations (5.4%). The squares are the violations of $\widehat{CoVaR}^{AB}_{GS|C,t}$, totally 74 violations (5.87%). The stars are the violations of $\widehat{VaR}_{GS,t}$, totally 137 violations (10.87%). $n = 1260$, $\tau = 5\%$.

Box Tests

- $\hat{\rho}_k$ be the estimated autocorrelation of lag k of violation $\{I_t\}$ and N be the length of the time series.
- Ljung-Box test:

$$\mathrm{LB}(m) = N(N+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{N-k} \tag{7}$$

- Lobato test:

$$\mathrm{L}(m) = N\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{\hat{v}_{kk}} \tag{8}$$

CaViaR Test

- Berkowitz, Christoffersen and Pelletier (2011): CaViaR performs best overall
- Test procedure:

$$I_t = \alpha + \beta_1 I_{t-1} + \beta_2 VaR_t + u_t,$$

where VaR_t can be replaced by $CoVaR_t$ in the case of conditional VaR. The residual u_t follows a Logistic distribution.

- The null hypothesis is $\beta_1 = \beta_2 = 0$.

Summary of Backtesting Procedure

- LB(1): Ljung-Box test of lag 1
- LB(5): Ljung-Box test of lag 5
- L(1): Lobato test of lag 1
- L(5): Lobato test of lag 5
- CaViaR-O: CaViaR test, all data 20060804-20110804
- CaViaR-C: CaViaR test, crisis data 20080915-20090315 (6 months after Lehman Brothers brankrupted)

Table 1: Goldman Sachs VaR/CoVaR(on C) backtesting p-values.

Measure	LB(1)	LB(5)	L(1)	L(5)	CaViaR-O	CaViaR-C
$\widehat{VaR}_{GS,t}$	0.0361	0.1323	0.0735	0.2614	$< 0.1\%$	0.0020
$\widehat{CoVaR}^{AB}_{GS\|SP,t}$	0.7174	0.3174	0.7341	0.6082	$< 0.1\%$	0.0097
$\widehat{CoVaR}^{PLM}_{GS\|SP,t}$	0.8332	0.3672	0.8396	0.6637	$< 0.1\%$	0.0211

Green, blue: significant at the 5, 1 percent levels.

Table 2: Bank of America VaR/CoVaR(on C) backtesting p-values.

Measure	LB(1)	LB(5)	L(1)	L(5)	CaViaR-O	CaViaR-C
$\widehat{VaR}_{GS,t}$	0.8418	0.0149	0.8449	0.0933	0.0037	0.0424
$\widehat{CoVaR}^{AB}_{GS\|SP,t}$	0.2185	0.0097	0.3094	0.1342	$< 0.1\%$	0.0192
$\widehat{CoVaR}^{PLM}_{GS\|SP,t}$	0.3099	0.0045	0.3922	0.0958	0.0069	0.1989

Green, blue: significant at the 5, 1 percent levels.

Table 3: J.P. Morgan VaR/CoVaR(on GS) backtesting p-values.

Measure	LB(1)	LB(5)	L(1)	L(5)	CaViaR-O	CaViaR-C
$\widehat{VaR}_{GS,t}$	0.3904	0.0038	0.4359	0.0052	$< 0.1\%$	0.0536
$\widehat{CoVaR}^{AB}_{GS\|SP,t}$	0.5800	0.9520	0.6404	0.9677	0.4265	0.0737
$\widehat{CoVaR}^{PLM}_{GS\|SP,t}$	0.4241	0.1475	0.4787	0.2930	0.0047	0.1782

Green, blue: significant at the 5, 1 percent levels.

Conclusions and Outlook

- Semiparametric PLM does well during financial crisis
- Nonlinear tail dependence is not negligible
- Multivariate nonlinear part in PLM

Quantile Regression in Risk Calibration

Shih-Kang Chao
Wolfgang Karl Härdle
Weining Wang

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. - Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de
http://www.case.hu-berlin.de

Macroprudential Risk Measures

- Marginal Expected Shortfall (MES): Portfolio $R = \sum_i w_i X_i$ where w_i: weights, X_i: asset return, $0 < \tau < 1$,

$$\text{MES}_\tau^i = \frac{\partial ES^\tau(R)}{\partial w_i} = -\mathsf{E}\left[X_i | R \leq -VaR_R^\tau\right]$$

- Distressed Insurance Premium (DIP): Huang et al. (2010) $L = \sum_{i=1}^N L_i$ total loss of a portfolio

$$DIP = \mathsf{E}^Q\left[L | L \geq L_{min}\right]$$

Return

Advantages of CoVaR

- Cloning Property: if dividing X_i into several clones, then the value of CoVaR conditioning on the individual large firm does not differ from the one conditioning on one of the clones
- Conservative Property: CoVaR conditioning on some bad event, the value would be more conservative than VaR
- Adaptive to the changing market conditions

Return

Check Function

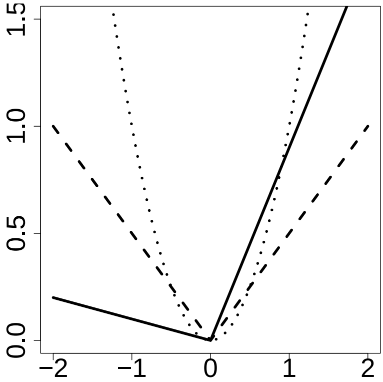


Figure 10: Solid line: $\tau = 0.9$. Dashed line: $\tau = 0.5$. Dotted line: $\rho(u) = u^2$ (OLS regression).

▸ LLQR

How to Bootstrap?

1) We have two asset returns sequence $\{Z_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$. $\{X_i\}_{i=1}^n$: n equally divided grid on $[0, 1]$. $n = 546$. Assume that Z is ordered by size and Y has been sorted by the order of Z.
2) Bivariate data: $\{(X_i, Y_i)\}_{i=1}^n$ Compute $l_h(x)$ of $Y_1, \ldots, Y_n$ and residuals $\hat{\varepsilon}_i = Y_i - l_h(X_i)$, $i = 1, \ldots, n$. $\tau = 5\%$.
   The bandwidth are $h = 0.1026$(GS-C), 0.2155(BOA-C) and 0.2188(JPM-GS).

3) Compute the conditional edf:

$$\hat{F}(t|x) = \frac{\sum_{i=1}^{n} K_h(x - X_i)\mathbf{1}\{\hat{\varepsilon}_i \leqslant t\}}{\sum_{i=1}^{n} K_h(x - X_i)}$$

with the quartic kernel

$$K(u) = \frac{15}{16}(1 - u^2)^2, \quad (|u| \leqslant 1).$$

4) Generate rv $\varepsilon^*_{i,b} \sim \hat{F}(t|x)$, $b = 1, \ldots, B$ and construct the bootstrap sample $Y^*_{i,b}, i = 1, \ldots, n, b = 1, \ldots, B$, $B = 500$, as follows:

$$Y^*_{i,b} = l_g(X_i) + \varepsilon^*_{i,b},$$

with $g = hn^{4/45} = 0.1796$(GS-C), 0.3774 (BOA-C), 0.3831(JPM-GS).

5) For each bootstrap sample $\{(X_i, Y^*_{i,b})\}_{i=1}^n$, compute l_h^* and the random variable

$$d_b \stackrel{\text{def}}{=} \sup_{x \in J^*} \left[\hat{f}\{l_h^*(x)|x\} \sqrt{\hat{f}_X(x)} |l_h^*(x) - l_g(x)| \right]. \qquad (9)$$

6) Calculate the $(1-\alpha)$ quantile d_α^* of $d_1, \ldots, d_B$.

7) Construct the bootstrap uniform confidence band centered around $l(z) = l_h(x)/\sqrt{\hat{f}_Z(z)}$, i.e.

$$l(z) \pm \left[\hat{f}\{l_h(x)|x\} \sqrt{\hat{f}_X(x)\hat{f}_Z(z)} \right]^{-1} d_\alpha^*$$

.

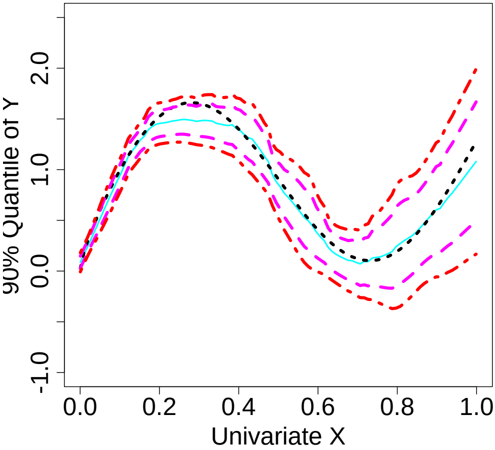


Figure 11: The real 0.9 quantile curve, 0.9 quantile estimate with corresponding 95% uniform confidence band from asymptotic theory and confidence band from bootstrapping.

How to Bootstrap?

1) Simulate $\{(X_i, Y_i)\}_{i=1}^n, n = 1000$ w.r.t. $f(x, y)$.

$$f(x, y) = f_{y|x}(y - \sin x)\mathbf{1}(x \in [0, 1]), \qquad (10)$$

where $f_{y|x}(x)$ is the pdf of $\mathrm{N}(0, x)$.

2) Compute $l_h(x)$ of $Y_1, \ldots, Y_n$ and residuals $\hat{\varepsilon}_i = Y_i - l_h(X_i),\ i = 1, \ldots, n.$
If we choose $p = 0.9$, then $\Phi^{-1}(p) = 1.2816$,
$l(x) = \sin(x) + 1.2816\sqrt{x}$ and the bandwidth is $h = 0.05$.

3) Compute the conditional edf:

$$\hat{F}(t|x) = \frac{\sum_{i=1}^{n} K_h(x - X_i)\mathbf{1}\{\hat{\varepsilon}_i \leqslant t\}}{\sum_{i=1}^{n} K_h(x - X_i)}$$

with the quartic kernel

$$K(u) = \frac{15}{16}(1 - u^2)^2, \quad (|u| \leqslant 1).$$

4) Generate rv $\varepsilon^*_{i,b} \sim \hat{F}(t|x)$, $b = 1, \ldots, B$ and construct the bootstrap sample $Y^*_{i,b}, i = 1, \ldots, n, b = 1, \ldots, B$ as follows:

$$Y^*_{i,b} = l_g(X_i) + \varepsilon^*_{i,b},$$

with $g = 0.2$.

5) For each bootstrap sample $\{(X_i, Y^*_{i,b})\}_{i=1}^n$, compute l^*_h and the random variable

$$d_b \stackrel{\text{def}}{=} \sup_{x \in J^*} \left[\hat{f}\{l^*_h(x)|x\} \sqrt{\hat{f}_X(x)} |l^*_h(x) - l_g(x)| \right]. \qquad (11)$$

6) Calculate the $(1-\alpha)$ quantile d^*_α of $d_1, \ldots, d_B$.

7) Construct the bootstrap uniform confidence band centered around $l_h(x)$, i.e. $l_h(x) \pm \left[\hat{f}\{l_h(x)|x\} \sqrt{\hat{f}_X(x)} \right]^{-1} d^*_\alpha$.

Convergence Rate (n small)

Table 4: Simulated coverage probabilities & areas of nominal asymptotic (bootstrap) 95% confidence bands with 500 repetition. $\tau = 0.9$.

n	Cov. Prob.
50	0.144 (0.642)
100	0.178 (0.742)
200	0.244 (0.862)

- For small n, bootstrap's » asymptotic's & not sacrifice much on the band's width
- Use larger bandwidth on both X & Y $(1/\hat{f}\{l_h(x)|x\})$

▸ Asymptotic Confidence Band

References

Acharya, V. V., Pedersen, L. H., Philippon, T., and Richardson, M.
Measuring systemic risk,
Working paper 10-02 (2010), Federal Reserve Bank of Cleveland.

Adrian, T. and Brunnermeier, M.
CoVaR,
Staff Reports 348 (2011), Federal Reserve Bank of New York

References

Berkowitz, J. W., P. Christoffersen and D. Pelletier
Evaluating Value-at-Risk Models with Desk-Level Data
Management Science, forthcoming

Cai, Z. and Wang, X.
Nonparametric estimation of conditional VaR and expected shortfall,
J. of Econometrics (2008) 147:120-130

References

- Carroll, R. and Härdle, W.
  Symmetrized nearest neighbor regression estimates
  Statistics and Probability Letters (1989), Vol. 26, pp. 271-292.

- Chernozhukov, V. and L. Umantsev,
  Conditional value-at-risk: Aspects of modeling and estimation
  Empirical Economics (2001) 26:271-292.

References

Engle, R. and Manganelli, S.
CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles,
J, of Business and Economic Statistics (2004) 22:367-381

Härdle, W. and S. Song
Confidence bands in quantile regression
Econometric Theory (2010) 26:1180:1200

References

Härdle, W., Y. Ritov and S. Song
Partial Linear Quantile Regression and Bootstrap Confidence Bands
J. of Multivariate Analysis, forthcoming 2012

Huang, X., Zhou, H. and Zhu, H.
Systemic risk contributions
Staff working papers 2011-08, The Federal Reserve Board

Liang, H., W. Härdle and R. J. Carroll
Estimation in a Semiparametric Partially Linear Errors-in-Variables Model
The Annals of Statistics (1999) 27(5): 1519-1535.

References

Schaumburg, J.
Predicting extreme VaR: Nonparametric quantile regression with refinements from extreme value theory
SFB Working Paper (2010)

Taylor, J. W.
Using Exponentially Weighted Quantile Regression to Estimate Value at Risk and Expected Shortfall
Journal of Financial Econometrics (2008) 6:382-406.

Yu, K. and Jones, M.C.
Local Linear Quantile Regression,
Journal of the American Statistical Association (1998) 98:228–237