On Extracting Information Implied in Options

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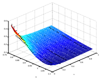
European (Plain Vanila) Options

European Options: financial derivatives with payoff:

$\max(S_T - K, 0)$ (Call Options) – Right to Buy
$\max(K - S_T, 0)$ (Put Options) – Right to Sell

T is the expiry time
K is the Strike price
S_t is the price process of underlying asset
$\tau = T - t$, time-to-maturity

Nothing is for free $\Rightarrow$ Option (Right) has a Option Price



European (Plain Vanila) Options – Questions

How to visualize **risk** on the option markets ?

Options are bets on the future states

$\Rightarrow$ market contain the participants' expectations

$\Rightarrow$ How to extract and visualize this information ?

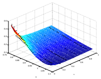
Implied Volatility

Implied Volatility (for European Options):

$$\widetilde{C}_t = C_t^{BS}(S_t, K, \tau, r, \sigma) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2)$$

where $d_1 = \frac{\ln(S_t/K) + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$
$\widetilde{C}_t$, observed market price, C_t^{BS} Black-Scholes Price
r interest rate, σ (unknown) volatility

$\sigma_t(K, \tau)$ is BS-IV for given K Strike, τ Time-to-maturity



Implied Volatility – Interpretation

- Empirical studies show violations of the assumptions of the BS model
- IV is basically understood as a transformation of option prices
- IV as estimate of future realized volatility
- IV rises if market drops

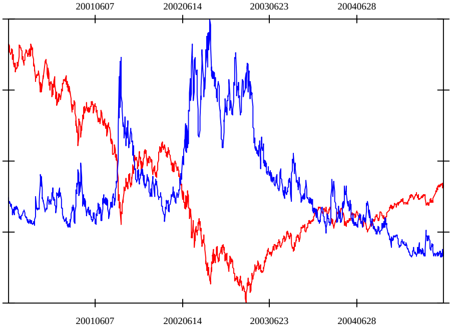


Figure 1: Illustration: Time plot of DAX centered closing level (red line) and scaled centered IV (blue) at ATM and maturity 45 days from June 1st 2000 to June 30th 2005. IV estimated by local polynomial estimator.

State Price Density, No arbitrage

Absence of arbitrage opportunities $\Rightarrow$ existence of equivalent martingale measure Q, with corresponding density q_{t,S_T} (SPD)
Price $\Pi_t(H)$ of a derivative with pay-off $H(S_T)$:

$$\Pi_t(H) = e^{-r\tau} \mathsf{E}_{\mathsf{Q}} \left(H | \mathcal{F}_t \right) = e^{-r\tau} \int_0^\infty H(s) q_{t,S_T}(s,\tau) ds \qquad (1)$$

[Breeden and Litzenberger (1978)] showed:

$$q_{t,S_T}(x,\tau) = e^{r\tau} \left. \frac{\partial^2 C_t(K,T)}{\partial K^2} \right|_{K=x} . \qquad (2)$$

SPD – Interpretation and Applications

- SPD visualize the expectation of ”market”/rep.agent”
- Construction of future confidence intervals

- Pricing of non-liquid derivatives
- Construction of trading strategies

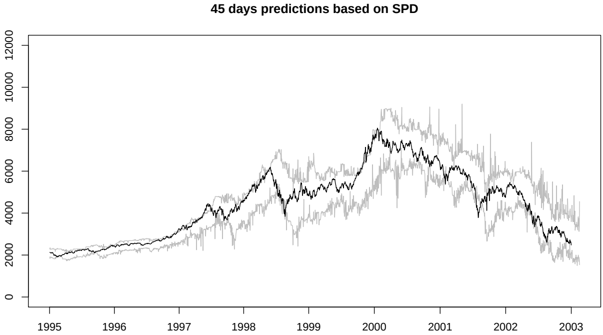


Figure 2: Confidence intervals for 45 days future DAX value based on the estimated SPD grey lines are the quantile lines (0.025 and 0.975). The black line is the true DAX level on the future date.

IV and SPD

IV has to be extracted from observed option prices, possibly affected by noise

SPD is a function of option prices

IV and SPD both with financial interpretation and are naturally connected

$\Rightarrow$ Aim: direct IV estimate by making use of the SPD

Outline of the Talk

1. Motivation ✓
2. Extracting the IV smile for fixed maturity
3. Estimating the IV-Surface
4. Summary and further remarks

Extracting the IV Smile for Fixed Maturity:

Observational model:

$$\tilde{\sigma}_i = \sigma(K_i) + \varepsilon_i. \tag{3}$$

$\tilde{\sigma}_i$ - IV calculated from observed option prices
$\sigma(K_i)$ – (true) IV function, ε_i – observational error

Observational error is caused by market market structures

We propose to estimate σ by local polynomial estimate and want to ensure positivity of the corresponding SPD

SPD Condition as Function of IV

State-price-density in terms of Call-price:

$$q_{t,S_T}(K,\tau) \stackrel{\text{def}}{=} e^{r\tau} \frac{\partial^2 C_t(K,T)}{\partial K^2} \tag{4}$$

State-price-density in terms of IV:

$$q_{t,S_T}(K,\tau) = F_t \sqrt{\tau} \varphi(d_1) \left\{ \frac{1}{K^2 \sigma \tau} + \frac{2 d_1}{K \sigma \sqrt{\tau}} \frac{\partial \sigma}{\partial K} + \frac{d_1 d_2}{\hat{\sigma}} \left(\frac{\partial \sigma}{\partial K} \right)^2 + \frac{\partial^2 \sigma}{\partial K^2} \right\} \tag{5}$$

SPD should be a p.d.f. $q_{t,S_T}(K,\tau) \geq 0$, $\int q_{t,S_T}(K,\tau) dK = 1$

IV Smoothing for Fixed Maturity

Local Quadratic Smoother:

$$\hat{\sigma}(K) = \underset{\alpha_0,\alpha_1,\alpha_2}{argmin} \sum_{i=1}^{n} \left\{ \tilde{\sigma}_i - \alpha_0 - \alpha_1(K_i - K) - \alpha_2(K_i - K)^2 \right\}^2 K_h(K - K_i) \quad (6)$$

$\alpha_0 = \hat{\sigma}(K_i)$, $\alpha_1 = \hat{\sigma}'(K_i)$, $2\alpha_2 = \hat{\sigma}''(K_i)$

$$F_t \sqrt{\tau} \varphi(d_1) \left\{ \frac{1}{K^2 \alpha_0 \tau} + \frac{2d_1}{K\alpha_0\sqrt{\tau}} \alpha_1 + \frac{d_1 d_2}{\alpha_0} (\alpha_1)^2 + 2\alpha_2 \right\} \geq 0 \quad (7)$$

with $d_1 = \frac{\ln(S_t/K) + (r + 1/2(\alpha_0)^2)\tau}{\alpha_0\sqrt{\tau}}$, $d_2 = d_1 - \alpha_0\sqrt{\tau}$

Constrained Optimization Problem

- Solution of (6) constrained by (7) for a grid of K yields to a system of separate nonlinear optimization problems
- Implemented by GAMS 22.0 using nonlinear solver MINOS visualization and further aux. calculations: MATLAB

Estimation from Contract Data

- Intra-day Data published by EUREX based on all settled contracts
- Challenge: the S_t is not constant over time (day)
- Solution: $\hat{\sigma}$ as function of κ, τ
  $\kappa = K/F_t$, $F_t = e^{r\tau} S_t$ is (futures) moneyness
- After some simple algebra:

$$q(\kappa,\tau) = \sqrt{\tau}\varphi(d_1)\left\{\frac{1}{\kappa^2\hat{\sigma}\tau} + \frac{2d_1}{\kappa\hat{\sigma}\sqrt{\tau}}\frac{\partial\hat{\sigma}}{\partial\kappa} + \frac{d_1 d_2}{\hat{\sigma}}\left(\frac{\partial\hat{\sigma}}{\partial\kappa}\right)^2 + \frac{\partial^2\hat{\sigma}}{\partial\kappa^2}\right\}. \quad (8)$$

- $\Rightarrow$ same type of constrained estimation as in daily data

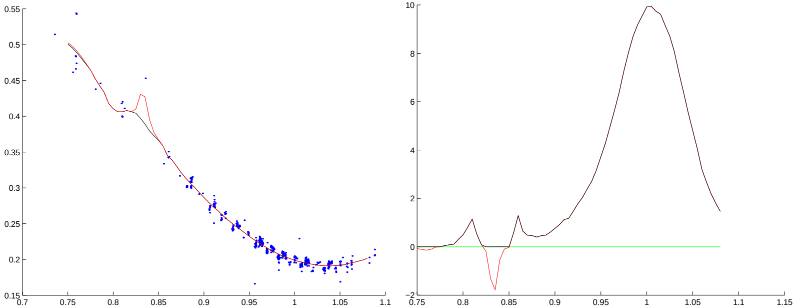


Figure 3: Left figure: Smoothed IV function black line (constrained) red line (unconstrained) and corresponding SPDs black line (constrained) and red line (unconstrained), Intraday Data, December 29, 2003, horizontal axis is moneyness level. Epanechnikov Kernel, $h = 0.045$.

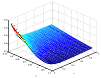
Smoothing the IV surface

- Aim is to estimate the IV function for arbitrary τ
  Estimate the entire IV Surface
- Challenge: no-arbitrage condition in time-to-maturity (calendar-arbitrage)

Maturity Direction – Calendar Arbitrage

Here the situation is much more difficult. Many proposals:

- [Kahale (2004)], [Fengler (2005)]
  Monotonicity of total variance
- [Benko, Härdle, Kneip (2006)]
  Linear interpolation in total variance
- [Härdle and Hlávka (2005)]
  Empirical argument: Lin. Interpolation of SPD variances.



- [Kahale (2004)] argues that total variance: $\sigma^2(K,\tau)\tau$
  should be monotonous in τ, assumes $r = 0$,
  Two step interpolation procedure proposed,
  input data needs to be arbitrage-free

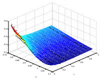
- [Fengler (2005)] argues that $\sigma^2(\kappa,\tau)$
  should be monotonous in τ, deterministic interest rate
  Two step smoothing procedure proposed,
  input data does not need to be arbitrage-free

Smoothing IV Surface

- Observational model:

$$\tilde{\sigma}_i = \sigma(K_i, \tau_i) + \varepsilon_i \ , \tag{9}$$

- Two-dim, local polynomial estimate:
  local quadratic in κ, linear in τ
- with non-negative corresponding SPD
  and monotonous corresponding total variance



Constrained 2-Dim Estimate

AIM:estimating the IV-surface $\hat{\sigma}(\kappa,\tau)$ for $\{\tau_1,\dots,\tau_L\}$:

SOLUTION: 2Dim. LocPol Estimator:

$$
\begin{aligned}
\hat{\sigma}(\kappa,\tau_1,\dots,\tau_L) \quad = \quad & \arg\min_{\alpha(l)} \sum_{l=1}^{L}\sum_{i=1}^{n} \mathcal{K}_H(\kappa-\kappa_i,\tau_l-\tau_i)\,\{\tilde{\sigma}_i-\alpha_0(l) \\
& -\alpha_1(l)(\kappa_i-\kappa)-\alpha_2(l)(\tau_i-\tau)-\alpha_{1,1}(l)(\kappa_i-\kappa)^2 \\
& -\alpha_{1,2}(l)(\kappa_i-\kappa)(\tau_i-\tau)\}^2
\end{aligned}
\qquad (10)
$$

subject to

$$
\begin{aligned}
q(k,\tau_l) \geq 0, &\qquad l=1,\quad \dots \quad, L \\
2\tau_l\alpha_0(l)\alpha_2(l)+\alpha_0^2(l) > 0 &\qquad l=1,\quad \dots \quad, L \\
\text{and } \alpha_0^2(l)\tau_l < \alpha_0^2(l')\tau_l', &\qquad \tau_l \quad < \quad \tau_l'
\end{aligned}
$$

(11)

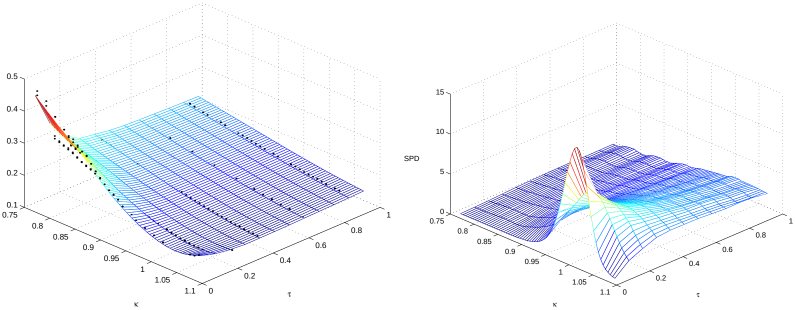
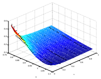


Figure 4: Left figure: Smoothed IV surface (constrained) left and corresponding family of SPDs right, Daily Data, February 2, 2006.

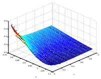


Summary

+ Smoothing for fixed maturity:
  solved using local quadratic smoother
  combining IV and SPD
+ Direct two dimensional estimate for IV smoothing
  utilizing SPD and recent results on calendar arbitrage
- Nonlinear constrained optimization

Outlook

- utilizing integral condition
- bandwidth choice (CV)
- confidence intervals (Wild bootstrap)



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