Value-at-Risk with Time Varying Copulae

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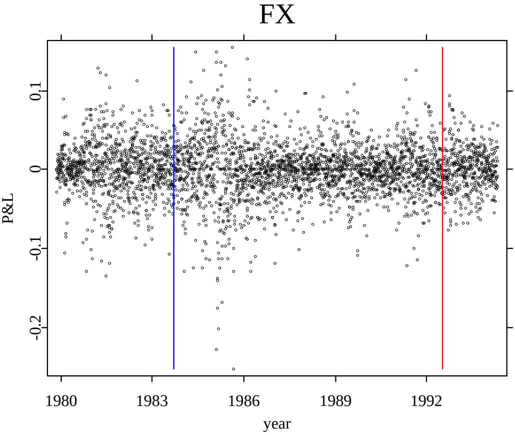


Figure 1: P&L, FX Portfolio (DEM/USD, GBP/USD), $w = (2, 1)^\top$.

Linear Portfolio

A linear portfolio has:

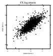
- positions $w = (w_1, \ldots, w_d)^\top$
- on assets $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$
- with risk factors $Z_t = (Z_{1,t}, \ldots, Z_{d,t})^\top$, $Z_t = \ln S_t$

and value

$$V_t = \sum_{j=1}^{d} w_i e^{Z_{j,t}}$$

The *profit and loss* (P&L) function:

$$L_{t+1} = (V_{t+1} - V_t)$$



The P&L can be expressed as

$$L_{t+1} = \sum_{j=1}^{d} w_j S_{j,t}(e^{X_{j,t+1}} - 1), X_{t+1} = (Z_{t+1} - Z_t)$$

The *Value-at-Risk* (*VaR*) is the α-quantile from F_L, the distribution from L_{t+1}:

$$VaR(\alpha) = F_L^{-1}(\alpha)$$

Log returns at 31.03.1983

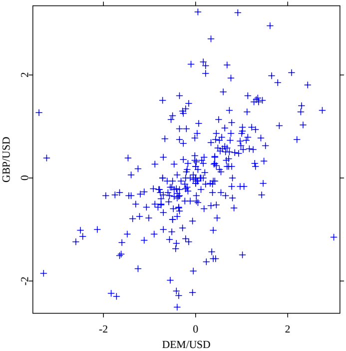


Figure 2: Standardised log returns DEM/USD and GBP/USD, last 250 realizations at 31.05.1983. mindep.xpl

Log returns at 14.04.1992

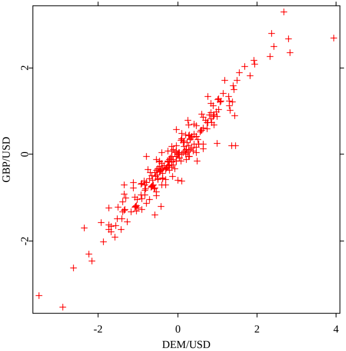
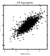
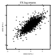


Figure 3: Standardised log returns DEM/USD and GBP/USD, last 250 realizations at 14.04.1992. maxdep.xpl



The VaR depends on
the distribution F_X of the risk factor increments $X = (X_1, \ldots, X_d)^\top$.

How does F_x and the dependency among $X_1, \ldots, X_d$ vary over time ?



Outline

Copulae

Theorem (Sklar's theorem)

For a distribution function F with marginals $F_{X_1} \dots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$ with

$$F(x_1, \dots, x_d) = C\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \tag{1}$$

If $F_{X_1}, \dots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \dots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \dots, F_{X_d}$.

For $u_j = F_{X_j}(x_j)$, $j = 1 \dots d$,

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}$$

the density function of $F(x_1, \dots, x_d)$ can be expressed as

$$f(x_1, \dots, x_d) = c\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \prod_{j=1}^{d} f_j(x_j)$$

where $f_j(x_j) = F'_{X_j}(x_j)$, $j = 1 \dots d$

Some examples of copulae:

1. **Normal copula**, d-dimensional with correlation matrix Σ

$$C(\mathbf{u};\Sigma) = \Phi_{\Sigma,d}(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_d))$$

   - Φ, univariate standard normal distribution
   - $\Phi_{\Sigma,d}$, d-dimensional normal distribution with correlation matrix Σ
   - $\mathbf{u} = (u_1,\dots,u_d)^\top$

2. Student's t-copula, d-dimensional with correlation matrix Σ

$$C(\mathbf{u};\Sigma,\nu) = T_{\Sigma,\nu}(T_\nu^{-1}(u_1),\dots,T_\nu^{-1}(u_d))$$

- T_ν, univariate Student's t distribution with ν degrees of freedom and
- $T_{\Sigma,\nu}$, d-dimensional standardized Student's t distribution with ν degrees of freedom and correlation matrix Σ

3. Ali-Mikhail-Haq copula, $-1 \le \theta < 1$

$$C(\mathbf{u};\theta) = \frac{\prod_{i=1}^{d} u_i}{1 - \theta \left(\prod_{i=1}^{d} 1 - u_i \right)}$$

4. Frank copula, $0 < \theta \le \infty$

$$C(\mathbf{u};\theta) = -\frac{1}{\theta} \ln \left\{ 1 + \frac{\prod_{i=1}^{d} (e^{-\theta u_i} - 1)}{e^{-\theta} - 1} \right\}$$

5. Gumbel copula, $1 \leq \theta \leq \infty$

$$C(\mathbf{u}; \theta) = \exp\left[- \left\{ \sum_{i=1}^{d} (\ln u_i)^{\theta} \right\}^{\theta^{-1}} \right]$$

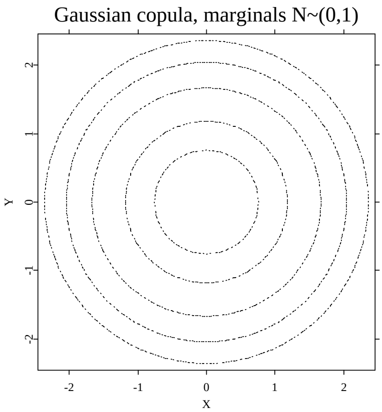
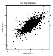


Figure 4: Contour plots of pdf from $F(x_1, x_2) = C(\Phi(x_1), \Phi(x_2))$ with Gaussian copula. contourG.xpl



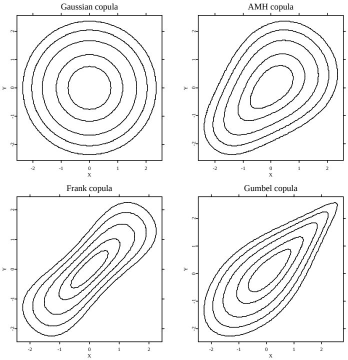
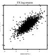


Figure 5: Contour plots of pdf from $F(x_1, x_2) = C(\Phi(x_1), \Phi(x_2))$ with Gaussian, AMH, Frank and Gumbel copulae. copplot.xpl, contourLine.xpl



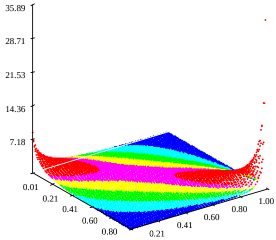


Figure 6: Density from Gumbel copula, $\theta = 2$. gumbelcol.xpl

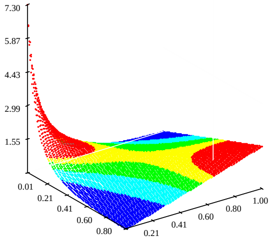
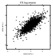


Figure 7: Density from AMH copula, $\theta = 0.9$. amhcol.xpl



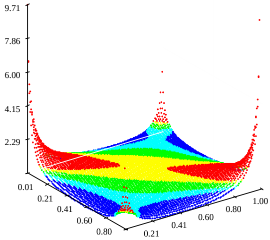
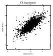


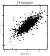
Figure 8: Density from t-copula, $\rho = 0.2$, $\nu = 3$. `tcol.xpl`



Value-at-Risk with Copulae

For a sample $\{X_t\}_{t=1}^{T}$

1. specification of marginal distributions $F_{X_j}(x_j;\delta_j)$
2. specification of copula $C(u_1,\dots,u_d;\theta)$
3. fit of the copula C
4. generation of Monte Carlo data $X_{T+1}{\sim}C\{F_1(x_1),\dots,F_d(x_d);\hat{\theta}\}$
5. generation of a sample of portfolio losses $L_{T+1}(X_{T+1})$
6. estimation of $\widehat{VaR}(\alpha)$, the empirical quantile at level α from $L_{T+1}(X)$.



For copulae $C(\cdot, \theta)$, $\theta \in \Theta$, the density of X is given by:

$$f(x_1, \ldots, x_d; \delta_1, \ldots, \delta_d, \theta) =$$

$$= c\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j)$$

where

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}$$

Inference for Margins

In the IFM (*inference for margins*) method, the log-likelihood function for each of the marginal distributions

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_i(x_{j,t}; \delta_j), j = 1, \ldots, d$$

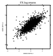
is maximized to obtain estimates $(\hat{\delta}_1, \ldots, \hat{\delta}_d)^\top$.

The function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} [\ln c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}]$$

is then maximized over θ to get the dependence parameter estimate $\hat{\theta}$. The estimates $\hat{\theta}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial \ell_1 / \partial \delta_1, \ldots, \partial \ell_d / \partial \delta_d, \partial \ell / \partial \theta) = 0$$



Static Approach

- DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- $T = 3719$
- copulae belong to the bivariate one-parametric Gumbel family

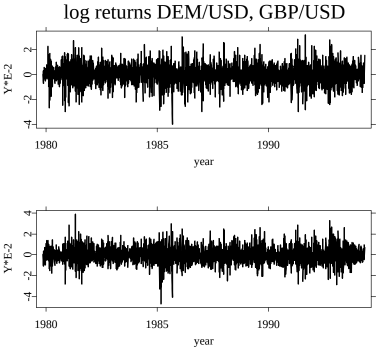


Figure 9: Log returns from DEM/USD (X_1) and GBP/USD (X_2).

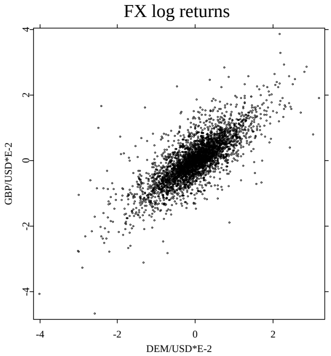


Figure 10: Scatterplot from log returns DEM/USD (X_1) and GBP/USD (X_2).

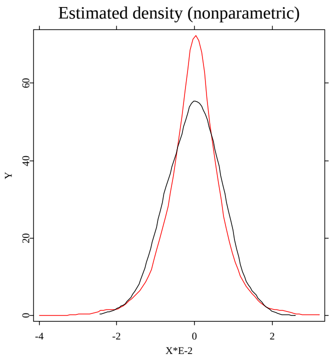
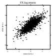


Figure 11: Kernel density estimator of the log returns from DEM/USD (red) and of the normal density (black). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.



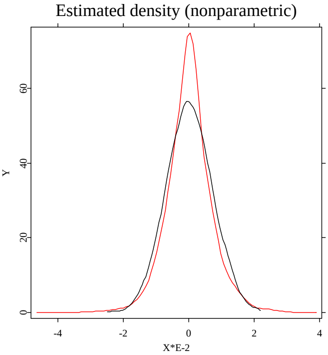


Figure 12: Kernel density estimator of the log returns from GBP/USD (red) and of the normal density (black). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.

Dependence

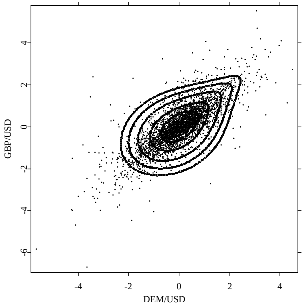
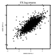


Figure 13: Standardised log returns DEM/USD and GBP/USD, fitted copula ($\hat{\theta} = 1.4461$) for $T = 3719$.

Value-at-Risk

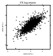
	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
$\widehat{VaR}(\alpha)$	-0.02436	-0.034115	-0.037921	-0.042611

Table 1: Estimated Value-at-Risk at 4 different levels, FX portfolio, $w = (1,1)^\top$.



Moving window

- DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994
- sample size $S = 3719$, time window $T = 250$, for $s = T + 1, \dots, S$
- using $\{X_t\}_{t=s-T}^{s}$
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- copulae belong to the bivariate one-parametric Gumbel family



Parameter $\hat{\sigma}_1$ from marginal distribution

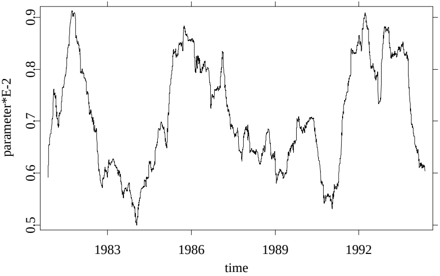


Figure 14: Estimated parameter from Normal marginal distribution $\hat{\sigma}_1$ for log returns from DEM/USD, $T = 250$.

Parameter $\hat{\sigma}_2$ from marginal distribution

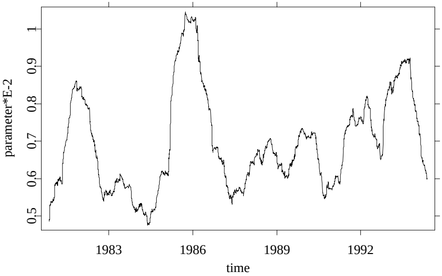


Figure 15: Estimated parameter from Normal marginal distribution $\hat{\sigma}_2$ for log returns from GBP/USD, $T = 250$.

Copula parameter $\hat{\theta}$

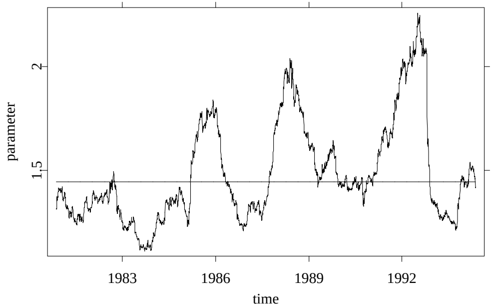


Figure 16: Gumbel dependence parameter $\hat{\theta}$ between DEM/USD and GBP/USD (standardised log returns). Estimated with Normal marginal distributions using IFM method, $T = 250$ (constant value for $T = 3719$).

	min	max	mean	median	std error
$\hat{\sigma}_1.10^3$	4.99	9.12	7.09	6.91	1.02
$\hat{\sigma}_2.10^3$	4.74	10.46	6.95	6.69	1.31
$\hat{\theta}$	1.11	2.25	1.48	1.42	0.24

Table 2: Descriptive statistics for estimated parameters $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\theta}$.

Minimal and maximal dependence

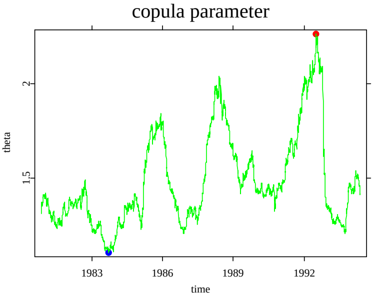


Figure 17: Minimal (blue), maximal (red) dependence parameter between standardised log returns DEM/USD and GBP/USD.

Minimal dependence

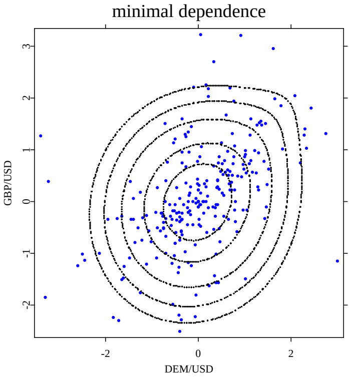


Figure 18: Standardised log returns DEM/USD and GBP/USD at minimal dependence (blue), fitted copula ($\hat{\theta} = 1.11$).

Maximal dependence

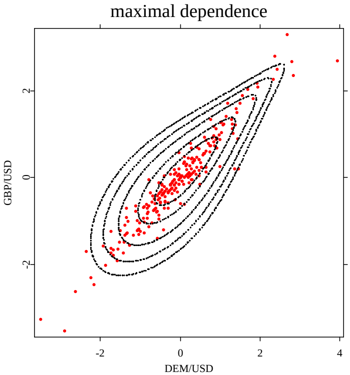
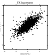


Figure 19: Standardised log returns DEM/USD and GBP/USD at maximal dependence (red), fitted copula ($\hat{\theta} = 2.25$).



MC sample, minimal dependence

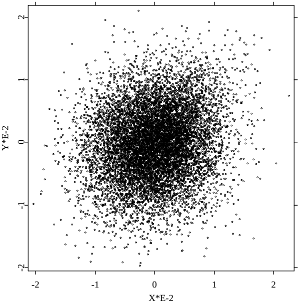


Figure 20: Monte Carlo sample of random variables $X \sim C\{\Phi_1(x_1), \Phi_2(x_2); \hat{\theta}\}$, minimal dependence ($\hat{\theta} = 1.11$).

Transformed MC sample, minimal dependence

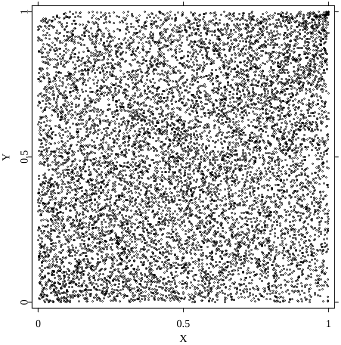


Figure 21: Monte Carlo sample of random variables transformed on the unit square, minimal dependence ($\hat{\theta} = 1.11$).

MC sample, maximal dependence

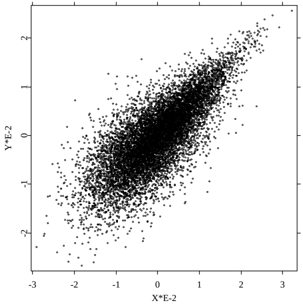


Figure 22: Monte Carlo sample of random variables $X \sim C\{\Phi_1(x_1), \Phi_2(x_2); \hat{\theta}\}$, maximal dependence ($\hat{\theta} = 2.25$).

Transformed MC sample, maximal dependence

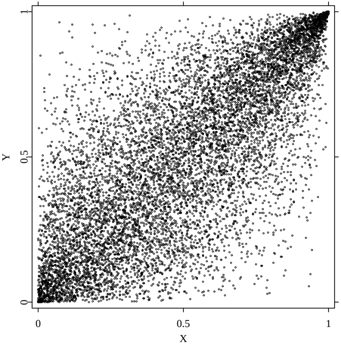


Figure 23: Monte Carlo sample of random variables transformed on the unit square, maximal dependence ($\hat{\theta} = 2.25$).

Backtesting

Evaluation:

- different portfolio compositions are used
- the VaR $\alpha = 0.05$, $\alpha = 0.01$, $\alpha = 0.005$ and $\alpha = 0.001$ is calculated
- *exceedance* for each P&L value smaller than VaR

Value-at-Risk

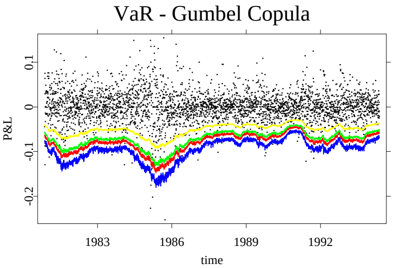


Figure 24: Value-at-Risk at levels $\alpha_1 = 0.05$ (yellow), $\alpha_2 = 0.01$ (green), $\alpha_3 = 0.005$ (red), and $\alpha_4 = 0.001$ (blue), P&L (black), $w = (2,1)^\top$, estimated at each time from a Monte Carlo sample of 10.000 P&L values.

Value-at-Risk (0.05) and exceedances

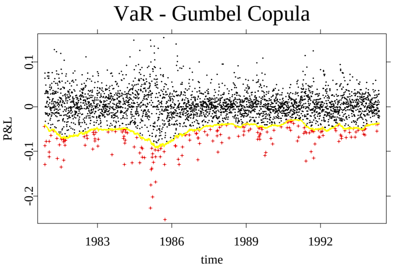


Figure 25: Value-at-Risk (yellow) at level $\alpha = 0.05$, P&L (black) and exceedances (red), $\hat{\alpha} = 0.0573$, $w = (2,1)^\top$. P&L samples generated with Gumbel copula.

Value-at-Risk (0.001) **and exceedances**

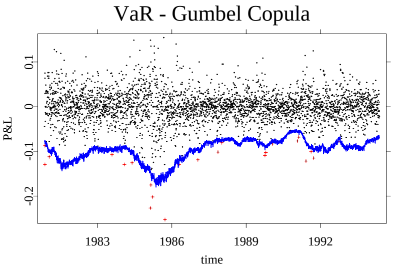


Figure 26: Value-at-Risk (blue) at level $\alpha = 0.001$, P&L (black) and exceedances (red), $\hat{\alpha} = 0.0069$, $w = (2, 1)^\top$. P&L samples generated with Gumbel copula.

Portfolio $w^\top$	level $\alpha(\times 10^2)$ 5	1	0.5	0.1
	empirical level $\hat{\alpha}(\times 10^2)$			
$(1, 1)$	6.05	2.45	1.75	0.83
$(1, 2)$	6.34	2.74	1.75	1.00
$(2, 1)$	5.73	2.24	1.58	0.69
$(2, 3)$	6.22	2.56	1.75	0.92
$(3, 2)$	5.99	2.30	1.55	0.74
$(-1, 2)$	1.64	0.37	0.20	0.11
$(1, -2)$	2.01	0.51	0.43	0.11
$(-2, 1)$	4.44	1.49	0.95	0.40
$(2, -1)$	4.09	1.35	1.09	0.49

Table 3: Gumbel copula, empirical levels $\hat{\alpha}$ for different FX portfolios.

Negative Log-returns

	level $\alpha (\times 10^2)$			
	5	1	0.5	0.1
Portfolio $w^\top$	empirical level $\hat{\alpha} (\times 10^2)$			
$(1, 1)$	5.25	1.82	1.15	0.63
$(1, 2)$	5.39	1.64	1.24	0.60
$(2, 1)$	5.27	1.79	1.27	0.66
$(2, 3)$	5.30	1.70	1.21	0.66
$(3, 2)$	5.27	1.78	1.26	0.66
$(-1, 2)$	1.41	0.29	0.23	0.05
$(1, -2)$	2.74	0.98	0.61	0.28
$(-2, 1)$	4.32	1.15	0.79	0.26
$(2, -1)$	4.49	1.67	1.24	0.69

Table 4: Gumbel copula on negative log-returns, empirical levels $\hat{\alpha}$ for different FX portfolios.

DAX-Dow Jones portfolio

- DAX and Dow Jones from 02.01.1997 to 30.12.2004
- sample size $S = 2022$, time window $T = 250$, for $s = T+1, \ldots, S$
- using $\{X_t\}_{t=s-T}^{s}$
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- copulae belong to the bivariate one-parametric Gumbel family

	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
Portfolio $w^\top$	empirical level $\hat{\alpha}(\times 10^2)$			
$(1,1)$	4.28	1.29	0.84	0.45
$(1,2)$	3.89	1.29	0.79	0.50
$(2,1)$	4.62	1.52	0.90	0.56
$(2,3)$	4.06	1.18	0.73	0.50
$(3,2)$	4.57	1.46	0.90	0.62
$(-1,2)$	5.07	1.52	0.84	0.39
$(1,-2)$	4.79	1.58	1.24	0.45
$(-2,1)$	4.96	1.46	0.95	0.39
$(2,-1)$	4.96	1.74	1.12	0.62

Table 5: Gumbel copula, empirical levels $\hat{\alpha}$ for different DAX Dow Jones portfolios.

DAX - Dow Jones: Value-at-Risk (0.05) and exceedances

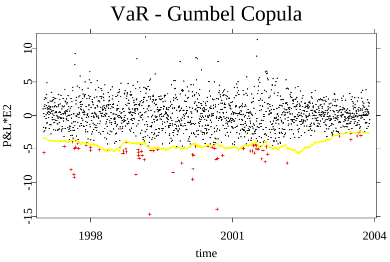


Figure 27: Value-at-Risk (yellow) at level $\alpha = 0.05$, P&L (black) and exceedances (red), $\hat{\alpha} = 038939$, $w = (1, 2)^\top$. P&L samples generated with Gumbel copula.

Adaptive Copulae

In the local homogeneity modelling the copula parameter is a piecewise constant function θ_t

- search for largest interval $I = [n - m, n[$ that does not contain a change point,

$$\theta_t = \theta_I, t \in I$$

- within I, θ_n can be estimated through

$$\tilde{\theta}_I = \arg\max_{\theta} L_I(\theta)$$

where $L_I(\theta) = \sum_{i \in I} \ell(x_i; \theta)$.

Determining I

The homogeneity interval I can be determined as follows

- select a set $\mathcal{I}$ of candidate intervals
- take the smallest $I \in \mathcal{I}$
- test homogeneity in I against change-point alternative
- if rejected at point $\nu \in I$, $\hat{I} = [\nu, n[$
- if not rejected, choose larger I

Change-point Test

- $\mathcal{T}(I)$ a family of internal points of I
- each $\tau \in \mathcal{T}(I)$ splits the interval I into sub-intervals $J = [n-\tau, n[$ and $J^c = [n-m, n-\tau[$
- likelihood ratio test statistic for change-point at τ

$$T_{I,\tau} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_I(\tilde{\theta}_I)$$

- change-point test

$$T_{I,\nu} = \max_{\tau} T_{I,\tau}$$

If $T_{I,\nu} \geq \lambda_I$, reject homogeneity and

- ν is change-point time
- $\hat{I} = [\nu, n[$ is the homogeneity interval
- $\tilde{\theta} = \arg\max_{\theta} L_{\hat{I}}(\theta)$ the estimated copula parameter.

Critical Value λ_I

Adaptive procedure, type I error (α): multiple testing problem

– for each I, define β_I and α_I such that

$$\sum_{I \in \mathcal{I}} \beta_I = \alpha$$

$$\alpha_I = \sum_{I' \in \mathcal{I}(I)} \beta_{I'}$$

where $\mathcal{I}(I) = \{I' : I' \in \mathcal{I}, I' \subseteq I\}$

- within I: change-point test at level α_I
- $n = 5000$ Monte Carlo simulations of T_I
- λ_I is $(1 - \alpha_I)$-quantile of computed test statistics T_I

Monte Carlo Simulation

Gumbel copulae simulated with parameter:

$$\theta_{1,t} = \begin{cases} 1 & : \quad 1 \le t \le 60 \\ 5 & : \quad 61 \le t \le 120 \\ 1 & : \quad 121 \le t \le 180 \end{cases}$$

and

$$\theta_{2,t} = \begin{cases} 1.5 & : \quad 1 \le t \le 260 \\ 6 & : \quad 261 \le t \le 320 \\ 3 & : \quad 321 \le t \le 380 \\ 1 & : \quad 381 \le t \le 440 \end{cases}$$

– set of candidate intervals

$$\mathcal{I} = \{I_k : I_k = [t - m_k, t[\}$$

$$m_k = [m_0 c^k], k = 0, 1, 2$$

– $[x]$ is the integer part of x

– defining β_{I_k} as

$$\beta_{I_k} = \frac{\alpha}{m_k} \left(\sum_{l=1}^{\infty} m_l^{-1} \right)^{-1} \approx \frac{\alpha(1 - c^{-1})}{c^k}$$

- defining α_{I_k} as

$$\alpha_{I_k} \approx \left(1 - c^{-(k+1)}\right)$$

- critical values λ_{I_k} are obtained through Monte Carlo simulation.
- values set to $m_0 = 30$, $c = 2$ and $\alpha = 0.05$

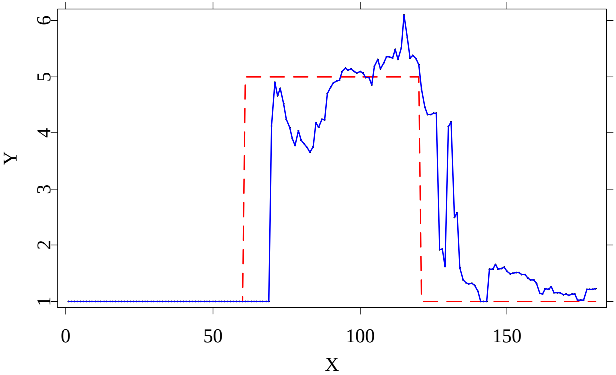
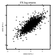


Figure 28: Real parameter $\theta_{1,t}$ (red) and estimated (blue). ADCOPsim1.xpl



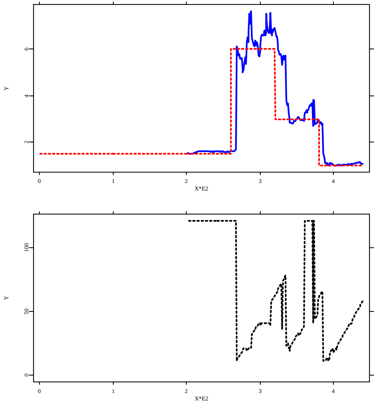


Figure 29: Real parameter $\theta_{2,t}$ (red), estimated (blue) and interval $\hat{I}$ (black).

ADCOPsim2.xpl

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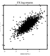

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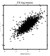

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