Value-at-Risk and Expected Shortfall when there is Long Range Dependence

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Financial Markets: Stylized Facts

- **Volatility clustering** - large changes that tend to be followed by large changes and vice versa.
- **Leptokurtosis** - returns that are heavy tailed distributed
- **Leverage effects**- changes in stock prices that tend to be negatively correlated with changes in volatility
- **Long range dependence**- existence of significant correlation structure in a time series at long lags, allowing for persistence or long-memory.

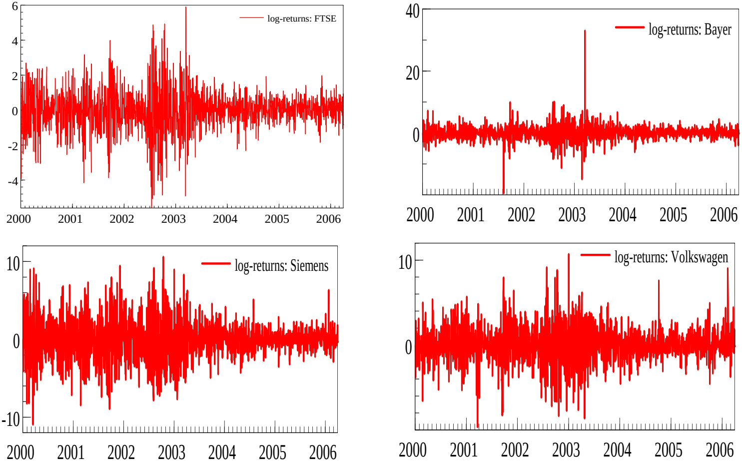

Figure 1: *Daily log-returns of the British FTSE* (04.01.2000 – 30.10.2006) *equity index and the German DAX stocks; Bayer, Siemens and Volkswagen,* (04.01.2000 – 30.10.2006)

Risk Management I

Classical Models

Heteroscedastic model: $r_t = \sigma_t \varepsilon_t, \quad t = 1, 2, \ldots$
r_t are log-returns, σ_t is standard deviation.

- RiskMetrics (Exponential Moving Average)
  $\sigma_t^2 = \omega + (1-\lambda)\varepsilon_{t-1}^2 + \lambda \sigma_{t-1}^2, \quad \varepsilon_t \sim N(0,1)$
- t-GARCH
  $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \quad \varepsilon_t \sim t(df)$

Main issue: Estimation of σ_t^2 and the distributional behaviour of ε_t

Example

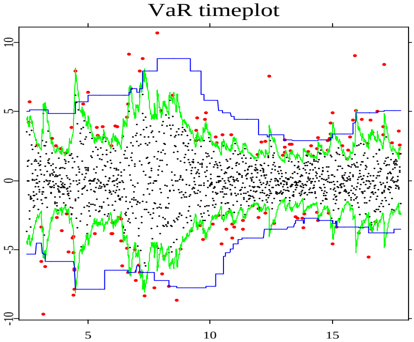

Figure 2: *Time plots of VaR forecasts at* 99% *and* 1% *level using* (*EDF*) *and* (*EMA*) *(with decay factor,* $\lambda = 0.94$ *and a window of* 250 *days) for the Volkswagen stock returns from* 3rd *January* 2000 - 10*th October* 2006. *Returns which exceed the VaR are marked in red. A total of 1781 data points.*

- financial asset may exhibit long range dependence on stock market volatility, [Ding et al. (1993)], [Henry (2002)]
- long term dependence has significant impact on the pricing of financial derivatives,
- has significant impact on forecasting market volatility,
- may in turn affect investment portfolio, bringing about losses or gains, [Bollerslev and Mikkelsen (1996)], [Herzberg and Sibbertsen (2004)]

Can long range dependence affect the measurement of market risk ?
Can we do better if long memory exist?

Risk Management II

Long Memory Volatility Models

$$\text{Skewed Student-}t\text{ distribution} + \begin{cases} \text{FIAPARCH} \\ \text{HYP-GARCH} \end{cases}$$

FIAPARCH- fractional integrated asymmetric power *ARCH*.

HYP-GARCH- hyperbolic *GARCH*

- Application to Value-at-Risk (VaR) and Expected Short Fall (ESF) model
- for daily trading portfolios of long and short positions

Outline

1. Motivation ✓
2. Long Memory Test and Models Specifications
3. Risk measures: e.g. Value-at-Risk and Expected Shortfall
4. Risk management: application with Long Memory Models.

Rescale Variance test

- centering the Kwiatkowski et al. (1992), ($KPSS$), based on partial sum of deviations from the mean

$$KPSS_T = \frac{1}{T^2 \widehat{\sigma}_T^2(q)} \sum_{k=1}^{T} \left\{ \sum_{j=1}^{k} (r_j - \overline{r}_T) \right\}^2 \tag{1}$$

$\widehat{\sigma}_T^2(q) = \widehat{\gamma}_0 + 2\sum_{j=1}^{q} \left(1 - \frac{j}{1+q}\right) \widehat{\gamma}_j$, the heteroscedastic and autocorrelation consistent (HAC) estimator of variance, $(q < T)$.
$\widehat{\gamma}_0$, variance estimate,
$\{\widehat{\gamma}_j\}_{j=1}^{q}$, the autocovariances up to order q.

Rescale Variance test

- Rescaled Variance test statistic (V/S):

$$V/S(q) = \frac{1}{T^2 \widehat{\sigma}_T^2(q)} \left[\sum_{k=1}^{T} \left\{ \sum_{j=1}^{k} (r_j - \overline{r}_T) \right\}^2 - \frac{1}{T} \left\{ \sum_{k=1}^{T} \sum_{j=1}^{k} (r_j - \overline{r}_T) \right\}^2 \right] \tag{2}$$

[Giraitis et al. (2003)]

FIAPARCH(p,d,q)

Conditional volatility specification:

$$\sigma_t^\delta = \omega + \left\{1 - [1-\beta(L)]^{-1}\alpha(L)(1-L)^d\right\}(|\varepsilon_t| - \gamma\varepsilon_t)^\delta \qquad (3)$$

$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} L^j$ where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the gamma function, [Tse (1998)].
δ plays roll of a Box-Cox transformation of $\sigma_t and \gamma$ reflects the leverage effect.

- allows for volatility clustering, leptokurtosis, long memory and
- features asymmetry in relationship between the conditional variance and the lagged squared innovations

HY-GARCH(p,α,d,q)

Extends the conditional variance of $FIGARCH(p,d,q)$ by introducing weights to the difference operator,

$$\sigma_t^2 = \frac{\omega}{1-\theta(L)} + \left[1 - \frac{\phi(L)\left\{1+\alpha(1-L)^d\right\}}{1-\theta(L)}\right]\varepsilon_t^2 \qquad (4)$$

where α are weights to $(1-L)^d$.

- for $\alpha = 0$, $GARCH$
- for $\alpha = d = 1$, $IGARCH$
- for $\alpha = 1$ ($\log\alpha = 0$), $FIGARCH$
- for $\alpha < 1$ ($\log\alpha < 0$), a stationary process

[Davidson (2004)]

(Standardized) skewed Student (skst) Distribution

The *skst* distribution has a density for $z_t \in \mathbb{R}$:

$$f_{skst}(z_t;\nu,\xi) = \begin{cases} \frac{2}{\xi+\frac{1}{\xi}} sg\left\{\xi(sz_t+m);\nu\right\} & \text{if} \quad z_t < -\frac{m}{s} \\ \frac{2}{\xi+\frac{1}{\xi}} sg\left\{\frac{(sz_t+m)}{\xi};\nu\right\} & \text{if} \quad z_t \geq -\frac{m}{s}, \end{cases} \tag{5}$$

$g(.;\nu)$ is the symmetric (unit variance) student density, ξ is the asymmetric coefficient, ν is the degree of freedom, m and s^2 are the mean and variance respectively of the non-standardized *skst*, [Lambert and Laurent (2001)], [Bauwens and Laurent (2005)]

- *skst* nests student-t for $\xi = 1$ and Normal for $\xi = 1$ and $\nu = 0$.

VaR-at-Risk (VaR)

- estimate the probability of a portfolio of assets losing a specific amount over a specified time period due to adverse movements in the underlying market factors, [Jorion (2001)].

If q_α is the α^{th} quantile of the distribution of ε_t, i.e. $P(\varepsilon_t < q_\alpha) = \alpha$.

$$P(r_t < \sigma_t q_\alpha | \mathcal{F}_{t-1}) = \alpha$$

$$VaR_{t,\alpha} = \sigma_t q_\alpha$$

$\widehat{\sigma}_t$ estimated by *FIAPARCH*, *HYGARCH* and q_α is quantile of *skst* distribution.

- $VaR_{t,\alpha}$ for long (VaR_L) and short (VaR_S) positions:

$$
\begin{aligned}
VaR_L &= \widehat{\mu}_t + \widehat{\sigma}_t . skst_{\alpha,\nu,\xi} \\
VaR_S &= \widehat{\mu}_t + \widehat{\sigma}_t . skst_{1-\alpha,\nu,\xi}
\end{aligned}
$$

$skst_{\alpha,\nu,\xi}$ and $skst_{1-\alpha,\nu,\xi}$ are the left and right quantiles respectively,
ν, degree of freedom and ξ the asymmetry coefficient
$\widehat{\mu}_t$, forecast of conditional mean
$\widehat{\sigma}_t$, forecast of conditional standard deviation.

Expected Short Fall (ESF)

- *VaR* can not aggregate risk in the sense of being subadditive on portfolios (in this case, risk is diversified).
- this shortcoming is addressed by (*ESF*), a "coherent risk measure", [Artzner et al. (1999)].
- *ESF* predicts the expected size of the loss given that the loss is greater than the *VaR*, [Scaillet (2004)].

$$ESF_t = \mathrm{E}(|L_t| > |VaR_t|),$$

L_t is the expected value of loss if a VaR_t violation occurs.

Backtesting VaR

- **unconditional coverage test**: [Kupiec (1995)]

Let $N = \sum_{t=1}^{T} I_t$ be the number of exceptions (I_t, indicator of exceptions) where $N \sim B(T, p)$, p is a pre-specified VaR level.

Let p_0 denote the modelled probability of failure.

Hypothesis for testing if failure rate equals expected one:

$$H_0 : p = p_0$$

Likelihood ratio statistic,

$$LR_1 = -2\log\left\{p^N(1-p)^{T-N}\right\} + 2\log\left\{(N/T)^N(1-N/T)^{T-N}\right\}$$

$LR_1 \sim \chi^2(1)$, [Jorion (2001)].

Backtesting VaR

- **conditional coverage test**: [Christoffersen (1998)]

Let $\pi_{ij} = P(1_t = j | 1_{t-1} = i)$ be the transition probability and $n_{ij} = \sum_{t=1}^{T} 1(1_t = j \textit{ and } 1_{t-1} = i)$, where $i, j = 0$ or 1.
Hypothesis of independence is given as:

$$H_0 : \pi_{00} = \pi_{10} = \pi, \quad \pi_{01} = \pi_{11} = 1 - \pi$$

Unconditional coverage and independence of failures.
Likelihood ratio statistic,

$$LR_2 = -2 \log \{\widehat{\pi}^{n_0} (1 - \widehat{\pi})^{n_1}\} + 2 \log \{\widehat{\pi}_{00}^{n_{00}} \widehat{\pi}_{01}^{n_{01}} \widehat{\pi}_{10}^{n_{10}} \widehat{\pi}_{11}^{n_{11}}\} \qquad (6)$$

where $\widehat{\pi}_{ij} = \frac{n_{ij}}{(n_{ij} + n_{i,1-j})}$, $n_j = n_{0j} + n_{1j}$ and $\widehat{\pi} = \frac{n_0}{(n_0 + n_1)}$.
Under H_0, $LR_2 \sim \chi^2(2)$.

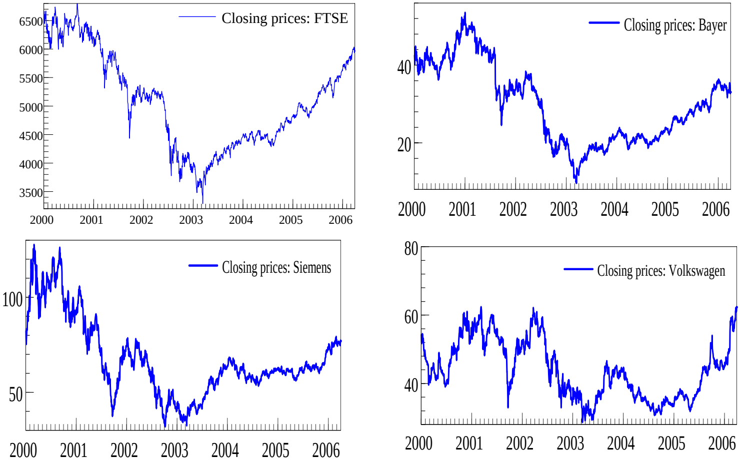

Figure 3: *Daily closing prices of the British FTSE* (03.01.2000 – 30.10.2006) *equity index and the German DAX stocks; Bayer, Siemens and Volkswagen*, (04.01.2000 – 30.10.2006).

Returns

Let S_t denote the stock price at time t.
The log-returns is expressed (in %) as,

$$r_t = 100 * \log\left(\frac{S_t}{S_{t-1}}\right),$$

the continuously compounded daily returns.

Data & descriptive statistics

	Stock index		Stocks	
	FTSE	Bayer	Siemens	VW
std. dev.	1.15	2.31	2.45	2.11
min.	-5.58	-19.42	-10.95	-9.65
max.	5.90	32.99	10.59	10.69
skew.	-0.16	1.10	0.07	-0.01
exc. kurt	2.96	27.71	1.79	2.38
$Q^2(24)$	2329.98 [0.0]	136.64 [0.0]	1724.63 [0.0]	794.20 [0.0]

Table 1: *Descriptive statistics for the daily log-returns for the FTSE stock index* (03.01.2000 – 30.10.2006), *a total of* 1725 *observations and DAX stocks; Bayer, Siemens and Volkswagen* (*VW*) *from* (04.01.2000 – 30.10.2006), *a total of* 1781 *observations.* $Q^2(24)$ *is the Box-Pierce statistic for remaining serial correlation in the squared standardized residuals using 24 lags, with p-values in brackets.*

Long memory tests

	$\lvert r_t \rvert$ FTSE	$\lvert r_t \rvert$ Bayer	$\lvert r_t \rvert$ Siemens	$\lvert r_t \rvert$ VW	r_t^2 FTSE	r_t^2 Bayer	r_t^2 Siemens	r_t^2 VW
$ARFIMA(1,d,1)$	0.50	0.43	0.49	0.42	0.41	0.23	0.38	0.35
V/S test: $m=5$	2.41	1.90	3.27	1.86	1.52	0.66	5.40	1.42
$m=10$	1.33	1.31	2.09	1.25	0.97	0.56	4.43	0.99
$m=32$	0.56	0.58	0.83	0.58	0.43	0.35	2.86	0.50
$m=40$	0.47	0.48	0.69	0.49	0.37	0.32	2.62	0.44
$m=110$	0.21	0.23	0.31	0.25	0.21	0.20	1.83	0.23

Table 2: *$ARFIMA(1,d,1)$ (memory parameter, d) and the Rescaled Variance tests on absolute, $\lvert r_t \rvert$ and square log-returns of the FTSE index and DAX stocks: Bayer, Siemens, Volkswagen (VW). m is the chosen bandwidth.*

- for $ARFIMA(1,d,1)$, $d \in (0, 0.5)$
- for V/S, evaluated statistics $>$ critical value, 0.1869 (at 5% sig. level)

Model performance

HYGARCH	*c*	β	R^2	*MSE*	*MAPE*	*LL*
FTSE	0.02	0.99	0.15	1.24	15300	7.22
Bayer	0.36	0.67	0.03	18.66	2479	7.62
Siemens	1.14	0.42	0.01	18.47	103.40	6.99
VW	2.20	0.22	0.002	59.37	145.50	7.69
FIAPARCH	*c*	β	R^2	*MSE*	*MAPE*	*LL*
FTSE	0.04	1.13	0.18	1.21	11600	7.22
Bayer	-0.94	1.50	0.11	17.02	1913	6.79
Siemens	0.50	0.80	0.03	17.76	95.47	6.69
VW	1.94	0.35	0.003	58.19	131.50	7.30

Table 3: *In sample one-step-ahead forecasting performance of the HYGARCH and FIAPARCH models on log returns. Forecast criteria are the Mincer-Zarnowitz regression (c, β are regression coefficients, R^2, the determination coefficient of regression for the model), MSE is the Mean Absolute Prediction Error, MAPE is the Root Mean Square Error and LL, the logarithmic Loss Function.*

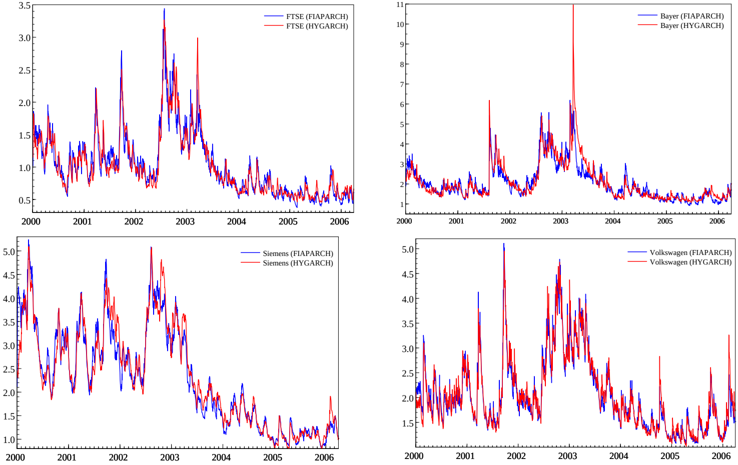

Figure 4: *Plots of the time path of conditional standard deviation for each log-return series based on the FIAPARCH and HYGARCH models under the skewed student-t distributed errors.*

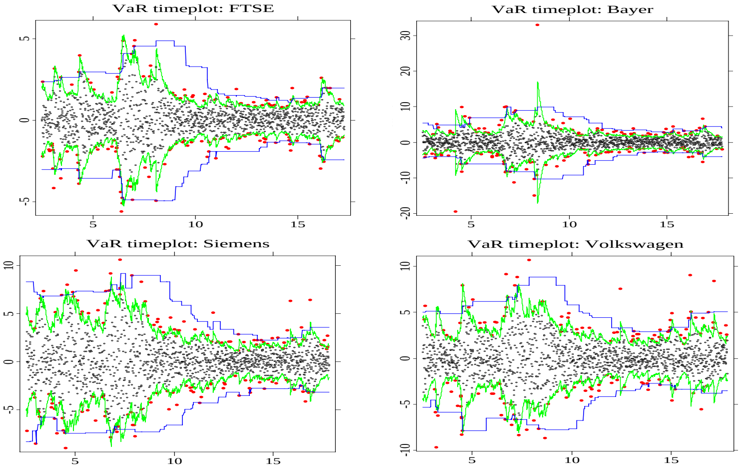

Figure 5: *Time plots of VaR forecasts at* 99% *and* 1% *levels with EDF (blue) and EMA (green) models (decay factor,* $\lambda = 0.94$*, window of* 250 *days). Returns that exceed VaR are in red.*

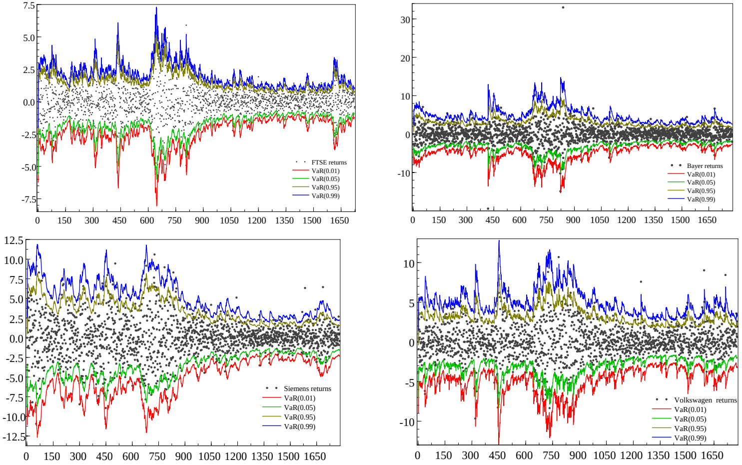

Figure 6: *FIAPARCH one-day-ahead VaR forecast plots for FTSE and DAX stocks log-returns. Quantiles are* $\alpha = 0.01, 0.05$ *and* $\alpha = 0.95, 0.99$ *for long and short positions respectively.*

In-sample *VaR* and *ESF* forecasts

FTSE	$h = 1$ day		$h = 5$ days	
Short position				
α Quantile	0.950	0.990	0.950	0.990
Failure rate ($\frac{N}{T}$)	0.948	0.990	0.948	0.991
Kupiec-LR_1	0.039	0.092	0.048	0.297
P-value	0.842	0.761	0.825	0.587
ESF	1.915	2.409	1.915	2.511
Long position				
α Quantile	0.050	0.010	0.050	0.010
Failure rate ($\frac{N}{T}$)	0.055	0.008	0.054	0.008
Kupiec-LR_1	0.916	0.307	0.761	0.297
P-value	0.338	0.579	0.382	0.585
ESF	-2.241	-2.587	-2.257	-2.587

Table 4: *In-sample Value-at-Risk (VaR) and Expected Shortfall (ESF) evaluation under the skewed student-t distribution - FIAPARCH for FTSE log-returns.*

Bayer	$h = 1$ day		$h = 5$ days	
Short position				
α Quantile	0.950	0.990	0.950	0.990
Failure rate ($\frac{N}{T}$)	0.951	0.992	0.952	0.992
Kupiec-LR_1	0.107	0.884	0.093	0.867
P-value	0.742	0.347	0.759	0.351
ESF	4.556	7.916	4.556	7.916
Long position				
α Quantile	0.050	0.010	0.050	0.010
Failure rate ($\frac{N}{T}$)	0.052	0.008	0.051	0.008
Kupiec-LR_1	0.186	0.470	0.056	0.457
P-value	0.665	0.492	0.811	0.498
ESF	-4.531	-7.131	-4.526	-7.131

Table 5: *In-sample Value-at-Risk (VaR) and Expected Shortfall (ESF) evaluation under the skewed student-t distribution - FIAPARCH for Bayer log-returns. $*$ is the* 5% *confidence level.*

Siemens	$h = 1$ day		$h = 5$ days	
Short position				
α Quantile	0.950	0.990	0.950	0.990
Failure rate ($\frac{N}{T}$)	0.952	0.992	0.953	0.992
Kupiec-LR_1	0.301	0.884	0.407	0.867
P-value	0.583	0.347	0.523	0.351
ESF	4.737	5.781	4.775	5.781
Long position				
α Quantile	0.050	0.010	0.050	0.010
Failure rate ($\frac{N}{T}$)	0.053	0.008	0.052	0.009
Kupiec-LR_1	0.417	0.190	0.314	0.182
P-value	0.518	0.662	0.574	0.669
ESF	-4.468	-5.075	-4.492	-5.075

Table 6: *In-sample Value-at-Risk (VaR) and Expected Shortfall (ESF) evaluation under the skewed student-t distribution - FIAPARCH for Siemens log-returns.*

Volkswagen	$h = 1$ day		$h = 5$ days	
Short position				
α Quantile	0.950	0.990	0.950	0.990
Failure rate ($\frac{N}{T}$)	0.951	0.992	0.951	0.992
Kupiec-LR_1	0.107	0.884	0.093	0.457
P-value	0.742	0.347	0.759	0.498
ESF	4.087	6.166	4.093	6.050
Long position				
α Quantile	0.050	0.010	0.050	0.010
Failure rate ($\frac{N}{T}$)	0.052	0.011	0.052	0.012
Kupiec-LR_1	0.186	0.549	0.206	0.564
P-value	0.665	0.458	0.649	0.452
ESF	-4.359	-5.738	-4.359	-5.738

Table 7: *In-sample Value-at-Risk (VaR) and Expected Shortfall (ESF) evaluation under the skewed student-t distribution - FIAPARCH for Volkswagen log-returns.*

Out-of-sample *VaR* and *ESF* forecasts

$h = 5$ days	FTSE		Bayer		Siemens		VW	
Short position	0.95	0.99	0.95	0.99	0.95	0.99	0.95	0.99
Failure rate ($\frac{N}{T}$)	0.95	0.99	0.94	0.99	0.95	0.97	0.93	0.96
Kupiec-LR_1	0.16	0.09	0.50	0.80	0.01	2.01	1.65	10.39
P-value	0.68	0.75	0.47	0.36	0.91	0.15	0.19	0.00*
ESF	1.59	1.71	3.45	4.32	3.00	4.46	4.12	5.14
Long position	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01
Failure rate ($\frac{N}{T}$)	0.06	0.02	0.04	0.02	0.04	0.01	0.02	0.00
Kupiec-LR_1	1.04	3.64	0.01	2.01	0.50	0.10	7.97	1.14
P-value	0.30	0.05*	0.91	0.15	0.47	0.74	0.00*	0.28
ESF	-1.68	-1.80	-2.86	-3.88	-2.89	-3.23	-3.73	-5.54

Table 8: *EWMA: backtesting results for five-day-ahead out-of-sample VaR and ESF forecasts for the FTSE and DAX stocks log-returns. * indicates a rejection of the model used.*

Out-of-sample *VaR* and *ESF* forecasts

$h = 5$ days	FTSE		Bayer		Siemens		VW	
Short position	0.95	0.99	0.95	0.99	0.95	0.99	0.95	0.99
Failure rate ($\frac{N}{T}$)	0.95	0.98	0.93	0.99	0.93	0.97	0.92	0.97
Kupiec-LR_1	0.01	0.10	0.56	0.09	0.50	2.00	3.26	5.60
P-value	0.91	0.74	0.45	0.75	0.45	0.15	0.07	0.01*
ESF	1.28	1.51	3.25	5.03	2.77	4.46	3.97	5.43
Long position	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01
Failure rate ($\frac{N}{T}$)	0.05	0.01	0.04	0.01	0.05	0.01	0.02	0.00
Kupiec-LR_1	0.03	0.10	0.50	0.10	0.03	0.10	2.87	1.14
P-value	0.85	0.74	0.47	0.74	0.85	0.74	0.09	0.28
ESF	-1.21	-1.24	-3.14	-4.54	-2.82	-3.37	-3.19	-5.54

Table 9: *FIAPARCH: backtesting results for one-day-ahead out-of-sample VaR and ESF forecasts for the FTSE and DAX stocks log-returns. * indicates a rejection of the model used.*

Conclusion

- we explore whether detecting and taking into account long memory in volatility can improve Value at Risk forecast.
- the *FIAPARCH* model perform very well for one day-day time horizon and reasonably well for the five-days time horizon
- Our results suggest that for proper risk valuation of options, the degree of persistence should be investigated and appropriate model that incorporate the existence of such characteristic be taken into account.

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