Backtesting Beyond VaR

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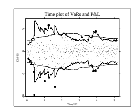
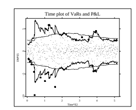


Agenda

- Regulators' Traffic Light Approach (TLA)
- Backtesting the whole forecast distribution
- Backtesting the Tail VaR (expected shortfall)
- Real Life Examples
- Conclusions & Outlook

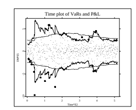


Framework

We generally adopt the JPM RiskMetriks delta-normal approach,i.e.

- all instruments are linear or assumed to be linearized
- the common distribution of risk factors $Y \in IR^d$ is a multivariate normal distribution
- the log price changes

$$Y_{t+1} = lnX_{t+1} - lnX_t \sim N_d(0, \Sigma_t),$$



Estimates for Σ_t: RMA and EMA

RMA : Rectangular Moving Average

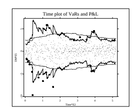
EMA : Exponential Moving Average

- $(n \times d)$ data matrix $\mathcal{X}_t = \{y_i\}_{i=t-n+1,\cdots,t}$, RMA defined by

$$\hat{\Sigma}_t = \frac{1}{n} \mathcal{X}_t^T \mathcal{X}_t$$

- $(n \times d)$ data matrix
$\tilde{\mathcal{X}}_t = \{\ diag(\lambda^d, \lambda^{d-1}, \cdots, \lambda, 1)^{1/2} y_i\}_{i=t-n+1,\cdots,t}$: EMA defined by

$$\hat{\Sigma}_t = (1-\lambda) \tilde{\mathcal{X}}_t^T \tilde{\mathcal{X}}_t$$



VaR for a single instrument

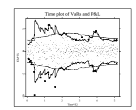
- L_{t+1} rv of (P&L) at time $t+1$, conditional forecast distribution $P_{t+1}; F_{t+1}$ the associated ccdf
- for a single asset with market value x_t and exposure $w_t = \lambda_t \, x_t$, $P_t = \mathcal{L}(L_{t+1} \mid \mathcal{H}_t) = \mathcal{L}(\lambda_t(X_{t+1} - x_t) \mid \mathcal{H}_t)$

$$\mathcal{L}\left(w_t \frac{X_{t+1} - x_t}{x_t} \mid \mathcal{H}_t\right) \approx$$

$$\mathcal{L}(w_t Y_{t+1} \mid \mathcal{H}_t) = N(0, w_t^2 \; \sigma_t^2)$$

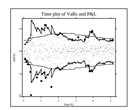
Approximation refers to

$$lnX_{t+1} - lnx_t = \frac{X_{t+1} - x_t}{x_t} + o(X_{t+1} - x_t)$$



VaR for a portfolio of instruments

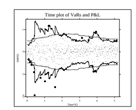
- generalization to linear portfolios is straightforward
- $w_t = (\lambda_t^1 x_t^1, \cdots, \lambda_t^d x_t^d)$ denotes a $d-$dimensional exposure vector
- $w_t^T Y_{t+1} \in \mathcal{P}_{t+1} = \{N(0, \sigma_t^2) : \sigma_t^2 \in [0, \infty)\}$, where $\sigma_t^2 = w_t^T \Sigma_t w_t$.



Mathematical target of VaR models:

$$P_t = \mathcal{L}(L_{t+1} \mid \mathcal{H}_t)$$

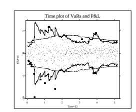
- good Backtesting strategies should stress **two** different issues of $\{P_t\}_{t=1}^{\tau}$
  1. Calibration: measures the quality of substantial components of the VaR model, e.g., adequate choice of risk factors
  2. Resolution: measures the statistical quality of the VaR model, e.g., adequate dynamics and probability models



For an adequate VaR model the realizations

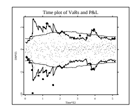
$F_t(L_t)$ should be iid $U[0,1]$

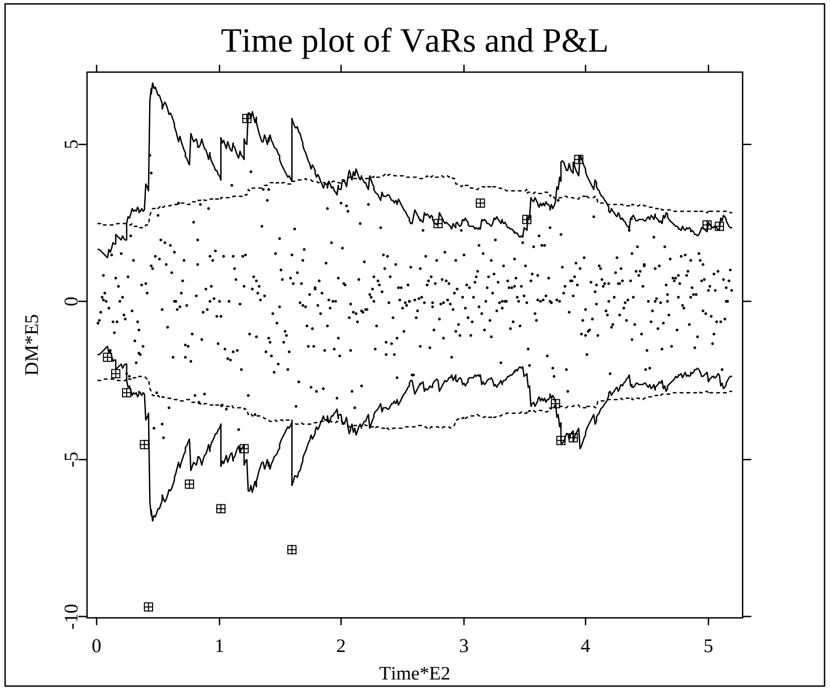
1. identically distributed $U[0,1]$ stands for a good calibration
2. independence stands for a good resolution



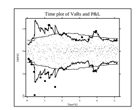
Shortcomings related to the TLA

- binomial or sign statistic is related to a very specific forecast task - limited calibration skills
- resolution properties ignored - no account for clusters of VaR exceedances
- no good support by graphical means at hand - timeplot insufficient
- the involved regulatory penalty function no strict proper scoring rule - even very conservative models are not penalized
- weak substantial interpretation for risk managers



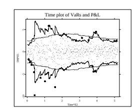


The dots show the observed change of the portfolio values, l_t. The dashed lines show the predicted VaRs based on RMA (99% and 1%).The solid lines show the same for EMA.



Lessons from the practice of Backtesting

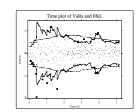
- exploratory means are often sufficient tools for analyzing backtesting data
- clean backtesting is indispensable
- Backtesting on sub portfolio level is essential
- analyze your position λ_t over time

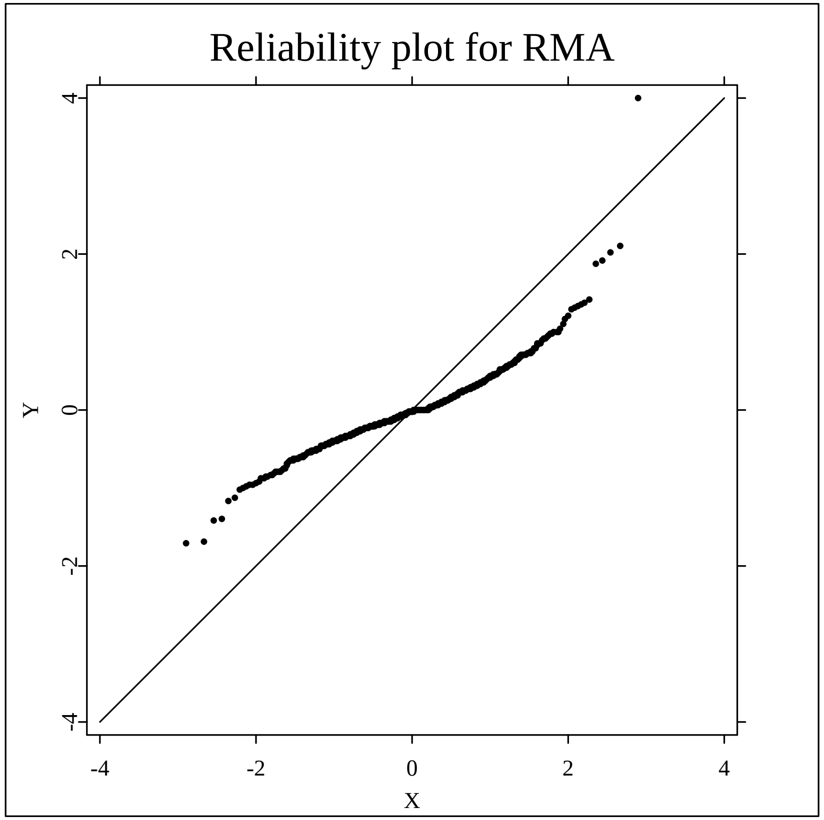


Refinement 1 for the TLA:

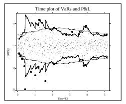
Empirical Calibration Curve

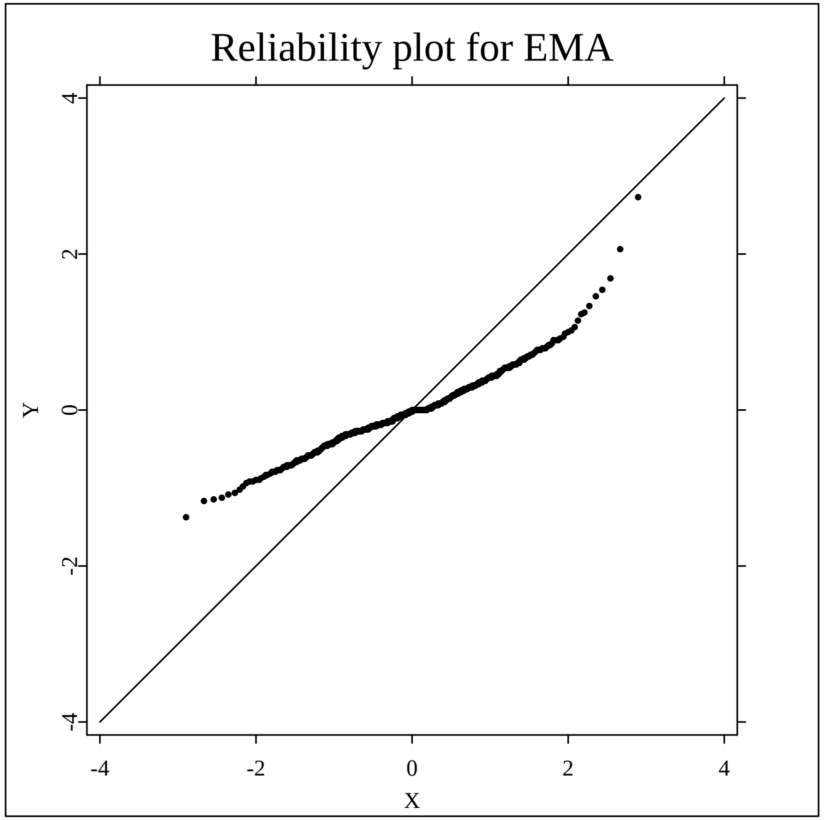
- within JP Morgan's Delta Normal Framework realisations $l_{t+1}/\widehat{VaR}_t$ should behave as *iid* $N(0, 2.33^{-1})$
- Q-Q-plot good exploratory tool to analyze calibration.



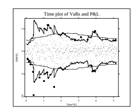


Q-Q plot of $l_{t+1}/\widehat{VaR}_t$ for RMA in 94.





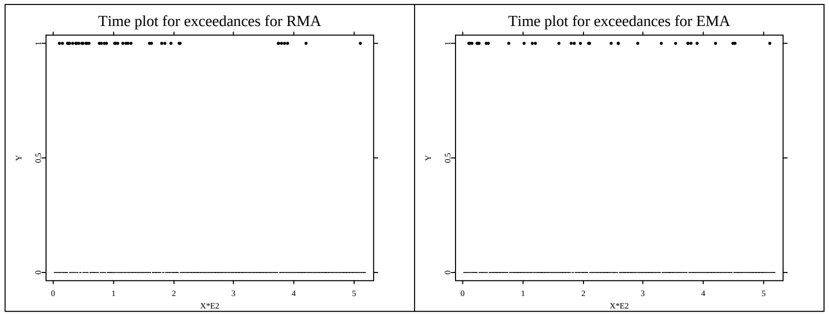
Q-Q plot of $l_{t+1}/\widehat{VaR_t}$ for EMA in 94.



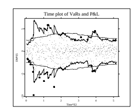
Refinement 2 for the TLA:

Timeplot of Exceedances

- $(t, I(l_{t+1} > \widehat{VaR}_t))_{t=1}^{260}$



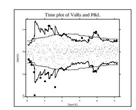
Timeplots of the exceedances over VaR of 80% level for RMA (left) and EMA. The better resolution of EMA is evident.



Refinement 3 for the TLA:

Backtesting standardized Tail-VAR

- Tail-VaR approx. for coherent risk measure
- substantial interpretation as self-insurance, risk neutral pricing for a reinsurance contract
- exceedance's height is incorporated



Delta normal World of JPM:

forecast distribution

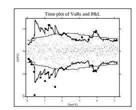
$$\mathcal{L}(L_{t+1} \mid \mathcal{H}_t) = N(0, \hat{\sigma}_t^2)$$

parameter of interest: stand. Tail-VaR

$$E(L_{t+1} \mid L_{t+1} > VaR_t) =$$

$$\sigma_t \; E(L_{t+1}/\sigma_t \mid L_{t+1}/\sigma_t > z_\alpha)$$

$$\vartheta = E(Z_{t+1} \mid Z_{t+1} > u) = \frac{\varphi(u)}{1 - \Phi(u)}$$



Estimator for ϑ

Mean

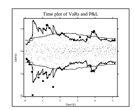
$$\hat{\vartheta} = \frac{\sum_{t=0}^{n} z_{t+1} \; I(z_{t+1} > u)}{\sum_{t=0}^{n} I(z_{t+1} > u)}$$

STD

$$\varsigma^2 = Var(Z_{t+1} \mid Z_{t+1} > u) = 1 + u \cdot \vartheta - \vartheta^2$$

Test statistic

$$T = \sqrt{N(u)} \left(\frac{\hat{\vartheta} - \vartheta}{\hat{\varsigma}} \right) \approx N(0,1)$$

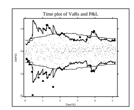


Method	$\vartheta = 1.4$	$\varsigma = 0.46$	T	sign.	nobs
EMA	$\hat{\vartheta} = 1.72$	$\hat{\varsigma} = 1.01$	2.44	0.75%	61
RMA	$\hat{\vartheta} = 1.94$	$\hat{\varsigma} = 1.3$	3.42	0.03%	68

Table 1. $H_0 \ : \vartheta \overset{(<)}{=} 1.4$

Method	$\vartheta = 1.47$	$\varsigma = 0.546$	T	sign.	nobs
EMA	$\hat{\vartheta} = 1.72$	$\hat{\varsigma} = 1.01$	2.01	2.3%	61
RMA	$\hat{\vartheta} = 1.94$	$\hat{\varsigma} = 1.3$	3.04	0.14%	68

Table 2. $H_0 \ : \vartheta \overset{(<)}{=} 1.47$

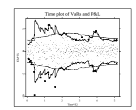


Method	$\vartheta = 1.4$	$\varsigma = 0.46$	T	sign.	nobs
EMA	$\hat{\vartheta} = 1.645$	$\hat{\varsigma} = 0.82$	2.31	1%	60
RMA	$\hat{\vartheta} = 1.83$	$\hat{\varsigma} = 0.93$	3.78	0.00%	67

Table 3. $H_0 : \vartheta \overset{(\le)}{=} 1.4$ - largest outlier excluded

Method	$\vartheta = 1.47$	$\varsigma = 0.546$	T	sign.	nobs
EMA	$\hat{\vartheta} = 1.645$	$\hat{\varsigma} = 0.82$	1.65	5%	60
RMA	$\hat{\vartheta} = 1.83$	$\hat{\varsigma} = 0.93$	3.1	0.15%	67

Table 4. $H_0 : \vartheta \overset{(\le)}{=} 1.47$ - largest outlier excluded



Outlook

- exploit the Panel structure of VaR models
- backtesting based on fixed events with varying probabilities in stead of fixed probabilities and varying intervals
- apply an economic motivated loss function
- analyze the (bivariate) structure of forecast-realization pairs (f_t, x_t)